STUDENT NUMBER: ______________________

TEACHER'S NAME: ______________________

BAULKHAM HILLS HIGH SCHOOL

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

2007

MATHEMATICS

EXTENSION 1

Time allowed – Two hours
(Plus five minutes reading time)

GENERAL INSTRUCTIONS:

- Attempt **ALL** questions.
- Start each of the 7 questions on a new page.
- All necessary working should be shown.
- Write your student number at the top of each page of answer sheets.
- At the end of the exam, staple your answers in order behind the cover sheet.

QUESTION 1 (START ON A NEW PAGE)

Marks

(a) Find $\int \frac{dx}{\sqrt{9-25x^2}}$. **2**

(b) Find the acute angle between the lines $y=3x+4$ *and* $2x+3y=6$. **2**

(c) When the polynomial $P(x)=2x^3-ax+1$ is divided by $x+1$, the remainder is 2. Find a. **2**

(d) Show that $\cot\theta - \cot 2\theta = \operatorname{cosec} 2\theta$ **3**

Hence find the exact value of $\cot 15^o$

(e) Use the substitution $u = 2x + 1$ to find $\int x\,(2x+1)^{10}\,dx$ **3**

QUESTION 2 (START ON A NEW PAGE)

(a) In the expansion of $\left(3x - \frac{1}{x^3}\right)^{12}$, find the term independent of x. **3**

(b) (i) Sketch $y = 2\sin^{-1}(x-3)$, stating clearly the domain and range. **3**

(ii) Find the exact gradient of the function at the point where $x = 3.5$ **2**

(c) A curve has parametric equations:

$x = 3\tan 2\theta$.

$y = \tan\theta$ **2**

Find its Cartesian equation.

(d) Find $\int \cos^2 2x\,dx$. **2**

QUESTION 3 (START ON A NEW PAGE) **Marks**

(a) Simplify $\dfrac{6^x+4^x}{3^x+2^x}$ **2**

(b) Solve the equation
$2x^3-17x^2+40x-16 = 0$ given that it has a double root which is an integer. **4**

(c) Prove by mathematical induction that for all integers n ≥ 1

$2\times 2^0+3\times 2^1+4\times 2^2+.................+(n+1)\times 2^{n-1} = n\times 2^n$ **3**

(d) Find the range of values of x if the series

$\dfrac{2}{1+3x}+\dfrac{6}{(1+3x)^2}+\dfrac{18}{(1+3x)^3}+.......$ has a limiting sum. **3**

QUESTION 4 (START ON A NEW PAGE)

(a) (i) Write $2\sqrt{3}\cos 2t-2\sin 2t$ in the form of $R\cos(2t+\alpha)$ **2**

(ii) A particle moves so that its displacement x metres is given by:

$x=2\sqrt{3}\cos 2t-2\sin 2t$

Show that the motion is simple harmonic and state its period and amplitude **3**

(b) The equation $x+2\tan x=0$ has a root near $\dfrac{3\pi}{4}$.

With $x=\dfrac{3\pi}{4}$ as a first approximation, find using Newton's method once, a second approximation to the root in terms of π **3**

(c) Show that $\dfrac{d}{d\theta}\left(\dfrac{1}{3}\tan^3\theta\right)=\sec^4\theta-\sec^2\theta$ **2**

Hence or otherwise find the value of $\displaystyle\int_0^{\frac{\pi}{4}}\sec^4\theta\ d\theta$ **2**

QUESTION 5 (START ON A NEW PAGE)

