BAULKHAM HILLS HIGH SCHOOL

TRIAL 2013
YEAR 12 TASK 4

Mathematics Extension 1

General Instructions
- Reading time – 5 minutes
- Working time – 120 minutes
- Write using black or blue pen
- Board-approved calculators may be used
- Show all necessary working in Questions 11-14
- Marks may be deducted for careless or badly arranged work

Total marks – 70
Exam consists of 10 pages.

This paper consists of TWO sections.

<u>Section 1</u> – Page 2-4 (10 marks)
Questions 1-10
- Attempt Question 1-10

<u>Section II</u> – Pages 5-9 (60 marks)
- Attempt questions 11-14

Table of Standard Integrals is on page 10

Section I - 10 marks
Use the multiple choice answer sheet for question 1-10

1. The value of $\lim_{x\to 0}\frac{\sin 7x}{3x}$ is

(A) $2\frac{1}{3}$

(B) $\frac{3}{7}$

(C) 0

(D) 1

2. What is the acute angle, to the nearest degree, between the lines $y = 7 - 4x$ and $2x - 3y - 6 = 0$.

(A) $26°$

(B) $48°$

(C) $70°$

(D) $75°$

3. Let α, β and γ be the roots of $2x^3 + x^2 - 4x + 9 = 0$.

What is the value of $\frac{1}{\alpha\beta} + \frac{1}{\alpha\gamma} + \frac{1}{\beta\gamma}$

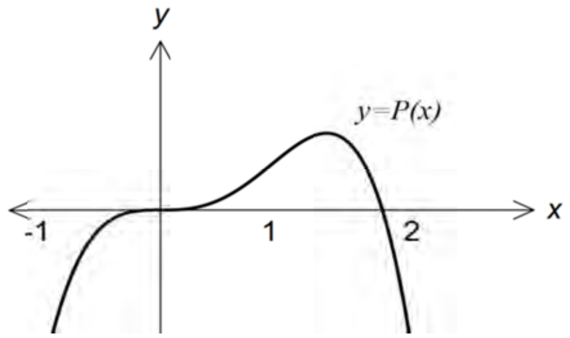
(A) $-\frac{1}{9}$

(B) $-\frac{1}{2}$

(C) $\frac{1}{9}$

(D) $\frac{1}{2}$

4. Which of the following could be the polynomial $y = P(x)$



(A) $P(x) = x^3(2 - x)$

(B) $P(x) = x^2(2 - x)^2$

(C) $P(x) = x^3(x - 2)$

(D) $P(x) = -x^3(x + 2)$

5. The coordinates of the points A and B are $(-1,1)$ and $(3,-1)$ respectively.
Find the coordinates of P that divides the interval AB externally in the ratio $1:3$.

(A) $\left(2, -\frac{1}{2}\right)$

(B) $\left(-1\frac{1}{2}, \frac{1}{2}\right)$

(C) $(-3,2)$

(D) $(4,-2)$

6. The number of different arrangements of the letters of the word **R E G I S T E R** which begin and end with the letter R is:

