Candidate Number: ....

BAULKHAM HILLS HIGH SCHOOL

Higher School Certificate

2010

Trial Examination

Mathematics

Extension I

General Instructions

- Exam time – 2 hours
- Reading time – 5 minutes
- Start each question on a new page
- All necessary working should be shown
- Write your student number at the top of each page of your answer booklet
- Board approved calculators may be used
- Write, using black or blue pen

Total Marks: 84

Attempt ALL questions

Question 1 (12 marks) **Marks**

a) Evaluate $\int_0^{\frac{\pi}{2}} \cos^2 x \, dx$ **2**

b) Solve the equation $\cos 2\theta = \sin\theta$ for $0 \le \theta \le 2\pi$ **3**

c) Solve the inequality $\dfrac{3x+4}{x-5} \ge 2$ **2**

d) By using the substitution $w = t^2 - 2$, evaluate $\displaystyle\int_{-1}^{14} \frac{w\,dw}{\sqrt{w+2}}$ **3**

e) A group of six goats is to be chosen from 10 goats.
In how many ways can the group be chosen if: **2**
i) 2 particular goats are included in the group
ii) 1 particular goat is excluded from the group

Question 2 (12 marks) - Start a new page

a) A (4,10), B (-3,1), C (5,7) are the vertices of triangle ABC and E is the midpoint of the side BC. **3**

Find the value of $\tan\theta$ where $\theta = \angle AEC$

b) If α β and γ are the roots of the equation $2x^3 + 5x - 3 = 0$ find the values of

i) $\alpha + \beta + \gamma$ **1**

ii) $\alpha^2 + \beta^2 + \gamma^2$ **1**

iii) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ **1**

c) P(x) is a monic polynomial of degree 3. P(x) has the quadratic factor $x^2 - 1$ and when P(x) is divided by $x - 2$ the remainder is -9.
Form an equation for P(x) and hence solve P(x) = 0 **2**

d) For the expansion $\left(x^2 + \dfrac{4}{x}\right)^{30}$ find which term

i) is independent of x **2**

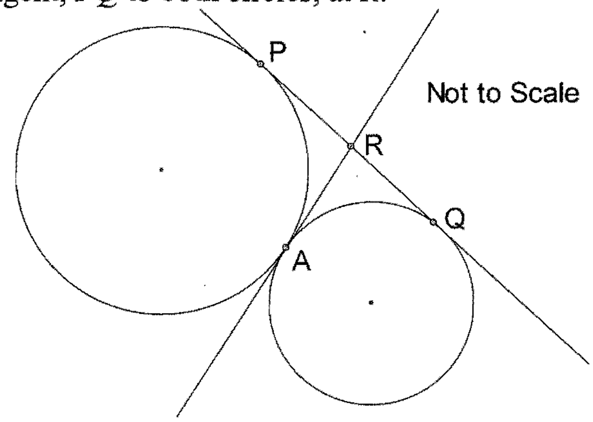
ii) has the greatest coefficient **2**

Question 3 (12 marks) - Start a new page **Marks**

a) Find $\frac{d}{dx}\left(\tan^{-1} x\right)^2$ and hence evaluate **3**

$$\int_{-1}^{\sqrt{3}} \frac{\tan^{-1} x}{1+x^2}\,dx$$

b) A tangent at the point of contact A of two circles (which touch externally) meets a common tangent, PQ to both circles, at R.



Prove that

i) R is the midpoint of PQ **2**

ii) PQ subtends a right angle at A **3**

c) By using the expansions of $\cos(A+B)$ and $\cos(A-B)$

i) find an expression for $\sin x \sin 3x$ **2**

ii) hence evaluate $\int_0^{\frac{\pi}{4}} \sin x \sin 3x\, dx$ **2**

Question 4 (12 marks) - Start a new page

a) If the chance that any one of 6 telephone lines is busy at any instant is $\frac{1}{3}$

i) What is the chance that exactly 4 of the lines are busy? **2**

ii) Determine the probability that at most two of the lines are busy **2**

b) The volume of a sphere is increasing at the rate of $5\text{cm}^3/\text{s}$. **3**
At what rate is the surface area increasing when the radius is 20cm.

