BAULKHAM HILLS HIGH SCHOOL

HIGHER SCHOOL CERTIFICATE

TRIAL EXAMINATION

2009

MATHEMATICS

EXTENSION 1

Time Allowed - Two hours
(Plus five minutes reading time)

General Instructions

- Attempt ALL questions
- Start each question on a new page
- All necessary working should be shown
- Write your student number at the top of each page of answer sheets
- Board approved calculators may be used
- Write using black or blue pen

Marks

Question 1

a) Evaluate $\sum_{n=0}^{4} (1-2n)$ **1**

b) Solve $\dfrac{x}{x-2} \geq 2$ **3**

c) Find the coordinates of the point P which divides the interval AB **externally** in the ratio 1 : 3, given A= (1,4) and B= (5,2) **2**

d) Evaluate

i) $\displaystyle\int_1^2 \frac{1}{\sqrt{4-x^2}}\, dx$ **2**

ii) $\displaystyle\int_{-1}^0 x\sqrt{1+x}\, dx$, using the substitution $u = 1 + x$ **4**

Question 2

a) Simplify $\dfrac{{}^nC_2}{{}^nC_1}$ **1**

b) A circular oil slick is spreading over a bay, such that its radius is increasing at a constant rate of 0.1 m/s

What is the radius when the area is increasing at $2\pi\ m^2/s$? **3**

c) Simplify $\sin 2\theta\ (\tan\theta + \cot\theta)$ **2**

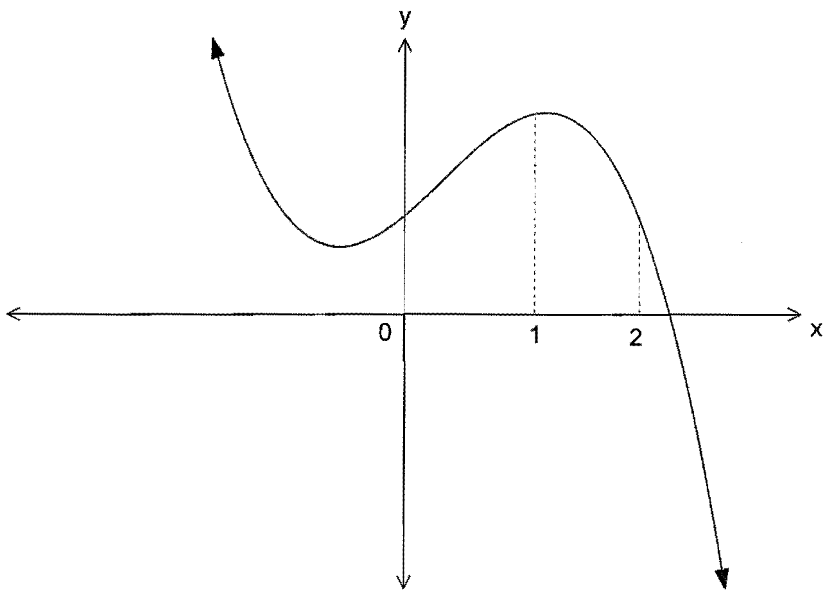
d) Consider the function $f(x) = 3\cos^{-1}\left(\dfrac{x}{2}\right)$

i) Sketch the graph $y = f(x)$ **3**

ii) Find the gradient of the tangent to the curve at the point on it where $x = \sqrt{3}$ **2**

Question 3

a) The polynomial $y = x^2 + 2x + 2 - x^3$ has only one root, as shown on the diagram below



Using one application of Newton's method and x=2 as the first approximation, find a better approximation this root. **3**

b) Solve for $0 \leq \theta \leq 2\pi$: $\cos 2\theta = \cos\theta$ **3**

c) Find the term independent of x in the expression of $\left(x - \frac{1}{2x^3}\right)^{20}$ **3**

d) A particle is moving with acceleration $\ddot{x} = -9x$ and is initially stationary at $x = 4$

i) Find v^2 as a function of x **2**

ii) What is the particle's maximum speed? **1**

Question 4

a) Find $\int \cos^2 2x \quad dx$ **3**

b) Prove, by Mathematical Induction that **4**

$$1 \times 2^2 + 2 \times 3^2 + 3 \times 4^2 + \dots\dots\dots + n(n+1)^2 = \frac{1}{12} n(n+1)(n+2)(3n+5)$$

for all positive integers n

c) i) Express $\cos\theta - \sqrt{3}\sin\theta$ in the form R cos $(\theta + \alpha)$ where $R > 0$ and $0 < \alpha < \pi/2$ **2**

ii) Solve for $0 \le \theta \le 2\pi$, $\cos\theta - \sqrt{3}\sin\theta = 1$ **2**