Marks

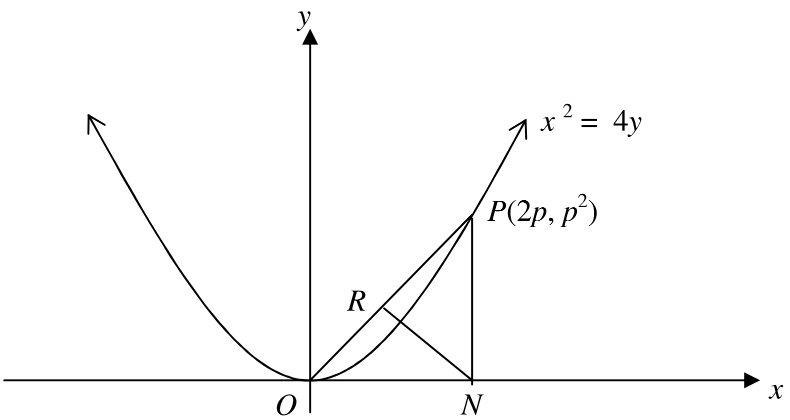
(a) Find $\frac{d}{dx}\log_{10}\left(x^2+1\right)$ **2**

(b) The acceleration $\ddot{x}\ m/s^2$ of a particle moving in a straight line is given by $\ddot{x} = 6\left(1-x^2\right)$. Initially the particle is at $x=-3$ and is moving with velocity $4\ m/s$. **(i)** By using the result $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$ show that

$$v^2 = 12x - 4x^3 - 56$$ **3**

(ii) Does the particle pass through the origin? Justify your answer.

(c)



$P(2p, p^2)$ is a variable point on the parabola $x^2 = 4y$. N is the foot of the perpendicular from P to the x axis. NR is perpendicular to OP

(i) Find the equation of OP **1**

(ii) Find the equation of NR **1**

(iii) Show that the R has coordinates $\left(\frac{8p}{p^2+4}, \frac{4p^2}{p^2+4}\right)$ **2**

(iv) Show that the locus of R is a circle and state its centre and radius **3**

QUESTION 6 (START ON A NEW PAGE)

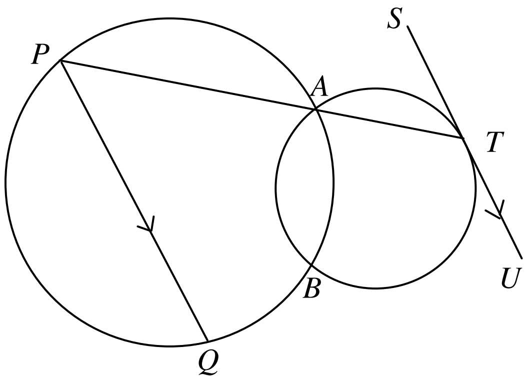
(a) Solve the equation $e^x - e^{-x} = 1$ for x in the exact form. **3**

(b) Given that $(1+ax)^7 + (1+bx)^7 = 2+21x+609x^2+$............. **4**
find the values of a and b

(c) Find the exact value of $\sin(2\tan^{-1}\frac{3}{5})$ **2**

(d) Two circles intersect at A and B. STU is a tangent and is parallel to PQ.

Prove that the points Q, B, T are collinear. **3**



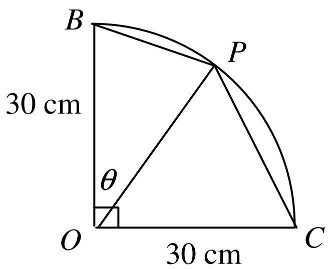
QUESTION 7 (START ON A NEW PAGE)

(a) If $f(x)=\ln(2x+3)$, find an expression for the inverse function $f^{-1}(x)$ **2**

(b) P rotates about O along the arc BC at a constant rate of $\frac{\pi}{60}$ radians/minute.