(A) $\frac{6!}{(2!)^2}$

(B) $\frac{8!}{(2!)}$

(C) $\frac{6!}{2!}$

(D) $\frac{8!}{2!2!}$

7. The inverse function of $g(x)$ where $g(x) = \sqrt{3x - 8}$ is

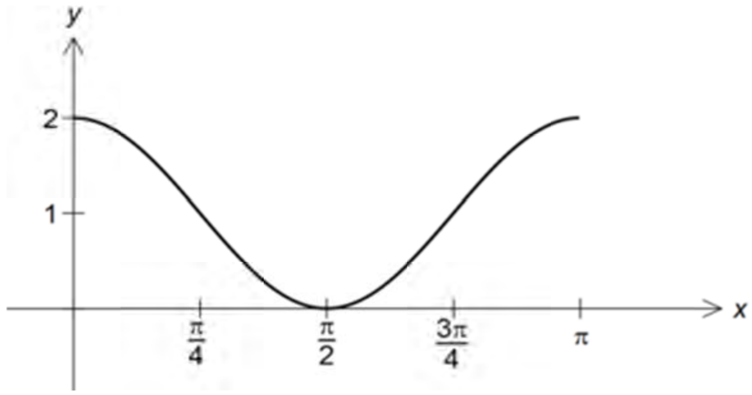
(A) $g^{-1}(x) = \frac{x^2+8}{3}$

(B) $g^{-1}(x) = (3x - 8)^2$

(C) $g^{-1}(x) = \sqrt{\frac{x}{3} + 8}$

(D) $g^{-1}(x) = \frac{x^2-8}{3}$

8. A possible equation for the curve drawn is:-



(A) $y = -1 + \cos 2x$

(B) $y = 1 + \cos 2x$

(C) $y = \cos(2x + 1)$

(D) $y = 1 - \cos 2x$

9. The middle term in the expansion $(2x - 4)^4$ is

(A) 81

(B) $216x^2$

(C) $384x^2$

(D) $-96x^3$

10. If $y = \sin^{-1}(2x)$, then $\frac{d^2y}{dx^2}$ equals

(A) $\frac{-8}{\sqrt{(1-4x^2)^3}}$

(B) $\frac{4x}{\sqrt{1-4x^2}}$

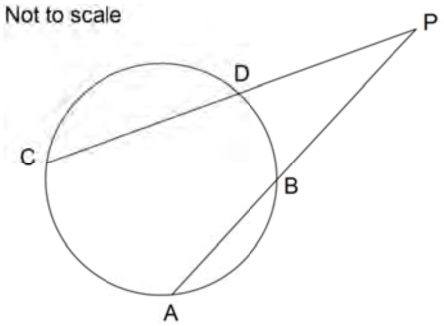
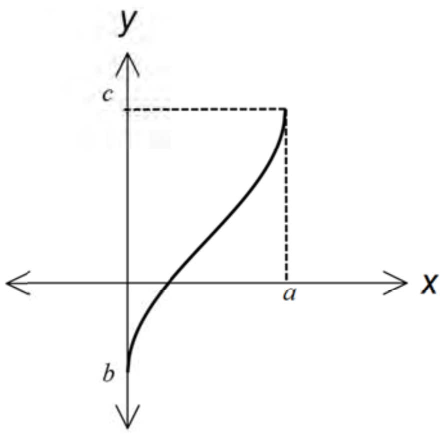
(C) $\frac{-8x}{\sqrt{(1-4x^2)^3}}$

(D) $\frac{8x}{\sqrt{(1-4x^2)^3}}$

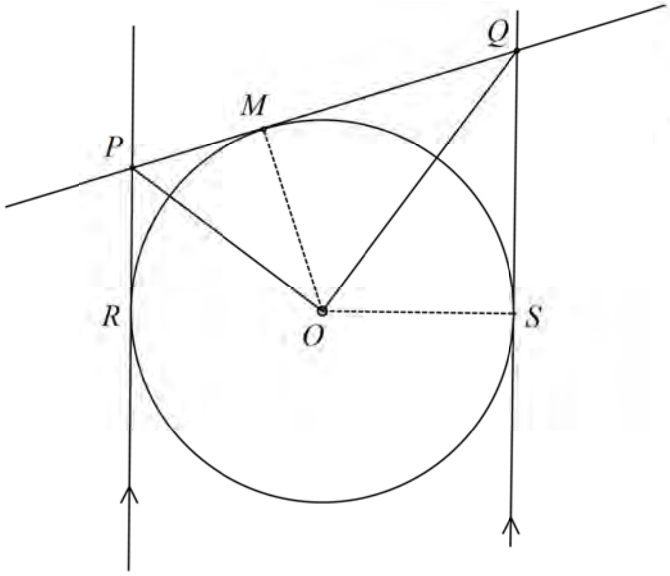
End of Section 1

Section II – Extended Response

Attempt questions 11-16.
Answer each question on a SEPARATE PAGE. Clearly indicate question number.
Each piece of paper must show your number.
All necessary working should be shown in every question.

Question 11 (15 marks)		**Marks**
a)	Solve for x $$\frac{2x+1}{1-x} \geq 1$$	**3**
b)	It is given that $y = 2\cos 3\left(x - \frac{\pi}{3}\right), 0 \leq x \leq 2\pi$ Find (i) the amplitude. (ii) the period.	 **1** **1**
c)	 Two chords CD and AB of a circle are extended to meet at P. If $AB = 7$cm, $BP = 3$cm, $DP = 5$cm find CP.	**1**
d)	The graph of $y = 1 + 2\sin^{-1}(2x - 1)$ is shown.  Determine the values of $a, b,$ and c	**2**
e)	Form the cartesian equation by eliminating the parameter, θ, from these parametric equations. $$x = 3\tan\theta \qquad y = 2\sec\theta$$	**2**
f)	Solve $\sin 2\theta + \cos\theta = 0$ for $0 \leq \theta \leq 2\pi$.	**3**
g)	The radius of a spherical balloon is increasing at the rate of 2cm/sec. Find the rate at which the volume of the balloon is increasing when the radius is 10cm (in terms of π).	**2**
	End of Question 11	

Question 12 (15 marks)		**Marks**
a)	Evaluate $\int_0^{\frac{\pi}{6}} 2\sin^2 x\, dx$	**2**
b)	By using the substitution $u = \sqrt{x}$, evaluate $$\int_1^4 \frac{dx}{x + \sqrt{x}}$$	**3**
c)	Two parallel tangents to a circle, centre O, are cut by a third tangent at P and Q.	