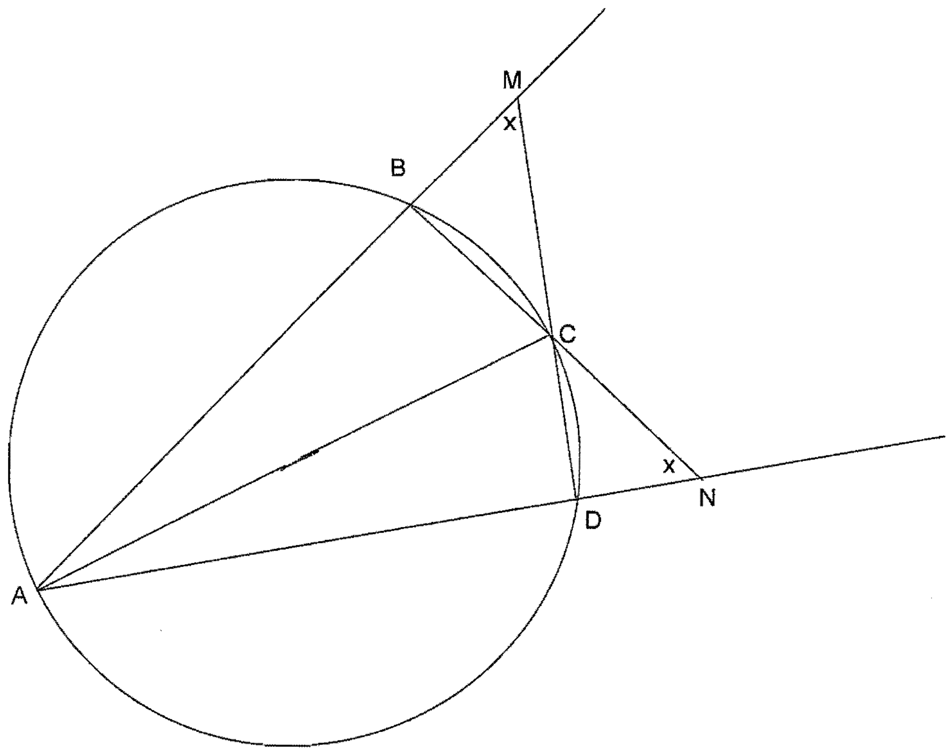
iii) What is the maximum value of $\cos\theta - \sqrt{3}\sin\theta$? **1**

Question 5

a) If $\alpha = \sin^{-1}\left(\frac{8}{17}\right)$ and $\beta = \tan^{-1}\left(\frac{3}{4}\right)$ calculate the exact value of sin $(\alpha - \beta)$ **3**

b) Find the greatest coefficient in the expansion of $(5 + 2x)^9$ **4**

c)



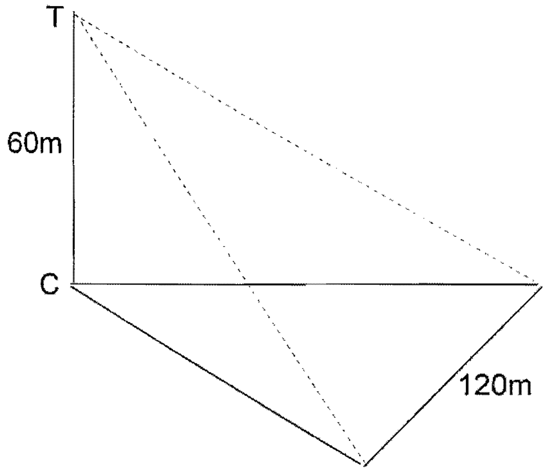
In the figure, *ABM, BCN* and *ADN* are straight lines and $\angle AMD = \angle BNA = x$