c) A particle moves in a straight line and its position in metres at anytime t seconds is given by $x = 3\cos 2t - 4\sin 2t$

i) by expressing the motion in terms of $A\cos(nt+\alpha)$ **3**
Show that the motion is simple harmonic.

ii) Find the particle's greatest speed **2**

Question 5 (12 marks) - Start a new page **Marks**

a) Use mathematical induction to prove the identity
$1.2 + 2.3 + 3.4 + \ldots\ldots. + n(n+1) = \frac{1}{3}n(n+1)(n+2)$ **6**

hence determine the limit of $\frac{1}{n^3}\sum_{k=1}^{n} k(k+1)$ as n approaches infinity

b) Newton's law of cooling states that the rate of change of the temperature θ of a body at any time t is proportional to the difference in temperature of the body and the temperature m of the surrounding medium, i.e. $\frac{d\theta}{dt} = k(\theta - m)$ where k is a constant.

i) Show that $\theta = m + Ae^{kt}$ where A is a constant, satisfies this equation. **1**

ii) If the temperature of the surrounding air is 40°C and the temperature of the body drops from 170°C to 105°C in 45 mins, find the temperature of the body in another 90 minutes. (to 2 decimal places) **3**

iii) Find the time taken for the temperature of the body to drop to 80°C (to the nearest minute) **2**

Question 6 (12 marks) - Start a new page

a) Solve $\log_e(\log_e x) = 0$ (in exact form) **1**

b) Find the equation of the normal at the point $T(2at, at^2)$ on the parabola $x^2 = 4ay$.
Hence determine the value(s) of t for the equations of the normals to this parabola to pass through the point $(-12a, 15a)$. **5**

c) The acceleration of a particle P is given by the equation
$\frac{d^2x}{dt^2} = 8x(x^2+1)$ where x is the displacement in cm, of P from a fixed point O after t seconds .
Initially, P is at the origin moving with velocity $-2cm/s$.

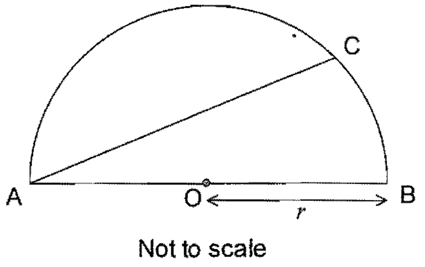
i) Show that the speed of the particle is $2(x^2 + 1)\ cm/s$ and hence find an expression for x in terms of t **5**

ii) Determine the displacement of P after $\frac{\pi}{8}$ seconds **1**

Question 7 (12 marks) - Start a new page **Marks**

a) Evaluate $\int_0^1 \frac{e^{\cos^{-1}x}}{\sqrt{1-x^2}}dx$ 2

b)



AB is the diameter of a semi circle with radius r

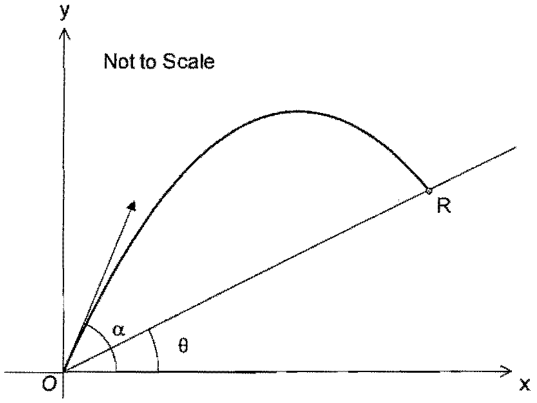
The chord AC divides the semicircle into two regions of equal area.

i) By letting $C\hat{A}B = \theta$ radians 2
Prove that $2\theta + \sin 2\theta = \frac{\pi}{2}$

ii) Show that $\theta = 0.4$ is a good approximation to the solution of $2\theta + \sin 2\theta = \frac{\pi}{2}$ 1

iii) Use Newton's method once to find an improved solution for the value of θ (to 2 significant figures) 2

c)



A stone is projected from O with velocity V at an angle α above the horizontal.