(i) Show that the area of $OBPC$ is given by $A=450(\sin\theta+\cos\theta)$ **2**

(ii) Find the rate at which the area is changing when $\theta=\frac{\pi}{6}$ **2**



(c) A projectile is fired from a point O on the ground with speed V m/s at an angle α. Given that the equations of motion are

$x=V\cos\alpha\ t$ and $y=V\sin\alpha\ t-\frac{gt^2}{2}$

(i) Find the time of flight of the projectile. **1**

(ii) The projectile is climbing at an angle of 45^o after a time T. Show that $T=\frac{V\sin\alpha-V\cos\alpha}{g}$ **2**

(iii) If T is $\frac{1}{3}$ of the time of flight, find α to the nearest degree. **3**

End of Paper

Ext 1 Trial 2007

Question 1

a) $\int \frac{dx}{\sqrt{9-25x^2}} = \int \frac{dx}{\sqrt{25(\frac{9}{25}-x^2)}}$ $= \frac{1}{5}\sin^{-1}\frac{5x}{3} + C$ ✓ ✓

(b) $m_1 = 3 \quad m_2 = -\frac{2}{3}$

$\tan\theta = \left|\frac{3 - -\frac{2}{3}}{1+3(-\frac{2}{3})}\right|$ ✓

$= \frac{11}{3}$

$\theta = 74°45'$ ✓

(c) $P(-1) = 2$

$2(-1)^3 - a(-1) + 1 = 2$ ✓

$-2 + a + 1 = 2$

$a = 3$ ✓

d) To prove $\cot\theta - \cot 2\theta = \operatorname{cosec} 2\theta$

Proof: $LHS = \frac{\cos\theta}{\sin\theta} - \frac{\cos 2\theta}{\sin 2\theta}$ ✓

$= \frac{\sin 2\theta\cos\theta - \cos 2\theta\sin\theta}{\sin\theta\sin 2\theta}$

$= \frac{\sin(2\theta-\theta)}{\sin\theta\sin 2\theta}$ ✓

$= \operatorname{cosec} 2\theta = RHS$

$\cot 15° - \cot 30° = \operatorname{cosec} 30°$

$\therefore \cot 15° = \cot 30° + \operatorname{cosec} 30°$

$= \sqrt{3} + 2$ ✓

(e) $\int \frac{u-1}{2} \cdot u^{10} \cdot \frac{du}{2}$ ✓ $\quad u = 2x+1$, $du = 2dx$, $x = \frac{u-1}{2}$

$= \frac{1}{4}\int u^{11} - u^{10}\, du$

$= \frac{1}{4}\left[\frac{u^{12}}{12} - \frac{u^{11}}{11}\right] + C$ ✓

$= \frac{1}{48}(2x+1)^{12} - \frac{1}{44}(2x+1)^{11} + C$ ✓

Question 2

(a) $\left(3x - \frac{1}{x^3}\right)^{12}$

$T_{k+1} = {}^{12}C_k\,(3x)^{12-k}\cdot\left(\frac{-1}{x^3}\right)^k$ ✓

$= {}^{12}C_k\,3^{12-k}(-1)^k \cdot x^{12-4k}$

For term indep of x: $12 - 4k = 0$, $k = 3$ ✓

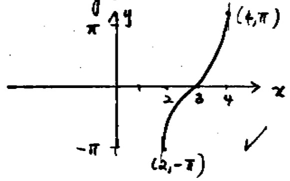
$\therefore T_4 = {}^{12}C_3\,3^9(-1)^3$

$= -220(3)^9$ or -4330260 ✓

(b) (i) $D: -1 \le x - 3 \le 1$

$2 \le x \le 4$ ✓

$R: -\pi \le y \le \pi$ ✓



✓

(ii) $y' = \frac{2}{\sqrt{1-(x-3)^2}}$ ✓

when $x = 3\frac{1}{2}$, $y' = \frac{2}{\sqrt{1-(\frac{1}{2})^2}}$

$= \frac{2}{\sqrt{3}/2} = \frac{4}{\sqrt{3}}$ or $\frac{4\sqrt{3}}{3}$ ✓

(c) $\frac{x}{3} = \tan 2\theta$, $y = \tan\theta$

$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$ ✓

$\frac{x}{3} = \frac{2y}{1-y^2}$

$x(1-y^2) = 6y$ ✓

(d) $\frac{1}{2}\int 1 + \cos 4x\, dx$ ✓

$= \frac{1}{2}\left[x + \frac{\sin 4x}{4}\right] + C$ ✓

Q3

(a) $\frac{6^x + 4^x}{3^x + 2^x} = \frac{2^x\cdot 3^x + 2^x\cdot 2^x}{3^x + 2^x}$ ✓