i) Copy the diagram and prove that $\angle ABC = \angle ADC$ **3**

ii) Hence, prove that *AC* is a diameter. **2**

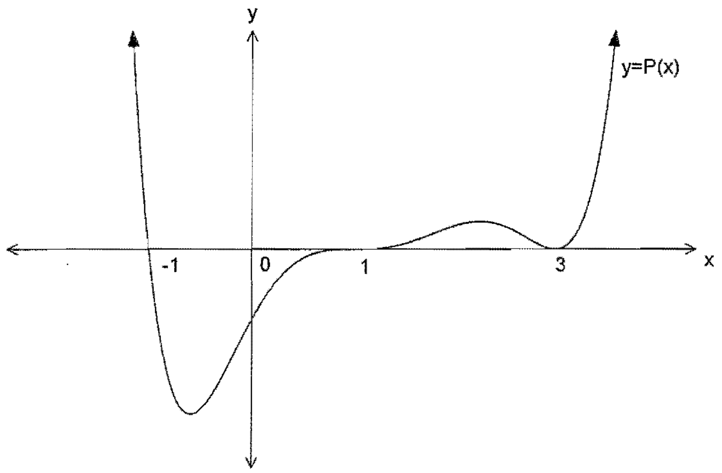
Question 6

a) The angles of elevation of the top of a tower TC, 60m high are measured from two points, *A* and *B*, which are 120m apart. *(A, B and C are all on level ground).* These angles of elevation are found to be 30° from A and 53° from B. **5**

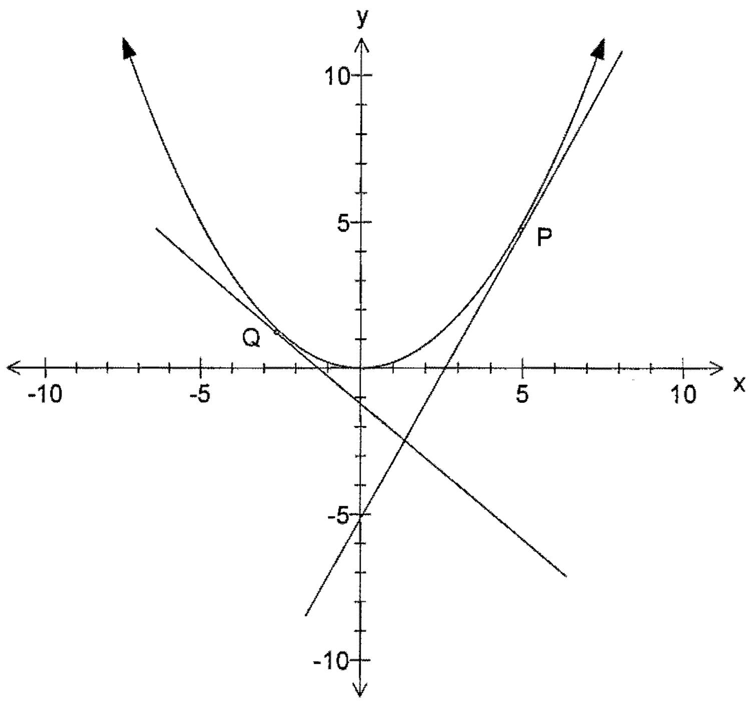


If *A* bears 038°T from the foot of the tower, find the possible bearings of *B* from the tower. Answer correct to nearest degree. (Copy the diagram first)

b) Write down a possible equation $y = P(x)$ for the polynomial function sketched below. **3**



c) $P(2ap,\ ap^2)$ and $Q(2aq,\ aq^2)$ are two points on the parabola $x^2 = 4ay$



The tangents at P and Q intersect at 45°

i) Show that the gradient of the tangent at P is p **1**

ii) Show that $|\ p - q\ | = |\ 1 + pq\ |$ **1**

iii) If $p = 2$, evaluate q **2**

Marks

Question 7

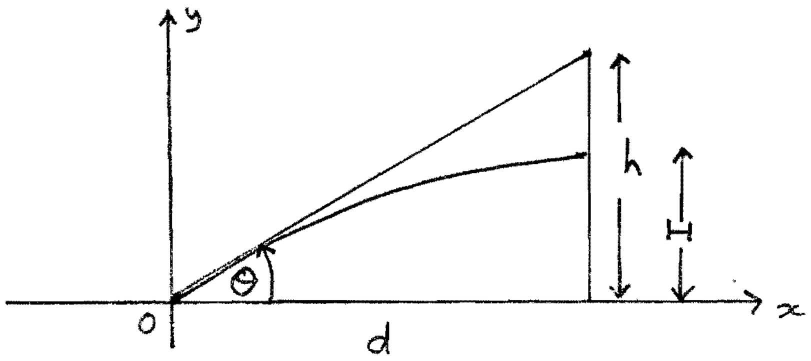
a) The roots of $x^3 + kx^2 - 54x - 216 = 0$ form a geometric progression. Find the roots. **4**

b) Evaluate $\lim_{x \to 0} \frac{\sin 3x}{5x}$ **1**

c) A target is hung on a wall at a height of h metres.