A straight road goes through O at an angle θ above the horizontal, where $\theta < \alpha$

The stone strikes the road at R.
Air resistance is to be ignored and the acceleration due to gravity is g

i) Given that the equations of motion of the stone are 1

$$\left.\begin{aligned} x &= vt\cos\alpha \\ y &= vt\sin\alpha - \frac{gt^2}{2} \end{aligned}\right\} \text{ Do NOT prove these results.}$$

Show that the Cartesian equation for the motion is

$$y = x\tan\alpha - \frac{gx^2\sec^2\alpha}{2v^2}$$

ii) If R is the point (x,y) express x and y in terms of RO and θ 2

Hence show that the range RO of the stone up the road is given by

$$RO = \frac{2v^2\cos\alpha\sin(\alpha-\theta)}{g\cos^2\theta}$$

iii) Find an expression for α when RO is a maximum and interpret this result. 2

End of Paper

Solutions Trial Ext. 2010. /84.

Question one.

a) $\int_0^{\frac{\pi}{2}} \cos^2 x\,dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} (1+\cos 2x)\,dx$ ①

$= \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}}$

$= \frac{1}{2}\left(\frac{\pi}{2} + \frac{\sin \pi}{2} - 0\right)$

$= \frac{\pi}{4}$ ①

b) $\cos 2\theta = \sin\theta \qquad 0 \le \theta \le 2\pi$

$1 - 2\sin^2\theta = \sin\theta$

$2\sin^2\theta + \sin\theta - 1 = 0$ ①

$(2\sin\theta - 1)(\sin\theta + 1) = 0$

$\sin\theta = \frac{1}{2}$ or $\sin\theta = -1$ ①

$\theta = \frac{\pi}{6}, \frac{5\pi}{6} \qquad \theta = \frac{3\pi}{2}$ ①

$\therefore \theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$

c) $\frac{3x+4}{x-5} \ge 2$

$\frac{3x+4-2x+10}{x-5} \ge 0$

$(x-5)^2 \frac{x+14}{x-5} \ge 0 (x-5)^2$

① some progression towards solution

$(x-5)(x+14) \ge 0$

$x \le -14, x \ge 5$

but $x \ne 5$

$\therefore x \le -14, x > 5$ ①

d) $\int_{-1}^{14} \frac{w\,dw}{\sqrt{w+2}}$ $\qquad w = t^2 - 2 \quad \frac{dw}{dt} = 2t$

$w = 14 \quad t^2 = 16 \quad t = 4 \qquad dw = 2t\,dt$

$w = -1 \quad t^2 = 1 \quad t = 1$ ① ① limits

$\therefore \int_1^4 \frac{(t^2-2) \times 2t\,dt}{\sqrt{t^2}} = 2\int_1^4 (t^2 - 2)\,dt$

$= 2\left[\frac{t^3}{3} - 2t\right]_1^4$ ①

$= 2\left(\left(\frac{64}{3} - 8\right) - \left(\frac{1}{3} - 2\right)\right) = 30$

e) $n = 10$

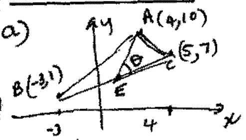
i) ${}^8C_4 = 70$ ①

ii) ${}^9C_6 = 84$ ①

12

Question Two

a)



midpt $\left(\frac{-3+5}{2}, \frac{1+7}{2}\right)$

$= (1, 4)$ ①

$m_{AE} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{10-4}{4-1} = 2 \qquad m_{EC} = \frac{7-4}{5-1} = \frac{3}{4}.$

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \left|\frac{2 - \frac{3}{4}}{1 + 2\times\frac{3}{4}}\right|$ ① (ignore finding θ)

$= \frac{1}{2}$ ①

b) $2x^3 + 5x - 3 = 0$

i) $\alpha + \beta + \gamma = \frac{-b}{a} = 0$ ① $\qquad \alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a} = \frac{5}{2}.$

$\alpha\beta\gamma = \frac{-d}{a} = \frac{3}{2}.$

ii) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$= 0 - 2\left(\frac{5}{2}\right)$

$= -5$ ①

iii) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \beta\alpha + \alpha\gamma}{\alpha\beta\gamma}$

$= \frac{5}{3}$ ①

c) $P(x) = (x^2 - 1)(x - \alpha)$

$P(2) = (4-1)(2-\alpha) = -9$

$6 - 3\alpha = -9$

$-3\alpha = -15$

$\alpha = 5$ ①

$\therefore P(x) = (x-1)(x+1)(x-5)$

$x = 1, -1, 5.$ ①

Quest 2. Cont.