Not to scale

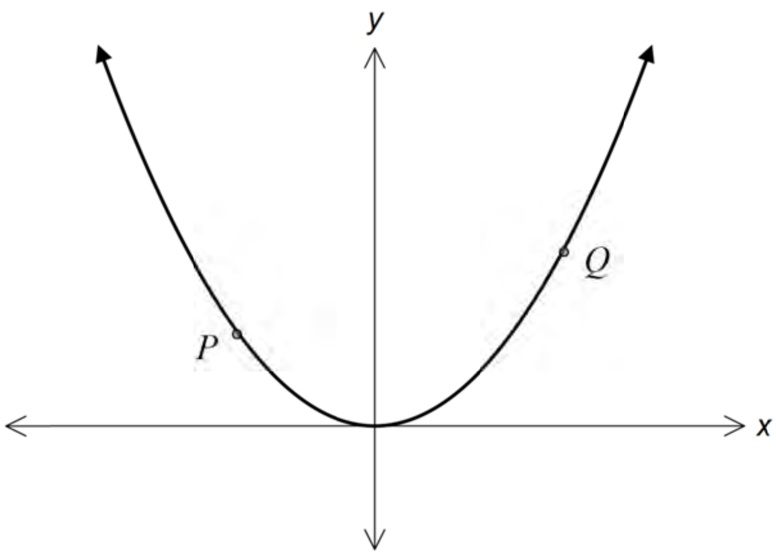
(i)	Copy or trace the diagram into your solution booklet.	
(ii)	Prove $\Delta MOQ \equiv \Delta SOQ$	**2**
(iii)	Prove that $\angle POQ = 90°$.	**2**
d) (i)	Express $x = 2\cos 3t - 5\sin 3t$ in the form $x = R\cos(3t + \alpha)$, where t is in radians.	**2**
(ii)	A particle moves in a straight line and its position at time t is given by $x = 2\cos 3t\ - 5\sin 3t$. Show that the particle is moving in simple harmonic motion.	**1**
e)	The coefficients of x^2 and x^{-1} in the expansion of $\left(ax - \frac{b}{x^2}\right)^5$ are the same. Show that $a + 2b = 0$, where a and b are positive integers.	**3**

End of Question 12

Question 13 (15 marks)

a)	In how many ways can a committee of 5 people be formed from a group of 9 people, if 2 people, Harry and Archie, refuse to serve together on the same committee.		2
b)	(i)	Show by a sketch, without using calculus, that the equation $e^{2x} + 4x - 5 = 0$ has only one root.	1
	(ii)	Show that this root lies between 0 and 1.	1
	(iii)	By taking $x = 0.5$ as a first approximation, use one application of Newton's method to find a better approximation of this root to two decimal places.	2
c)	A particle moves in a straight line with velocity v m/sec and acceleration given by $\ddot{x} = 2e^{x}$, where x is the displacement from O. The initial velocity is -2 m/sec at the origin.		
	(i)	Prove that $v^2 = 4e^{x}$.	2
	(ii)	Hence find the displacement in terms of t.	3
d)	When a body falls, the rate of change of its velocity, v, is given by $\frac{dv}{dt} = -k(v - 500)$ where k is a constant.		
	(i)	Show that $v = 500 - 500e^{-kt}$ is a possible solution to this equation.	1
	(ii)	The velocity after 5 seconds is 21m/sec , find the value of k to 3 significant figures.	1
	(iii)	Find the velocity after 20 seconds.	1
	(iv)	Explain the effect on the velocity as t becomes large.	1