A small cannon, which fires a lead slug, is located on the floor, d metres from the wall. The initial velocity, V, at which the slug is fired is adjustable.

The cannon is aimed at the bullseye on the target at an angle of elevation of θ degrees. At the instant the cannon is fired, the target is released and falls vertically downwards under the force of gravity, g



Given that $\ddot{x} = 0$ and $\ddot{y} = -g$:

i) Show that the position of the lead slug at time t is given by **2**

$$x = Vt\cos\theta \text{ and } y = \frac{-gt^2}{2} + Vt\sin\theta$$

ii) Show that the slug hits the wall at a vertical height of **2**

$$H = \frac{-gd^2\sec^2\theta}{2V^2} + d\tan\theta$$

iii) Experiments with the cannon show that the slug always hits the bullseye, regardless of the initial velocity. Explain why this is always so. **3**

End of examination

Ext 1 09 Trial

(Final copy)

SOLUTIONS.

Q1.

a) $1 + (-1) + (-3) + (-5) + (-7)$

$= \underline{-15}$ (1)

b) $\frac{x}{x-2} \times (x-2)^2 \geq 2(x-2)^2$ (1)

(NOTE: $x \neq 2$)

$x^2 - 2x \geq 2x^2 - 8x + 8$

$0 \geq x^2 - 6x + 8$

$0 \geq (x-4)(x-2)$

$(x-4)(x-2) \leq 0$ (1)

Roots: $x = 4, 2$

Soln:

$\underline{2 < x \leq 4}$ (1)

c) $A(1,4)$ $B(5,2)$

$1 : -3$

$x = \frac{1(5) - 3(1)}{1-3}$, $y = \frac{1(2) - 3(4)}{1-3}$

$= \frac{2}{-2}$ $= \frac{-10}{-2}$

$= -1$ $= 5$

$\therefore \underline{P = (-1, 5)}$ (1) x-value (1) y-value

d) i) $\left[\sin^{-1}\frac{x}{2} \right]_1^2$ (1)

$= \sin^{-1} 1 - \sin^{-1}\frac{1}{2}$

$= \frac{\pi}{2} - \frac{\pi}{6}$

$= \frac{\pi}{3}$ (1)

ii) $\int_{-1}^{0} x\sqrt{1+x}\, dx$

$u = 1 + x$ $x = u - 1$

$du = dx$

If $x = -1$, $u = 0$

If $x = 0$, $u = 1$ (1)

$= \int_0^1 (u-1) \cdot u^{1/2} \, du$ (1)

$= \int_0^1 u^{3/2} - u^{1/2} \, du$

$= \left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} \right]_0^1$ (1)

$= \left(\frac{2}{5} - \frac{2}{3} \right) - (0 - 0)$

$= -\frac{4}{15}$. (1)

Q2. [11 marks]

a) $\frac{n!}{(n-2)!\,2!} \div \frac{n!}{1!\,(n-1)!}$

$= \frac{n!}{2!\,(n-2)!} \times \frac{(n-1)!}{n!}$

$= \frac{n-1}{2!}$

$= \frac{n-1}{2}$ (1)

b) $A = \pi r^2$ $\frac{dA}{dr} = 2\pi r$

$\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$ (1)

$2\pi = 2\pi r \times 0.1$ (1)

$2\pi = 0.2\pi r$

$\underline{r = 10 \text{ m}}$ (1)

c) $2\sin\theta\cos\theta \left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} \right)$ (1)

$= 2\sin^2\theta + 2\cos^2\theta$

$= 2(\sin^2\theta + \cos^2\theta)$

$= 2 \times 1$

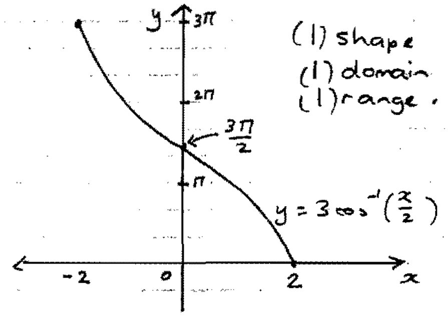
$= \underline{2}$. (1)

d) $f(x) = 3\cos^{-1}\left(\frac{x}{2}\right)$

(i) D: $-1 \leq \frac{x}{2} \leq 1$

$-2 \leq x \leq 2$

R: $0 \leq f(x) \leq 3\pi$



(1) shape
(1) domain
(1) range.