d) $\left(x^2 + \frac{4}{x}\right)^{30} \qquad T_{k+1} = {}^nC_k a^{n-k} . b^k$

$= {}^{30}C_k x^{60-2k} . 4^k . x^{-k}$

$= {}^{30}C_k 4^k . x^{60-3k}$

i) independent of x ie. x^0

$\therefore 0 = 60 - 3k.$ ① statement to solve

$k = 20$ ①

the term is T_{21} or 21st term.

ii) greatest co-eff.

$\frac{T_{k+1}}{T_k} = \frac{n-k+1}{k} \cdot \frac{b}{a}$

$= \frac{30-k+1}{k} \cdot \frac{4}{x^3}$ ①

Coeff $= \frac{30-k+1}{k} \cdot 4 > 1$

$124 - 4k > k$

$124 > 5k.$

$k < 24\frac{4}{5}$

$\therefore k = 24$ ①

the term is T_{25} or 25th term

12

Question 3.

a) $\frac{d}{dx}(\tan^{-1}x)^2 = 2\tan^{-1}x \times \frac{1}{1+x^2}$

$= \frac{2\tan^{-1}x}{1+x^2}$ ①

$\int_{-1}^{\sqrt{3}} \frac{\tan^{-1}x}{1+x^2}\,dx = \frac{1}{2}\int_{-1}^{\sqrt{3}} \frac{d}{dx}(\tan^{-1}x)^2\,dx$

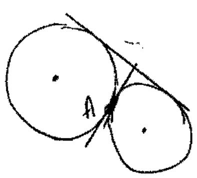
$= \frac{1}{2}\left[(\tan^{-1}x)^2\right]_{-1}^{\sqrt{3}}$ ①

$= \frac{1}{2}\left((\tan^{-1}\sqrt{3})^2 - (\tan^{-1}-1)^2\right)$

$= \frac{1}{2}\left(\frac{\pi}{3}^2 - \left(-\frac{\pi}{4}\right)^2\right)$

$= \frac{7\pi^2}{288}$ ①

b)



i) $RP = RA$ (tangents from R to circle centre B)

$RQ = RA$ (tangents from R to circle centre C) ①

$\therefore RP = RQ$ (both equal to RA) ①

$\therefore$ R is the midpoint of PQ.

ii) since $PR = RA = RQ$ (from above) ①

$\therefore$ radii of circle centre R ①

then $PQ = PR + RQ$ (a diameter) ①

$\therefore P\hat{A}Q = 90$ (∠ in a semicircle with centre R) ①

— — or — — — or

in △ APR

$PR = RA$ (above)

$\therefore$ △APR is isosceles (2 sides equal)

$\therefore R\hat{P}A = R\hat{A}P$ (base ∠'s equal) ①

$= x$

in △ ARQ

$AR = RQ$ (above)

$\therefore$ △ARQ is isos (2 sides equal)

$\therefore R\hat{A}Q = A\hat{Q}R$ (base ∠s equal) ①

$= y$

now in △ PAQ

$x + x + y + y = 180$ (∠ sum of △)

$2x + 2y = 180$

$\therefore x + y = 90°$

$= P\hat{A}Q$ ①

c i) $\cos(3x + x) = \cos 3x\cos x - \sin 3x\sin x$

$\cos(3x - x) = \cos 3x\cos x + \sin 3x\sin x$

$\therefore \cos(3x - x) - \cos(3x + x)$ ①

$= 2\sin 3x\sin x$

①

$\therefore \sin 3x\sin x = \frac{1}{2}(\cos(2x) - \cos(4x))$

Quest 3

c) cont

ii) $\int_0^{\frac{\pi}{4}} \sin x \sin 3x \, dx = \frac{1}{2}\int_0^{\frac{\pi}{4}} \cos 2x - \cos 4x \, dx$

$= \frac{1}{2}\left[\frac{1}{2}\sin(2x) - \frac{1}{4}\sin(4x)\right]_0^{\frac{\pi}{4}}$ ①

