BAULKHAM HILLS HIGH SCHOOL

2015 HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 120 minutes
- Write using black or blue pen
- Board-approved calculators may be used
- Show all necessary working in Questions 11-14
- Marks may be deducted for careless or badly arranged work

Total marks – 70
Exam consists of 11 pages.

This paper consists of TWO sections.

<u>Section 1</u> – Page 2-4 (10 marks)
Questions 1-10

- Attempt Questions 1-10
  Allow about 15 minutes for this section.

<u>Section II</u> – Pages 5-10 (60 marks)

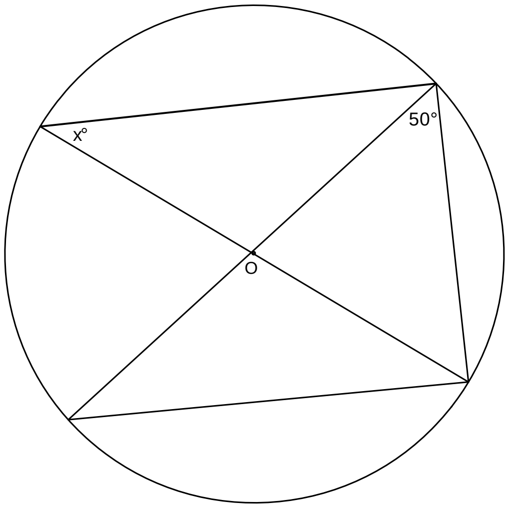
- Attempt questions 11-14
  Allow about 1 hour and 45 minutes for this section.

Table of Standard Integrals is on page 11

Section I - 10 marks

Use the multiple choice answer sheet for question 1-10

1. If O is the centre of the circle, the value of x in the following diagram is:



(A) 25°

(B) 40°

(C) 50°

(D)) 80°

2. The point P divides the interval AB externally in the ratio 3 : 2. If A(-2,2) and B(8,-3) what is the y coordinate of the point P?

(A) -13

(B) -1

(C) 4

(D) 28

3. How many distinct arrangements of the letters of the word $ALGEBRA$ are possible in a straight line if the A's are separated.

(A) 720

(B) 1800

(C) 2160

(D) 2520

4. The polynomial $P(x) = x^3 - 6x^2 - 2x + k$ has a factor of $x + 2$. What is the value of k?

(A) -28

(B) -20

(C) 20

(D) 28

5. The acute angle between the lines $4x + y = 2$ and $y = 2x - 1$ to the nearest degree is :

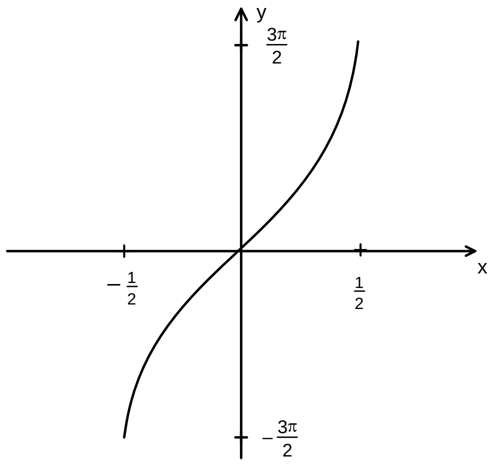
(A) 12°

(B) 13°

(C) 40°

(D) 41°

6. The equation of an inverse trig function drawn below is :



(A) $y = \frac{1}{3}\sin^{-1}\frac{x}{2}$

(B) $y = \frac{1}{3}\sin^{-1}2x$

(C) $y = 3\sin^{-1}\frac{x}{2}$

(D) $y = 3\sin^{-1}2x$

7. A particle moving in simple harmonic motion with displacement x and velocity v, is given by $v^2 = 9(16 - x^2)$. What is its amplitude (A) and its period (T) ?