(ii) $\frac{dy}{dx} = 3 \cdot \frac{-1}{\sqrt{4-x^2}}$ (1)

$= \frac{-3}{\sqrt{4-3}}$

$= \underline{-3}$ (1)

Q3.

a) $a_1 = a_0 - \frac{f(a_0)}{f'(a_0)}$

$= 2 - \frac{f(2)}{f'(2)}$ (1)

$a_0 = 2$

$f(2) = 4+4+2-8 = 2$

$f'(x) = 2x+2-3x^2$

$f'(2) = 4+2-12 = -6$

$= 2 - \frac{2}{-6}$ (1)

$= 2\frac{1}{3}$ (1)

b) $2\cos^2\theta - 1 = \cos\theta$

$2\cos^2\theta - \cos\theta - 1 = 0$

$(2\cos\theta + 1)(\cos\theta - 1) = 0$ (1)

$\cos\theta = -\frac{1}{2}$ $\quad \cos\theta = 1$

$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}, 0, 2\pi$

(1) (1)

c) $T_{k+1} = {}^{20}C_k \cdot x^{20-k} \cdot (2x^3)^{-k} \cdot (-1)^k$ (1)

$= {}^{20}C_k \cdot x^{20-k} \cdot 2^{-k} \cdot x^{-3k} \cdot -1$

$= -{}^{20}C_k \cdot x^{20-4k} \cdot 2^{-k}$

indep. of x when $20-4k=0$ ie. $k=5$ (1)

Term indep. of x is:

$T_6 = -{}^{20}C_5 \cdot x^0 \cdot 2^{-5} = -{}^{20}C_5 \cdot 2^{-5}$ (1)

(or -484.5)

d) (i) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \ddot{x}$

$= -9x$

$t=0$, $x=4$, $v=0$

$v^2 = 2\int -9x \cdot dx$

$v^2 = -9x^2 + c$ (1)

$0 = -9(16) + c$

$c = 144$ (1)

$v^2 = -9x^2 + 144$

(ii) ~~Max speed when $\ddot{x}=0$~~

~~ie. when $x=0$~~

Max $v^2 = 144$

$\therefore$ Max $|v| = 12$. — (1)

Q4.

a) $\int \cos^2 2x \cdot dx$

$= \int \frac{1}{2}(1+\cos 4x) \cdot dx$ (1)

$= \frac{1}{2}\left(x + \frac{1}{4}\sin 4x\right) + c$ (2)

b) If $n=1$ LHS $= 1\times 2^2 = 4$

RHS $= \frac{1}{12} \cdot 1 \cdot 2 \cdot 3 \cdot 8 = 4$

$\therefore$ LHS = RHS, so true for $n=1$. (1) Prove for $n=1$

Assume true for $n=k$

ie Assume

$1\times 2^2 + 2\times 3^2 + 3\times 4^2 + \ldots + k(k+1)^2 = \frac{1}{12}k(k+1)(k+2)(3k+5)$

Need to prove true for $n=k+1$

ie. Prove (1) Correct assume.. AND need to prove..

$1\times 2^2 + 2\times 3^2 + 3\times 4^2 + \ldots + k(k+1)^2 + (k+1)(k+2)^2$

$= \frac{1}{12}(k+1)(k+2)(k+3)(3k+8)$

LHS $= \frac{1}{12}k(k+1)(k+2)(3k+5) + (k+1)(k+2)^2$

by assumption

$= \frac{1}{12}(k+1)(k+2)\left(k(3k+5) + 12(k+2)\right)$ (1)

$= \frac{1}{12}(k+1)(k+2)(3k^2+17k+24)$ (1)