$= \frac{1}{2}\left(\left(\frac{1}{2}\sin\frac{\pi}{2} - \frac{1}{4}\sin\pi\right) - \left(\frac{1}{2}\sin 0 - \frac{1}{4}\sin 0\right)\right)$

$= \frac{1}{2}\left(\frac{1}{2} - 0\right)$

$= \frac{1}{4}.$ ①

[12]

Question four.

a) $p = \frac{1}{3}$ $q = \frac{2}{3}$ $n = 6$

i) 4 lines busy $= {}^6C_4\left(\frac{1}{3}\right)^4 \times \left(\frac{2}{3}\right)^2$ ①

$= \frac{20}{243}$ ①

ii) $P\left(\begin{smallmatrix}\text{at most}\\ \text{2 lines busy}\end{smallmatrix}\right) = P(0) + P(1) + P(2)$ ①

$= {}^6C_0\left(\frac{2}{3}\right)^6 + {}^6C_1\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^5 + {}^6C_2\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^4$ ← or

$= 0.6808\ldots$

or $\frac{496}{729}$ ①

b.) $V_{sp} = \frac{4}{3}\pi r^3$ $SA = 4\pi r^2$ $\frac{dv}{dt} = 5$

$\frac{dv}{dr} = 4\pi r^2$ $\frac{dA}{dr} = 8\pi r$

$\frac{dv}{dt} = \frac{dv}{dr} \cdot \frac{dr}{dt}$ ① $\frac{dA}{dr} = \frac{dA}{dr} \cdot \frac{dr}{dt}$ ①

$5 = 4\pi r^2 \times \frac{dr}{dt}$ $\frac{dA}{dr} = 8\pi r \times \frac{5}{4\pi r^2}$

$\frac{dr}{dt} = \frac{5}{4\pi r^2}$ $= \frac{10}{r} = \frac{1}{2}$

∴ SA is increasing at a rate of $0.5 \text{ cm}^2/\text{s}$. ①

c) $x = 3\cos 2t - 4\sin 2t$

i) $x = A\cos(nt + \alpha)$ $n = 2$

$A = \sqrt{16 + 9}$ $\tan\alpha = \frac{4}{3}$ (could be left, $\tan^{-1}\frac{4}{3}$)

$= 5$ $\alpha = 0.927$ (radians)

∴ $x = 5\cos(2t + 0.927)$ ①

$\dot{x} = -10\sin(2t + 0.927)$

$\ddot{x} = -20\cos(2t + 0.927)$ ①

$= -4x$ ① ∴ in SHM with $n = 2$.

ii) greatest speed ⇒ $\ddot{x} = 0$ or $x = 0$

$5\cos(2t + 0.927) = 0$

$\cos(2t + 0.927) = 0$

$2t + 0.927 = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

$2t = 0.643$

$t = 0.3218\ldots$ ①

sub in to $\dot{x}$

$\dot{x} = -10\sin(2 \times 0.3218 + 0.927)$

$= -10 \times 1$

$= -10$ m/sec ①

∴ max speed $= |v| = \underline{\underline{10 \text{ m}}}/\text{sec}$

Question Five

a) $1.2 + 2.3 + 3.4 + \ldots n(n+1)$

$S_n = \frac{1}{3}n(n+1)(n+2)$

test $n=1$

S_1: LHS $= 1(1+1)$ RHS $= \frac{1}{3}(2)(3)$

$= 2$ $= \frac{1}{3} \times 6$

$= 2$ ①

∴ LHS = RHS true for $n=1$

assume true for $n=k$

ie $S_k = 1.2 + 2.3 + 3.4 + \ldots k(k+1)$

$= \frac{1}{3}k(k+1)(k+2)$

Prove true for $n = k+1$.

ie. $S_{k+1} = S_k + T_{k+1}$

Quest 5 cont

a) cont.