End of Question 13

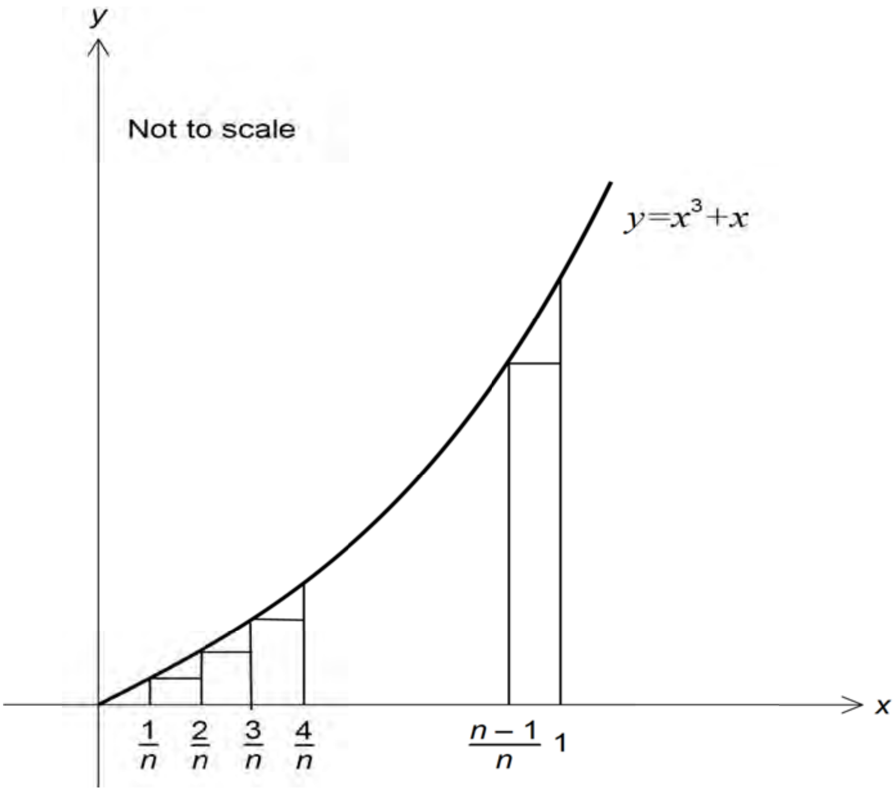
Question 14 (15 marks)		**Marks**
a)	A particle is projected with an initial velocity of 60m/sec at an angle of 45° to the horizontal. The equations of motion are: $x = 30\sqrt{2}\,t$ and $y = 30\sqrt{2}\,t - 5t^2$ (Do not prove these) (i) How high is the particle after it has travelled 120m horizontally? (ii) What is the horizontal range of the particle?	 **2** **1**
b)	Two points P and Q move on the parabola $x^2 = 4ay$ so that their x values differ by a.  (i) If P has coordinates (x_1, y_1), show that the midpoint, C, of PQ is $\left(\frac{2x_1 + a}{2}, \frac{2x_1^2 + 2ax_1 + a^2}{8a}\right)$ (ii) Show that the locus of C is another parabola.	 **2** **2**
	Question 14 continues on page 6	

Question 14 (continued)

c) (i) Prove by mathematical induction **3**

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$

(ii) The curve $y = x^3 + x$ is drawn and divided into n intervals between 0 and 1. The inner rectangles are completed as shown. **3**



If A_n is the area of these rectangles show that

$$A_n = \frac{1}{n^4}\sum_{r=1}^{n-1} r^3 + \frac{1}{n^2}\sum_{r=1}^{n-1} r$$

(iii) Given that the area bounded by the curve, between $x = 0$ and $x = 1$, and the x axis is $\lim_{n\to\infty} A_n$, find a value for A_n using the result in part (i). **2**

End of Exam

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \quad if \quad n < 0$$

$$\int \frac{1}{x} dx = \ln x, \qquad x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos axdx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin axdx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 axdx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan axdx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln(x + \sqrt{x^2 - a^2}), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln(x + \sqrt{x^2 + a^2})$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Question 11.

a. $\frac{2x+1}{1-x} \geq 1$

various methods

example

$\frac{2x+1-1+x}{1-x} \geq 0$ ✓working

$(1-x)^2 \times \frac{3x}{1-x} \geq 0 \times (1-x)^2$

$(1-x)(3x) \geq 0$

$0 \leq x \leq 1$

but $x \neq 1$ ✓

$\therefore 0 \leq x < 1$ ✓

(3)

b. i) amplitude $= 2$ ✓

ii) period $= \frac{2\pi}{3}$ ✓

(2)

c. $CP \times DP = AP \times BP$

$CP \times 5 = 10 \times 3$

$CP = 6\text{cm}$ ✓

(1)

d. Domain: $-1 \leq 2x - 1 \leq 1$

$0 \leq x \leq 1$

Range: $-\frac{\pi}{2} \leq \frac{y-1}{2} \leq \frac{\pi}{2}$

$1-\pi \leq y \leq 1+\pi$

$\therefore a = 1$ ✓

$b = 1-\pi \qquad c = 1+\pi$ ✓ for both

(2)