$= \frac{2^x(3^x + 2^x)}{3^x + 2^x}$

$= 2^x$ ✓

(b) $P(x) = 2x^3 - 17x^2 + 40x - 16$

$P'(x) = 6x^2 - 34x + 40$

For double root $P'(x) = 0$, $P(x) = 0$

$2(3x^2 - 17x + 20) = 0$ ✓

$(3x-5)(x-4) = 0$

$x = 4$ or $5/3$ ✓

root is integer $\therefore x = 4$

$P(4) = 0$

$(x-4)^2(2x-1) = 0$ ✓

$\therefore x = 4, 4, \frac{1}{2}$ ✓

OR Let roots be α, α, β

$2\alpha + \beta = \frac{17}{2}$

$\alpha^2 + 2\alpha\beta = 20$ ✓

$\alpha^2 + 2\alpha(\frac{17}{2} - 2\alpha) = 20$

$\alpha^2 + 17\alpha - 4\alpha^2 = 20$

$3\alpha^2 - 17\alpha + 20 = 0$ ✓

$(3\alpha - 5)(\alpha - 4) = 0$

α integral $\therefore \alpha = 4$ ✓

$\therefore$ Roots are $4, 4, (\frac{17}{2} - 8)$

ie $4, 4, \frac{1}{2}$ ✓

(c) To prove

$2\times 2^0 + 3\times 2^1 + \ldots + (n+1)2^{n-1} = n.2^n$

Proof: test $n = 1$

$LHS = 2\times 2^0 = 2$

$RHS = 1.2^1 = 2$ ✓

$\therefore$ true for $n = 1$

Assume true for $n = k$

ie $2\times 2^0 + 3\times 2^1 + \cdots + (k+1)2^{k-1} = k.2^k$

Prove true for $n = k+1$

ie $2\times 2^0 + 3\times 2^1 + \ldots + (k+2)2^k$

$= k.2^k + (k+2)2^k$ ✓

$= 2^k(k + k + 2)$

$= 2^k(2k+2)$

$= 2^k.2(k+1)$ ✓

$= (k+1)2^{k+1}$

$\therefore$ If true for $n = k$, it will be true for $n = k+1$

Since true for $n = 1$, it will be true for $n = 2, 3, \ldots$ ie for all $n \ge 1$

(Lose 1 if conclusion not there)