(A) A=3 T= $\frac{\pi}{2}$

(B) A=3 T= $\frac{2\pi}{3}$

(C) A=4 T = $\frac{\pi}{2}$

(D) A=4 T = $\frac{2\pi}{3}$

8. $\int \frac{1}{\sqrt{25-4x^2}}dx =$

(A) $\frac{1}{2}\sin^{-1}\frac{2x}{5}+c$

(B) $\frac{1}{2}\sin^{-1}\frac{4x}{25}+c$

(C) $\frac{1}{4}\sin^{-1}\frac{2x}{5}+c$

(D) $\frac{1}{4}\sin^{-1}\frac{4x}{25}+c$

9. The derivative of $\tan^{-1}x^4$ is:

(A) $\frac{1}{1+x^8}$

(B) $\frac{4x^3}{1+x^8}$

(C) $4(\tan^{-1}x)^3$

(D) $\frac{4(\tan^{-1}x)^3}{1+x^2}$

10. The solution to $|2x-1| \leq |x-2|$ is

(A) $x \leq -1$

(B) $x \geq 1$

(C) $-1 \leq x \leq 1$

(D) $x \leq -1 \; or \; x \geq 1$

End of Section 1

Section II – Extended Response

All necessary working should be shown in every question.

Question 11 **(15 marks)** - Start on the appropriate page in your answer booklet		**Marks**
a)	Solve $\frac{2}{3x-1} \leq 1$	**3**
b)	Find $\int \frac{2x\ dx}{(2x+1)^2}$ using the substitution $u = 2x + 1$	**3**
c)	Evaluate $\lim_{x \to 0} \frac{\sin \frac{1}{2} x}{3x}$	**2**
d)	Find the constant term in the expansion $\left(2x + \frac{3}{x^3}\right)^8$.	**2**
e)	(i) Show that a root of the continuous function $f(x) = x + \frac{1}{2}\sin 2x - \frac{\pi}{4}$ lies between 0.4 and 0.5.	**1**
	(ii) Hence use one application of Newton's method with an initial estimate of $x = 0.4$ to find a closer approximation for the root to 2 significant figures.	**2**
f)	Solve $\sin 2\theta = \cos\theta$ for $0 \leq \theta \leq 2\pi$	**2**
	End of Question 11	

Question 12 **(15 marks)** - Start on the appropriate page in your answer booklet		**Marks**
a)	Evaluate $\int_0^{\frac{\pi}{8}} \sin^2 2x \ dx$	**3**
b)	(i) From a group of 6 boys and 6 girls, 8 are chosen at random to form a group. How many different groups of 8 people can be formed?	**1**
	(ii) How many of these groups consist of 4 boys and 4 girls?	**1**
	(iii) 4 boys and 4 girls are chosen and placed around a circle. What is the probability that the boys and girls alternate?	**2**
c)	The rate of change of the temperature (T) of an object is proportional to the difference between the temperature of the object and the temperature of the surrounding medium(C), ie $\frac{dT}{dt} = k(T - C)$ An object is heated and placed in a room of temperature $20°C$ to cool. After 10 minutes its temperature is 36°C. After 20 minutes the temperature is 30°C.	
	(i) Show $T = C + Ae^{kt}$ is a solution to the differential equation above.	**1**
	(ii) Find the value of A and the value of k to 3 decimal places.	**3**
	(iii) What was the temperature of the object when it was first placed in the room?	**1**
d)	Prove $(1)(3)(5)$ … … … … … … … …(2n+1) $= \frac{(2n+1)!}{2^n n!}$ for $n \geq 0$ by mathematical induction.	**3**
	End of Question 12	

Question 13 **(15 marks)** - Start on the appropriate page in your answer booklet	**Marks**

a) The polynomial $P(x) = x^3 - 6x^2 + kx - 8$ has roots α, β and γ.