e. $\frac{x}{3} = \tan\theta \qquad \frac{y}{2} = \sec\theta$ ✓ working

connector: $\tan^2\theta + 1 = \sec^2\theta$

$\therefore \frac{x^2}{9} + 1 = \frac{y^2}{4}$ ✓

or $4x^2 - 9y^2 = -36$

(2)

f. $\sin 2\theta + \cos\theta = 0 \qquad 0 \leq \theta \leq 2\pi$

$2\sin\theta\cos\theta + \cos\theta = 0$ ✓

$\cos\theta(2\sin\theta + 1) = 0$

$\therefore \cos\theta = 0$ ✓ $\qquad \sin\theta = -\frac{1}{2}$ ✓

$\theta = \frac{\pi}{2}, \frac{3\pi}{2} \qquad \theta = \frac{7\pi}{6}, \frac{11\pi}{6}$

$\therefore \theta = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$

(3)

g. $V = \frac{4}{3}\pi r^3 \qquad \frac{dr}{dt} = 2\text{cm/sec}$

$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

$= 4\pi r^2 \times 2$ ✓

at $r = 10$

$\frac{dV}{dt} = 800\pi \text{ cm}^3/\text{sec}$ ✓

(2)

/15

Multiple choice

1. A
2. C
3. C
4. A
5. C
6. C
7. A
8. B
9. C
10. D

Question 12

a. $\int_0^{\frac{\pi}{6}} 2\sin^2 x\,dx = \int_0^{\frac{\pi}{6}} (1-\cos 2x)\,dx$

$= \left[x - \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{6}}$ ✓

$= \left(\frac{\pi}{6} - \frac{1}{2}\sin\frac{\pi}{3} - 0\right)$

$= \frac{\pi}{6} - \frac{\sqrt{3}}{4}$ ✓ or $\frac{2\pi - 3\sqrt{3}}{12}$

(2)

b. $\int_1^4 \frac{dx}{x+\sqrt{x}}$

$u = x^{\frac{1}{2}}$ $\qquad$ at $x = 4 \quad u = 2$

$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$ ✓ $\qquad x = 1 \quad u = 1$

$2u\,du = dx$.

$= \int_1^2 \frac{2u\,du}{u^2 + u}$

$= \int_1^2 \frac{2\,du}{u+1}$ ✓

$= \left[2\ln(u+1)\right]_1^2$

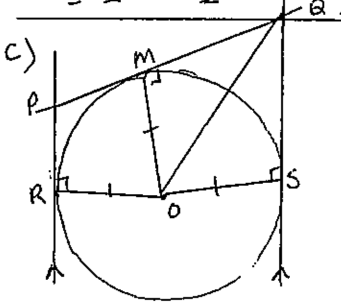
$= 2\ln 3 - 2\ln 2$

$= 2\ln\frac{3}{2}$ ✓

(3)

> Suggested solutions and marking scale for Trial yr12 Ext1 2013.
> • variations do occur

c)



ii) Aim: prove $\triangle MOQ \equiv \triangle SOQ$.

Method:

In $\triangle MOQ$ and $\triangle SOQ$.

$MO = OS$ (radii)

OQ is common

$Q\hat{M}O = Q\hat{S}O$ (radii to tangent is ⊥)

$= 90°$

$\therefore \triangle MOQ \equiv \triangle SOQ$ (R.H.S)

(2) marks – complete proof

(1) mark – 2 reasons.

(2)

Quest 12 Cont.

iii) An example

Similar to (ii) $\triangle MOP \equiv \triangle ROP$ (R.H.S)

let $M\hat{O}P = y$

$= R\hat{O}P$ (matching $\angle$s in congruent $\triangle$s)

and let $R\hat{P}O = x$

$= M\hat{P}O$ (matching $\angle$s in congruent $\triangle$s)

Since tangent PQ meets RP and SQ (parallel tangents)

$R\hat{P}Q + S\hat{Q}M = 180$ (cointerior $\angle$s, PR∥QS)

$\therefore 2x + 2y = 180$

$x + y = 90$ --- ①

now $P\hat{O}M = 90 - x$ ($\angle$ sum of $\triangle$ POM)

$Q\hat{O}M = 90 - y$ ($\angle$ sum of $\triangle$ QOM)

then $P\hat{O}Q = 90 - x + 90 - y$ (addition of adjacent $\angle$s)

$= 180 - (x+y)$

using ① $= 180 - 90$

$= 90°$ as required.

* needed to establish ROS was a straight line!