now $T_{k+1} = (k+1)(k+2)$

$S_{k+1} = \frac{1}{3}(k+1)(k+2)(k+3)$

and

$S_k + T_{k+1} = \frac{1}{3}k(k+1)(k+2) + (k+1)(k+2)$ ①

$= \frac{1}{3}k(k+1)(k+2) + \frac{3}{3}(k+1)(k+2)$

$= \frac{(k+1)(k+2)}{3}(k+3)$

$= \frac{1}{3}(k+1)(k+2)(k+3) = S_{k+1}$ ①

∴ assumed true for $n=k$ proved true for $n=k+1$ ①

Since true for $n=1$ now true by M.I. for $n = 1+1 = 2$, etc

for all positive integers

now $\frac{1}{n^3}\sum_{k=1}^{n} k(k+1) = \frac{1}{n^3}(1(2) + 2(3) + 3(4) + \ldots n(n+1))$ ①

$= \frac{1}{n^3}\left(\frac{1}{3}n(n+1)(n+2)\right)$ from above

∴ $\lim_{n\to\infty} \frac{1}{n^3}\sum_{k=1}^{n} k(k+1) = \lim_{n\to\infty} \frac{1}{3} \times \frac{1}{n^2}(n+1)(n+2)$

$= \lim_{n\to\infty} \frac{1}{3} \times \frac{(n+1)}{n} \times \frac{(n+2)}{n}$

$= \lim_{n\to\infty} \frac{1}{3}\left(1 + \frac{1}{n}\right)\left(1 + \frac{2}{n}\right)$

$= \frac{1}{3} \times 1$ as $\lim_{n\to\infty}\frac{1}{n} = 0$ & $\lim_{n\to\infty}\frac{2}{n} = 0$

$= \frac{1}{3}.$ ①

b) $\theta = m + Ae^{kt}$

i) $\frac{d\theta}{dt} = kAe^{kt}$

but $Ae^{kt} = \theta - m$ (from above)

∴ $\frac{d\theta}{dt} = k(\theta - m)$ as required. ①

~~Que 1b~~
sub for t into

~~$x+ty = at^3+2at$~~
~~at $t=1$~~
~~$x+y = a+2a$~~
~~$x+y = 3a$~~

~~at $t=-4$.~~
~~$x-4y = a\times(-4)^3+2a\times -4$~~
~~$x-4y = -72a$~~

~~at $t=3$~~
~~$x+3y = 27a+6a$~~
~~$x+3y = 33a$~~

taken out.

c) $\dfrac{d^2x}{dt^2} = 8x(x^2+1)$ $t=0$ $V=-2$ cm/s, $x=0$

since $\dfrac{d^2x}{dt^2} = \dfrac{d}{dx}\tfrac{1}{2}v^2$

$\therefore \dfrac{d}{dx}\tfrac{1}{2}v^2 = 8x^3+8x$ ①

$\tfrac{1}{2}v^2 = 2x^4+4x^2+c$

$x=0, v=-2$

$\therefore \tfrac{1}{2}\times 4 = 0+0+c$

$c = 2$

$\therefore \tfrac{1}{2}v^2 = 2x^4+4x^2+2$

$v^2 = 4x^4+8x^2+4$

$= 4(x^4+2x^2+1)$

$= 4(x^2+1)^2$

$v = \pm 2(x^2+1)$ ①

$\therefore$ speed $= |v| = 2(x^2+1)$ cm/sec.

now $\dfrac{dx}{dt} = \pm 2(x^2+1)$ (± as not speed)

$\dfrac{dt}{dx} = \dfrac{\pm 1}{2(1+x^2)}$ → (from initial conditions can assume $\dfrac{dt}{dx} = \dfrac{-1}{2(1+x^2)}$) ①

$t = \pm\tfrac{1}{2}\tan^{-1}x + c$

$t=0$ $x=0$

$0 = \pm\tfrac{1}{2}\tan^{-1}0 + c$

$c = 0$

$2t = \pm\tan^{-1}x$

then $2t = -\tan^{-1}x$ ①

$\therefore \tan(\pm 2t) = x$

ie $\pm\tan(2t) = x$.

considering both cases. (not required)

$x = \tan 2t$ $x = -\tan 2t$

$\dot{x} = 2\sec^2 2t$ $\dot{x} = -2\sec^2 2t$

at $t=0$ $\dot{x} = -2$

$-2 = 2\sec^2 0$ $-2 = -2\sec^2 0$

$-2 = 2\times 1$ $-2 = -2$

-2 ∴ not possible

$\therefore x = -\tan 2t$. ①

ii) at $t = \frac{\pi}{8}$

$x = -\tan 2\times\frac{\pi}{8}$

$= -\tan\frac{\pi}{4}$

$= -1$ ①

$\therefore$ P is 1 cm to the left of the origin.