(d) $r = \frac{3}{1+3x}$

For limiting sum $|r| < 1$

$\left|\frac{3}{1+3x}\right| < 1$ ✓

$3 < |1+3x|$

$1 + 3x > 3$ or $-1 - 3x > 3$

$x > \frac{2}{3}$ ✓ $\qquad x < -\frac{4}{3}$ ✓

Q4

a) (i) $2\sqrt{3}\cos 2t - 2\sin 2t$

$\equiv R\cos(2t + \alpha)$

$\equiv R\cos 2t\cos\alpha - R\sin 2t\sin\alpha$

$R\cos\alpha = 2\sqrt{3}$

$R\sin\alpha = 2$

$\therefore R = \sqrt{(2\sqrt{3})^2 + 2^2} = 4$

$\tan\alpha = \frac{1}{\sqrt{3}}$, $\alpha = \frac{\pi}{6}$

$\therefore 2\sqrt{3}\cos 2t - 2\sin 2t \equiv 4\cos(2t + \frac{\pi}{6})$ ✓ ✓

(ii) $x = 2\sqrt{3}\cos 2t - 2\sin 2t$

$= 4\cos(2t + \frac{\pi}{6})$

$\dot{x} = -8\sin(2t + \frac{\pi}{6})$

$\ddot{x} = -16\cos(2t + \frac{\pi}{6})$

$= -4x$ ✓

of the form $-n^2x$

∴ S.H.M

period $= \frac{2\pi}{2} = \pi$ ✓

amplitude $= 4$ ✓

(b) $f(x) = x + 2\tan x$

$f'(x) = 1 + 2\sec^2 x$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$= \frac{3\pi}{4} - \frac{\frac{3\pi}{4} + 2\tan\frac{3\pi}{4}}{1 + 2\sec^2\frac{3\pi}{4}}$

$= \frac{3\pi}{4} - \frac{\frac{3\pi}{4} - 2}{1 + 2(-\sqrt{2})^2}$ ✓ ✓

$= \frac{3\pi}{4} - \frac{\frac{3\pi}{4} - 2}{5}$

$= \frac{3\pi}{5} + \frac{2}{5}$ ✓

(c) $\frac{d}{dx}\left(\frac{1}{3}\tan^3\theta\right)$

$= \frac{1}{3} \cdot 3\tan^2\theta \sec^2\theta$ ✓

$= (\sec^2\theta - 1)\sec^2\theta$ ✓

$= \sec^4\theta - \sec^2\theta$

$\therefore \int_0^{\pi/4} \sec^4\theta \, d\theta$

$= \left[\frac{1}{3}\tan^3\theta\right]_0^{\pi/4} + \int_0^{\pi/4} \sec^2\theta \, d\theta$ ✓

$= \frac{1}{3} \cdot 1^3 + \left[\tan\theta\right]_0^{\frac{\pi}{4}}$

$= \frac{1}{3} + 1 - 0$

$= \frac{4}{3}$ ✓

Q5

(a) $\frac{d}{dx}\log_{10}(x^2+1)$

$= \frac{d}{dx}\frac{\ln(x^2+1)}{\ln 10}$

$= \frac{1}{\ln 10} \cdot \frac{2x}{x^2+1}$ ✓ ✓

(b) $\ddot{x} = 6(1 - x^2)$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 6 - 6x^2$

$\frac{v^2}{2} = 6x - 2x^3 + C$ ✓

$x = -3, v = 4 \quad \therefore 8 = -18 + 54 + C$

$-28 = C$ ✓

$\therefore v^2 = 12x - 4x^3 - 56$

(ii) If $x = 0$, $v^2 = -56$ not possible

∴ No ✓

(c) (i) OP: $y = \frac{p^2}{2p}x \quad \therefore y = \frac{p}{2}x$ ✓

(ii) $y - 0 = -\frac{2}{p}(x - 2p)$

$y = -\frac{2}{p}x + 4$ ✓

(iii) Sub (i) into (ii)

$\frac{p}{2}x = -\frac{2}{p}x + 4$

$p^2x = -4x + 8p$ ✓

$x(p^2+4) = 8p$

$x = \frac{8p}{p^2+4}$

$y = \frac{p}{2} \cdot \frac{8p}{p^2+4} = \frac{4p^2}{p^2+4}$ ✓

iv) $\frac{x}{y} = \frac{2}{p} \quad \therefore p = \frac{2y}{x}$ ✓

Sub in x $\quad x = \frac{8\left(\frac{2y}{x}\right)}{\frac{4y^2}{x^2} + 4} = \frac{16yx}{4y^2+4x^2}$