* various methods. ① mark connecting (ii)
② marks – clear proof

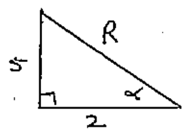
②

d) $x = 2\cos 3t - 5\sin 3t$

$x = R\cos 3t\cos\alpha - R\sin 3t\sin\alpha$.

$R\cos\alpha = 2$ $\quad$ $R\sin\alpha = 5$

$\cos\alpha = \frac{2}{R}$ $\quad$ $\sin\alpha = \frac{5}{R}$.



$R = \sqrt{29}$ ✓

$\tan\alpha = \frac{5}{2}$

$\alpha = 1.19$ (2dp)

or $= \tan^{-1}\left(\frac{5}{2}\right)$

$\therefore x = \sqrt{29}\cos(3t + 1.19)$ ✓ or $\sqrt{29}\cos(3t + \tan^{-1}\frac{5}{2})$

②

Question 12 Cont.

d (ii) $x = \sqrt{29}\cos(3t + 1.19)$

$\dot{x} = -3\sqrt{29}\sin(3t + 1.19)$

$\ddot{x} = -9\sqrt{29}\cos(3t + 1.19)$

✓ $= -9 \times x$ $\quad$ now defn of SHM is $\ddot{x} = -n^2 x$

$\therefore$ the particle is moving in SHM with $n = 3$.

①

e) $\left(ax - \frac{b}{x^2}\right)^5 = (ax - bx^{-2})^5$

$T_{k+1} = {}^nC_k\, a^{n-k} \cdot b^k$ for $(a+b)^n$

$= {}^5C_k (ax)^{5-k} \cdot (-1)^k \cdot (bx^{-2})^k$ ✓

$= {}^5C_k\, a^{5-k} \cdot (-1)^k \cdot b^k \cdot x^{5-k} \cdot x^{-2k}$

to find k.

for x^2: $2 = 5 - 3k$
$3k = 3$
$k = 1$

Coef of $T_2 = {}^5C_1 a^4 (-b)^1$
$= -5a^4 b$

for x^{-1}: $-1 = 5 - 3k$
$3k = 6$
$k = 2$

Coef of $T_3 = {}^5C_2 a^3 b^2$
$= 10a^3 b^2$ ✓

since coefs of x^2 and x^{-1} are equal

$\therefore -5a^4 b = 10a^3 b^2$

$10a^3b^2 + 5a^4 b = 0$ ✓

$5a^3 b(2b + a) = 0$

$\therefore 2b + a = 0$ as required.

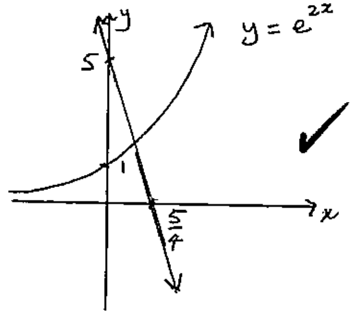
③

Question 13.

a) no. of ways $= {}^9C_5 - {}^7C_3$
$= 126 - 35$
$= 91$ ✓

(2)

b). i)



✓

$e^{2x} + 4x - 5 = 0$
let $y = e^{2x} = 5 - 4x$.
$\therefore y = e^{2x}$, $y = 5 - 4x$

or 1 mark for the correct graph of $y = e^{2x} + 4x - 5$

ii) $f(x) = e^{2x} + 4x - 5$

$f(0) = e^0 - 5$
$= -4 < 0$ ✓

$f(1) = e^2 + 4 - 5$
$= 6.39 > 0$

$\therefore$ a root lies between 0 and 1 as $f(x)$ neg to pos

iii) let $x_1 = 0.5$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$f(0.5) = -0.2817\ldots$
$f'(x) = 2e^{2x} + 4$
$f'(0.5) = 9.436\ldots$

$= 0.5 - \frac{-0.2817\ldots}{9.436\ldots}$ ✓

$= 0.529\ldots$
$= 0.53$ (2dp) ✓

(4)

c. i) $\ddot{x} = 2e^x$ $\quad t = 0$, $v = -2$ m/sec $\quad x = 0$.

since $\ddot{x} = \frac{d}{dx}(\frac{1}{2}v^2)$

then $\frac{d}{dx}(\frac{1}{2}v^2) = 2e^x$

$\frac{1}{2}v^2 = 2\int e^x dx$

$\frac{1}{2}v^2 = 2e^x + C$

at $v = -2$ $x = 0$

$2 = 2\times 1 + C$

1 mark working towards

1 mark

$C = 0$ $\quad \therefore v^2 = 4e^x$.