[12]

Question 7.

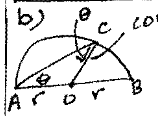
a) $\displaystyle\int_0^1 \frac{e^{\cos^{-1}x}}{\sqrt{1-x^2}}\,dx = \left[-e^{\cos^{-1}x}\right]_0^1$ ①

$= -e^{\cos^{-1}1} + e^{\cos^{-1}0}$

$= -e^0 + e^{\frac{\pi}{2}}$

$= e^{\frac{\pi}{2}} - 1$ ①

b)



construct ~~CB~~

$\triangle AOC$ is isosceles (radii - 2 sides equal)

$\therefore C\hat{O}B = 2\theta$ (ext ∠ to △)

$A\hat{O}C = \pi - 2\theta$ ①

Area

i) $\tfrac{1}{2}$ Semi circle $= A_\triangle + A_{\text{sector}}$ ①

$\frac{\pi r^2}{4} = \tfrac{1}{2}r^2\sin(\pi-2\theta) + \tfrac{1}{2}r^2 2\theta$

$\frac{\pi}{4} = \tfrac{1}{2}\left(\sin(\pi-2\theta)+2\theta\right)$

$\frac{\pi}{2} = \sin 2\theta + 2\theta$.

ii) $\theta = 0.4$ ①

$f(\theta) = 2\theta + \sin 2\theta - \frac{\pi}{2} = 0$

$f(0.4) = 0.8 + \sin 0.8 - \frac{\pi}{2} = -0.053\ldots$

$\doteqdot 0$

Quest 5 cont.

b) ii) $m = 40$ $\theta = 170$ $t=0$ to $\theta = 105$ $t = 45$

$\therefore \theta = ?$ when $t = 90+45$

$= 135$

$\theta = m + Ae^{kt}$

at $t=0$

$170 = 40 + Ae^0$ ①

$A = 130$

$\theta = 40 + 130e^{kt}$

at $t=45$

$105 = 40 + 130e^{45k}$

$65 = 130e^{45k}$

$\ln\frac{65}{130} = 45k\ln e$

$k = \left(\ln\frac{65}{130}\right) \div 45$

$\doteqdot -0.0154\ldots$ ①

at $t=135$

$\theta = 40 + 130e^{135k}$

$= 56.25°C$ ①

iii) $t = ?$

$80 = 40 + 130e^{kt}$

$40 = 130e^{kt}$ ①

$\ln\frac{4}{13} = kt\ln e$

$t = \ln\left(\frac{4}{13}\right) \div k$

$= 76.53$

$= 77$ min. ①

[12]

Question 6

a) let $M = \log_e x$

$\therefore \log_e M = 0$

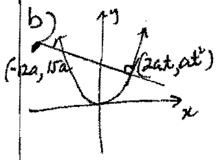
$e^0 = M$

$M = 1$

$\therefore \log_e x = 1$

$e^1 = x$. ①

b)



$x^2 = 4ay$ $T(2at, at^2)$

$y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{2x}{4a}$

at $x = 2at$ $m = t$ ①

For the normal $m_2 = -\frac{1}{t}$

equation: $y - y_1 = m(x-x_1)$

$y - at^2 = -\frac{1}{t}(x - 2at)$

$ty - at^3 = -x + 2at$ ①

$\therefore x + ty = at^3 + 2at$

passes thru $(-12a, 15a)$

$\therefore -12a + 15at = at^3 + 2at$ ①

$at^3 - 13at + 12a = 0$

$t^3 - 13t + 12 = 0$

Find factor: try 1, -1, etc.