Locus is $4x^2 + 4y^2 = 16y$ } ✓

$x^2 + (y-2)^2 = 4$

centre $(0,2)$, $r = 2$ ✓

Q6

a) $e^x - \frac{1}{e^x} = 1$

$e^{2x} - e^x - 1 = 0$ ✓

$u = e^x \quad u^2 - u - 1 = 0$

$e^x = u = \frac{1 \pm \sqrt{1 - 4.1.(-1)}}{2}$

$e^x > 0 \quad \therefore e^x = \frac{1+\sqrt{5}}{2}$ ✓

$x = \ln\left(\frac{1+\sqrt{5}}{2}\right)$ ✓

(b) $(1+ax)^7 + (1+bx)^7$

$= 1 + {}^7C_1 ax + {}^7C_2 a^2x^2 \ldots$

$+ 1 + {}^7C_1 bx + {}^7C_2 b^2x^2 \ldots$

$\therefore 7a + 7b = 21$ or $a + b = 3$ ✓

${}^7C_2 a^2 + {}^7C_2 b^2 = 609$ or $a^2 + b^2 = 29$ ✓

$a = 3 - b$

$9 - 6b + b^2 + b^2 = 29$

$2b^2 - 6b - 20 = 0$

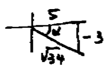
$b^2 - 3b - 10 = 0$

$(b-5)(b+2) = 0$

$\therefore b = 5$ or -2 any pair

$a = -2$ or 5 ✓✓

(c) $\sin\left(2\tan^{-1}\left(-\frac{3}{5}\right)\right)$
α



$= 2\sin\alpha\cos\alpha$

$= 2\left(-\frac{3}{\sqrt{34}}\right) \cdot \frac{5}{\sqrt{34}} = -\frac{30}{34} = -\frac{15}{17}$ ✓ ✓

(d) Join AB, QB, BT

Let $\angle APQ = \alpha$

✓ $\angle ABQ = 180° - \alpha$ (opp. ∠s of cyclic quad. supp.)

✓ $\angle STA = \angle APQ = \alpha$ (alt. ∠s on // lines)

✓ $\angle ABT = \angle STA = \alpha$ (∠ between tangent and chord = ∠ in alt seg.)

$\therefore \angle ABQ + \angle ABT$

$= 180° - \alpha + \alpha$

$= 180°$

∴ QBT is a straight line

ie Q, B, T are collinear

Q7 (a) Let $x = \ln(2y+3)$

$e^x = 2y + 3$ ✓

$f^{-1}(x) = y = \frac{e^x - 3}{2}$ ✓

(b) (i) Area of OBPC

= area of ΔOPB + area of ΔOPC

$= \frac{1}{2} \cdot 30^2 \sin\theta + \frac{1}{2} 30^2 \sin(90° - \theta)$ ✓

$= 450(\sin\theta + \cos\theta)$ ✓

(ii) $\frac{dA}{d\theta} = 450(\cos\theta - \sin\theta)$

$\frac{d\theta}{dt} = \frac{\pi}{60}$ rad/min ✓

$\therefore \frac{dA}{dt} = 450\left(\cos\frac{\pi}{6} - \sin\frac{\pi}{6}\right) \times \frac{\pi}{60}$ cm²/min

$= 450\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right) \times \frac{\pi}{60}$ cm²/min

$= \frac{15\pi}{4}(\sqrt{3} - 1)$ or 8.624 cm²/min

either one ✓

(c) (i) Time of flight when $y = 0$

$t\left(V\sin\alpha - \frac{gt}{2}\right) = 0$

$\therefore t = \frac{2V\sin\alpha}{g}$ ✓

(ii) $\tan 45° = \frac{y}{x}$

$1 = \frac{V\sin\alpha - gT}{V\cos\alpha}$ ✓

$V\cos\alpha = V\sin\alpha - gT$ ✓

$\therefore T = \frac{V\sin\alpha - V\cos\alpha}{g}$

(iii) $\frac{V\sin\alpha - V\cos\alpha}{g} = \frac{1}{3} \cdot \frac{2V\sin\alpha}{g}$ ✓

$3\sin\alpha - 3\cos\alpha = 2\sin\alpha$

$\tan\alpha = 3$ ✓

$\alpha = 72°$ ✓