(2)

Quest 13 Cont.

ii) $v^2 = 4e^x$

$v = \pm\sqrt{4e^x}$
$= \pm 2e^{\frac{x}{2}}$

now $v < 0$ initially ✓
$\therefore v = -2e^{\frac{x}{2}}$

In terms of t

$\frac{dx}{dt} = -2e^{\frac{x}{2}}$

$\frac{dt}{dx} = -\frac{1}{2}e^{-\frac{x}{2}}$

$t = e^{-\frac{x}{2}} + C$

at $t = 0$ $x = 0$

$\therefore 0 = e^0 + C$

$C = -1$ ✓

$\therefore t = e^{-\frac{x}{2}} - 1$

$t + 1 = e^{-\frac{x}{2}}$

$\ln(t+1) = -\frac{x}{2}$ (as $\ln e = 1$)

$\therefore x = -2\ln(t+1)$ ✓

(3)

d) an example

i) $V = 500 - 500e^{-kt}$

$\frac{dV}{dt} = k\,500e^{-kt}$

but $\therefore 500e^{-kt} = 500 - V$.

$= k(500 - V)$
$= -k(V - 500)$ as required

✓ any solid method

ii) $v = 0$ $t = 0$ $\quad V = 21$ m/sec $t = 5$

$V = 500 - 500e^{-kt}$

$21 = 500 - 500e^{-5k}$

$-479 = -500e^{-5k}$

$0.958 = e^{-5k}$

$\ln(0.958) = \ln e^{-5k}$

$\therefore k = \frac{\ln 0.958}{-5}$ ✓

$= 0.008581\ldots$
$= 0.00858$ (3 sig fig)

iii) $V = ?$ when $t = 20$ sec

$V = 500 - 500e^{-20k}$
$= 78.8546$ ✓
$\doteqdot 78.9$ m/sec (1dp)

iv) as $t \to \infty$ $\quad e^{-kt} = \frac{1}{e^{kt}} \to 0$

$\therefore V \to 500$ m/sec. ✓

(4)

Question 14.

a) $x = 30\sqrt{2}\,t \qquad y = 30\sqrt{2}\,t - 5t^2$

i) $x = 120\text{m} \quad t = ?$

$120 = 30\sqrt{2}\,t$

$t = \frac{4}{\sqrt{2}}$ sec ✓

height: $y = 30\sqrt{2}\,t - 5t^2$

$= 30\sqrt{2}\left(\frac{4}{\sqrt{2}}\right) - 5\left(\frac{4}{\sqrt{2}}\right)^2$

$= 120 - 40$

$= 80\text{ m}$ ✓

ii) horizontal range: $y = 0 \quad t = ?$

$30\sqrt{2}\,t - 5t^2 = 0$

$5t\,(6\sqrt{2} - t) = 0$

$t = 0 \quad t = 6\sqrt{2}$.

range: $x = 30\sqrt{2} \times 6\sqrt{2}$

$= 360\text{m}$ ✓

Therefore this particle will travel 360m horizontally.

(3)

b) i) $P\,(x_1, y_1) \quad Q\,(x_1 + a, y_2)$ since $x^2 = 4ay$

$y = \frac{x^2}{4a}$

$y_2 = \frac{(x_1+a)^2}{4a}$ ✓

midpt C $\left(\frac{x_1 + x_1 + a}{2},\ \frac{\frac{x_1^2}{4a} + \frac{(x_1+a)^2}{4a}}{2}\right)$

$= \left(\frac{2x_1 + a}{2},\ \frac{x_1^2 + x_1^2 + 2ax_1 + a^2}{8a}\right)$ ✓

$= \left(\frac{2x_1 + a}{2},\ \frac{2x_1^2 + 2ax_1 + a^2}{8a}\right)$

(2)

Quest. 14. Cont.

b ii)

$x = \frac{2x_1 + a}{2}$

$2x = 2x_1 + a$

$x_1 = \frac{2x - a}{2}$

$y = \frac{2x_1^2 + 2ax_1 + a^2}{8a}$

$y = \frac{2\left(\frac{2x-a}{2}\right)^2 + 2a\left(\frac{2x-a}{2}\right) + a^2}{8a}$ ✓

$= \frac{\frac{2(4x^2 - 4ax + a^2)}{4\,2} + 2ax - a^2 + a^2}{8a}$

$= \frac{4x^2 - 4ax + a^2 + 4ax}{16a}$

$= \frac{4x^2 + a^2}{16a}$. ✓

$\therefore 16ay = 4x^2 + a^2$

$4x^2 = 16ay - a^2$

$\therefore$ a parabola (the vertex is $(0, \frac{a}{16})$)