$P(t) = t^3 - 13t + 12$ ①

$P(1) = 1 - 13 + 12 = 0 \therefore (t-1)$ is a factor

$$
\begin{array}{r}
t^2 + t - 12 \\
t-1\,\big)\,\overline{t^3 - 13t + 12} \\
t^3 - t^2 \\
\hline
t^2 - 13t \\
t^2 - t \\
\hline
-12t + 12 \\
-12t + 12 \\
\hline
0
\end{array}
$$

$\therefore (t-1)(t+4)(t-3) = 0$

$\therefore t = 1, -4, 3$ ①

Quest 7. cont

b)(iii)

$f(\theta) = \sin 2\theta + 2\theta - \frac{\pi}{2}$

$f'(\theta) = 2\cos 2\theta + 2$

$f'(0.4) = 2\cos 0.8 + 2$

$\therefore \theta_2 = \theta_1 - \frac{f(\theta_1)}{f'(\theta_1)}$ ①

$= 0.4 - \frac{\sin 0.8 + 0.8 - \frac{\pi}{2}}{2\cos 0.8 + 2}$

$= 0.4157\ldots$ any approx ①

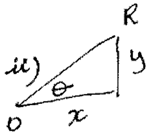
c) $x = vt\cos\alpha \qquad y = vt\sin\alpha - \frac{gt^2}{2}$

i) $t = \frac{x}{v\cos\alpha} \Rightarrow y = v\left(\frac{x}{v\cos\alpha}\right)\sin\alpha - \frac{g}{2}\left(\frac{x}{v\cos\alpha}\right)^2$

① $= x\tan\alpha - \frac{gx^2}{2v^2\cos^2\alpha}$

$y = x\tan\alpha - \frac{gx^2\sec^2\alpha}{2v^2}$

ii)



$\cos\theta = \frac{x}{RO}$

$x = RO.\cos\theta$ ① either

$\sin\theta = \frac{y}{RO}$

$y = RO.\sin\theta$

sub in to cartesian equation

$RO.\sin\theta = RO.\cos\theta\tan\alpha - \frac{g(RO.\cos\theta)^2.\sec^2\alpha}{2v^2}$

$RO.\sin\theta = \frac{RO.\cos\theta\sin\alpha}{\cos\alpha} - \frac{g.RO^2\cos^2\theta}{2v^2\cos^2\alpha}$

$RO.\sin\theta.\cos^2\alpha = RO.\cos\theta\sin\alpha\cos\alpha - \frac{g.RO^2\cos^2\theta}{2v^2}$

$\div RO$

$\sin\theta.\cos^2\alpha = \cos\theta\sin\alpha\cos\alpha - \frac{g.RO.\cos^2\theta}{2v^2}$

$\frac{g\cos^2\theta}{2v^2}.RO = \cos\theta\sin\alpha\cos\alpha - \sin\theta\cos^2\alpha$

$= \cos\alpha(\sin\alpha\cos\theta - \cos\alpha\sin\theta)$ ①

$= \cos\alpha.\sin(\alpha-\theta)$

$RO = \frac{2v^2\cos\alpha\sin(\alpha-\theta)}{g\cos^2\theta}$

c iii)

$\frac{dRO}{d\alpha} = \frac{2v^2(-\sin\alpha\sin(\alpha-\theta) + \cos\alpha\cos(\alpha-\theta))}{g\cos^2\theta}$

$= \frac{2v^2(\cos(\alpha+\alpha-\theta))}{g\cos^2\theta}$

$= \frac{2v^2\cos(2\alpha-\theta)}{g\cos^2\theta} = 0$

$\therefore \cos(2\alpha-\theta) = 0$

$2\alpha - \theta = \frac{\pi}{2}$

$\alpha = \frac{1}{2}\left(\frac{\pi}{2}+\theta\right)$ ①

Test max

$\frac{d^2RO}{d\alpha^2} = \frac{2v^2\times -2\sin(2\alpha-\theta)}{g\cos^2\theta}$

$= \frac{-4v^2\sin(2\alpha-\theta)}{g\cos^2\theta}$

at $\alpha = \frac{1}{2}\left(\theta+\frac{\pi}{2}\right)$

$\frac{d^2RO}{d\alpha^2} = \frac{-4v^2\sin\left(2\times\frac{1}{2}\left(\theta+\frac{\pi}{2}\right)-\theta\right)}{g\cos^2\theta}$

$= \frac{-4v^2\times 1}{g\cos^2\theta} < 0 \therefore$ max.

Since RO is a max for $\alpha = \frac{1}{2}\left(\theta+\frac{\pi}{2}\right)$

① α bisects the angle $\left(\theta+\frac{\pi}{2}\right)$

the end! 12