$= \frac{1}{12}(k+1)(k+2)(k+3)(3k+8)$

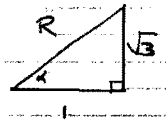
= RHS

$\therefore$ If true for $n=k$, then also true for $n=k+1$.
Now statement is true for $n=1$.
$\therefore$ Also true for $n = 2, 3, 4 \ldots$
By induction, true for all positive integers n.
[-1 if conclusion inappropriate]

c) (i) $R\cos(\theta+\alpha) = R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

$R\cos\alpha = 1/R$

$R\sin\alpha = \sqrt{3}/R$



$\alpha = \frac{\pi}{3}$, $R = 2$. $\quad \alpha \leftarrow (1)$, $R \leftarrow (1)$

$\therefore \cos\theta - \sqrt{3}\sin\theta = 2\cos\left(\theta+\frac{\pi}{3}\right)$

(ii) $2\cos\left(\theta+\frac{\pi}{3}\right) = 1 \qquad 0 \le \theta \le 2\pi$

$\frac{\pi}{3} \le \theta + \frac{\pi}{3} \le \frac{7\pi}{3}$

$\cos\left(\theta+\frac{\pi}{3}\right) = \frac{1}{2}$

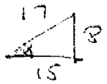
$\theta + \pi/3 = \pi/3, \frac{5\pi}{3}, \frac{7\pi}{3}$

$\theta = 0, \frac{4\pi}{3}, 2\pi$. (2)

(iii) 2 (1)

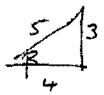
Q5.

a) $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$ (1)



$= \frac{8}{17}\cdot\frac{4}{5} - \frac{15}{17}\cdot\frac{3}{5}$ (2)

-1 per error



$= \frac{-13}{85}$

~~b) T_{k+1}~~

b) Ratio of terms, $\frac{T_{k+1}}{T_k} = \frac{n-k+1}{k}\cdot\frac{b}{a}$

$= \frac{10-k}{k}\cdot\frac{2x}{5}$

Ratio of coeff's: $\frac{10-k}{k}\cdot\frac{2}{5} > 1$ for coeffs. increasing (1)

$20 - 2k > 5k$

$20 > 7k$ (1)

$k < 2\frac{6}{7} \qquad k = 2 \ldots$ greatest such integer (1)

Max coeff. (is in T_3) $= {}^9C_2 . 5^7 . 2^2$ (1)

$= 11\,250\,000$

c) (i) $\angle BCM = \angle DCN$ (vertically opp. $\angle$s equal) (1)

$\angle ABC = x + \angle BCM$
$\angle ADC = x + \angle DCN$ } ext. $\angle$ of Δ = sum of int. opp. $\angle$s (1)

$\therefore \angle ABC = \angle ADC$ (addition of equals). (1)

(ii) $\angle ABC + \angle ADC = 180$ (opp. $\angle$s of cyclic quad. supplementary) (1)

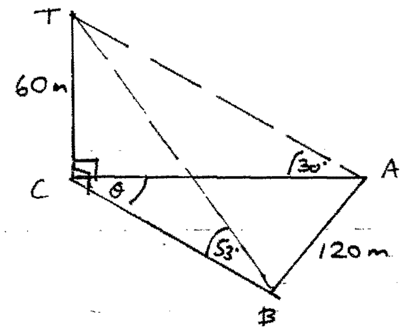
but $\angle ABC = \angle ADC$ from (i)

$\therefore \angle ABC = \angle ADC = 90°$ (1)

$\therefore$ AC is a diameter ($\angle$ in semicircle is 90°).

Q6.

a)



$\tan 30° = \frac{60}{AC}$ $\quad AC = \frac{60}{\tan 30°} \doteqdot 103.92$

$\tan 53° = \frac{60}{BC}$ $\quad BC = \frac{60}{\tan 53} \doteqdot 45.21$ (1)

$$\cos\theta = \frac{AC^2 + BC^2 - AB^2}{2 \cdot AC \cdot BC}$$

$$= \frac{103.92^2 + 45.21^2 - 120^2}{2 \times 103.92 \times 45.21}$$ (1)

$= -0.1657$

$\theta = 100°$ (nearest °). (1)

B may be 100° clockwise from A

or 100° anticlockwise from A

∴ Bearing of B = 138°T or 298°T (1) both bearings

b) $y = (x+1)(x-1)^3(x-3)^2$ (3)

c) (i) $y = \frac{x^2}{4a}$ $\quad \frac{dy}{dx} = \frac{2x}{4a} = \frac{2(2ap)}{4a}$ at P.