(2)

c.) i) $\sum_{r=1}^{n} r^3 = \frac{1}{4} n^2 (n+1)^2$

test true for $n=1$: LHS $= 1^3 = 1$ RHS $= \frac{1}{4} \times 1\,(1+1)^2 = \frac{1}{4} \times 4 = 1$

$\therefore$ LHS = RHS — true for $n = 1$

assume true for $n = k$, $\sum_{r=1}^{n} r^3 = 1 + 2^2 + 3^3 + \cdots n^3 = \frac{1}{4}(n^2)(n+1)^2$

$S_k = 1 + 2^2 + 3^3 + \cdots k^3 = \frac{1}{4} k^2 (k+1)^2$

prove true for $n = k+1$: ie. $S_k + T_{k+1} = S_{k+1}$

$S_{k+1} = \frac{1}{4}(k+1)^2(k+2)^2$

$LHS = \frac{1}{4}k^2(k+1)^2 + (k+1)^3$

$= \frac{(k+1)^2}{4}\left(k^2 + 4(k+1)\right)$

must have $= \frac{1}{4}(k+1)^2(k+2)^2$

$= RHS.$

* ∴ If true for $n=k$ now proved true for $n=k+1$

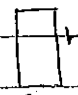
Since true for $n=1$ now true for $n=1+1=2$, $n=3$ and so on by the principles of M.I for all n.

(3) marks - correct

(2) marks - one error (1) Mark - 2 errors.

ii) $y = x^3 + x$

intro



$h = \left(\frac{1}{n}\right)^3 + \frac{1}{n}$. Area $= \frac{1}{n} \times \left(\frac{1}{n^3} + \frac{1}{n}\right)$ ✓

width $= \frac{1}{n}$

2nd strip $h = \left(\frac{2}{n}\right)^3 + \frac{2}{n}$ last strip $h = \left(\frac{n-1}{n}\right)^3 + \frac{n-1}{n}$.

$A = \frac{1}{n}\left(\left(\frac{1}{n}\right)^3 + \frac{2}{n}\right)$ $A = \frac{1}{n} \times \left(\left(\frac{n-1}{n}\right)^3 + \frac{n-1}{n}\right)$

✓working towards An.

∴ Total Area $A_n = \frac{1}{n} \times \left[\left(\frac{1}{n^3} + \frac{1}{n}\right) + \left(\left(\frac{2}{n}\right)^3 + \frac{2}{n}\right) + \left(\left(\frac{3}{n}\right)^3 + \frac{3}{n}\right) + \cdots + \left(\left(\frac{n-1}{n}\right)^3 + \frac{n-1}{n}\right)\right]$

∴ two parts $A_n = \frac{1}{n}\sum_{r=1}^{n-1}\left(\frac{r}{n}\right)^3 + \frac{1}{n}\sum_{r=1}^{n-1}\frac{r}{n}$ ✓ splitting

$= \frac{1}{n^4}\sum_{r=1}^{n-1}(r)^3 + \frac{1}{n^2}\sum_{r=1}^{n-1} r$

iii) Area $= \lim_{n\to\infty} A_n$ and $\sum_{r=1}^{n} r^3 = \frac{1}{4}(n)^2(n+1)^2$ from (i)

$A_n = \frac{n^2(n+1)^2}{4} \cdot \frac{1}{n^4} + \frac{1}{n^2}\sum_{r=1}^{n-1} r$

now $\sum_{r=1}^{n-1} r$ is a series (AP)

$1 + 2 + 3 \cdots n-1$

$\therefore S_n = \frac{n}{2}(a+l)$

$= \frac{n-1}{2}(1+n-1)$

$= \frac{(n-1)}{2}(n)$

$\therefore A_n = \frac{n^2(n+1)^2}{4n^4} + \frac{1}{n^2} \times \frac{(n-1)}{2}n$

$= \frac{1}{4} \cdot \frac{(n+1)^2}{n^2} + \frac{1}{2} \cdot \frac{(n^2-n)}{n^2}$

$\lim_{n\to\infty} A_n = \lim_{n\to\infty}\left[\frac{1}{4}\left(1 + \frac{2}{n} + \frac{1}{n^2}\right) + \frac{1}{2}\left(1 - \frac{1}{n}\right)\right]$

$= \frac{1}{4} + \frac{1}{2}$ ✓✓ as $\frac{1}{n}, \frac{1}{n^2} \to 0$ as $n \to \infty$

$= \frac{3}{4} u^2$

* needed to show working that achieves $\frac{1}{4}$ and $\frac{1}{2}$.