$= p$. (1)

(ii) At P : $m_1 = p$ from above

At Q : $m_2 = q$ similarly

∴ $\tan 45° = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$ [illegible]

$$1 = \left|\frac{p-q}{1+pq}\right|$$ (1) (must show tan 45°).

$|1+pq| = |p-q|$

(iii) $|1+2q| = |2-q|$

$1 + 2q = 2 - q$ or $1 + 2q = -2 + q$

$3q = 1$ $\quad q = -3$ (1)

$q = \frac{1}{3}$ (1)

Q7.

a) Let roots $= \frac{\alpha}{r}, \alpha, \alpha r$

Sum $\frac{\alpha}{r} + \alpha + \alpha r = -k$

Prod. $\alpha^3 = 216$

$\alpha = 6$ (1)

Since $\alpha = 6$ is a root,

$216 + 36k - 324 - 216 = 0$

$36k = 324$

$k = 9$ (1)

∴ $\frac{6}{r} + 6 + 6r = -9$

$6 + 6r + 6r^2 = -9r$

$6r^2 + 15r + 6 = 0$

$2r^2 + 5r + 2 = 0$

$(2r+1)(r+2) = 0$

$r = -\frac{1}{2}$ or -2 (1)

∴ Roots $\frac{6}{r}, 6, 6r$

$= -3, 6, -12$ (1)

b) $\frac{3}{5}$ (1)

c) (i)

$\ddot{x} = 0$

$\dot{x} = \int 0 \, dt$

$= c$

$t=0$, $\dot{x} = V\cos\theta$

$\therefore \dot{x} = V\cos\theta$

$x = \int V\cos\theta \, dt$

$= (V\cos\theta)t + c$

$t=0$, $x=0$: $0 = 0 + c$

$c = 0$

$x = Vt\cos\theta$

(1)

$\ddot{y} = -g$

$\dot{y} = \int -g \, dt$

$= -gt + c$

$t=0$, $\dot{y} = V\sin\theta$

$V\sin\theta = 0 + c$

$\dot{y} = -gt + V\sin\theta$

$y = \int -gt + V\sin\theta \, dt$

$= -\frac{1}{2}gt^2 + (V\sin\theta)t + c$

$t=0$, $y=0$: $0 = 0 + 0 + c$

$y = -\frac{1}{2}gt^2 + Vt\sin\theta$ (1)

(ii) The slug hits the wall when $\begin{cases} x = d \\ y = H \end{cases}$:

$d = Vt\cos\theta$

$t = \frac{d}{V\cos\theta}$ —— (1)

Then

$H = -\frac{1}{2}g \cdot \left(\frac{d}{V\cos\theta}\right)^2 + V \cdot \frac{d}{V\cos\theta} \cdot \sin\theta$

$= -\frac{1}{2} \cdot g \cdot \frac{d^2}{V^2\cos^2\theta} + d \cdot \tan\theta$

$= \frac{-gd^2}{2V^2\cos^2\theta} + d\tan\theta$ (1)

ie. $H = \frac{-gd^2}{2V^2} \cdot \sec^2\theta + d\tan\theta$

(iii) Target falls vertically from rest

Initial: $t=0$, $y=h$, $\dot{y}=0$, $\ddot{y}=-g$

$\therefore$ $\ddot{y} = -g$

$\dot{y} = -gt + c$

$t=0$, $\dot{y}=0$: $0 = 0 + c$

$c = 0$

$\dot{y} = -gt$

$y = -\frac{1}{2}gt^2 + c$

$t=0$, $y=h$: $h = 0 + c$

$c = h$

$y = -\frac{1}{2}gt^2 + h$ ← height of target. (1)

(or $y = h - \frac{1}{2}gt^2$)

When the slug hits the wall, $t = \frac{d}{V\cos\theta}$

At this time, $y = -\frac{1}{2}g \cdot \left(\frac{d}{V\cos\theta}\right)^2 + h$

$= \frac{-gd^2}{2V^2\cos^2\theta} + h$

$= \frac{-gd^2}{2V^2} \cdot \sec^2\theta + h$ (1)

but $\tan\theta = \frac{h}{d}$, $\therefore h = d\tan\theta$

$y = \frac{-gd^2}{2V^2} \cdot \sec^2\theta + d\tan\theta$ (1)

ie. $y = H$

Height of target → y; ← Height of slug

$\therefore$ It will always hit.