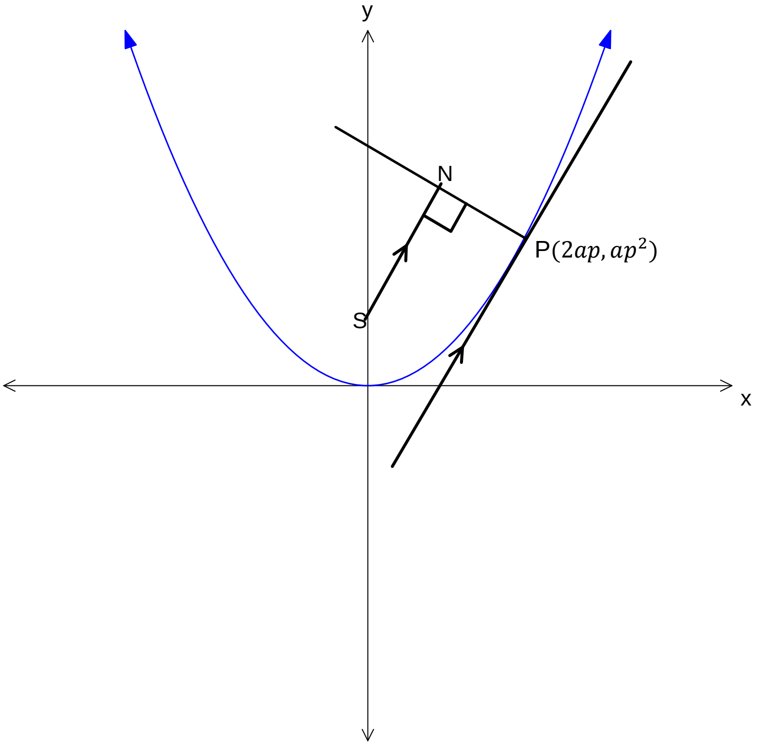
Find :

(i) $\alpha + \beta + \gamma$ **1**

(ii) $\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2$ **2**

(iii) k if $P(x)$ has a triple root. **2**

b) *PN* is the normal to the parabola $x^2 = 4ay$ at the point P $(2ap, ap^2)$. The normal intersects the line *SN* which is parallel to the tangent at *P*. *S* is the focus of the parabola.



(i) Show the equation of the normal *PN* is $x + py = 2ap + ap^3$. **2**

(ii) Find the equation of the line *SN*. **1**

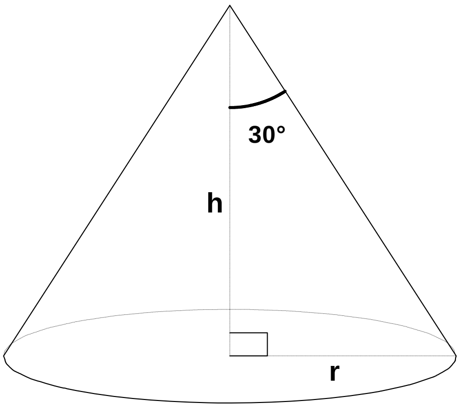
(iii) Show that *N* has coordinates $(ap, ap^2 + a)$. **2**

(iv) Find the equation of the locus of *N* as *P* moves on the parabola. **2**

Question 13 continues on the following page

c) Sand is falling on the ground forming a conical pile whose semi apex angle is 30°.

The volume of the pile is increasing at a rate of $\frac{\pi}{100} m^3/s$. $(V = \frac{1}{3}\pi r^2 h)$



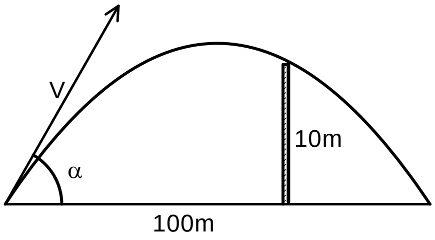
(i) Show that the volume of the pile is given by: $V = \frac{\pi h^3}{9}$ **1**

(ii) Find the rate at which the height of the pile is increasing when the height of the pile is 2 metres. **2**

End of Question 13

Question 14 (15 marks) - Start on the appropriate page in your answer booklet

a)



A projectile is fired from the ground with an angle of projection given by $\alpha = \tan^{-1}\frac{3}{4}$ and initial velocity V.

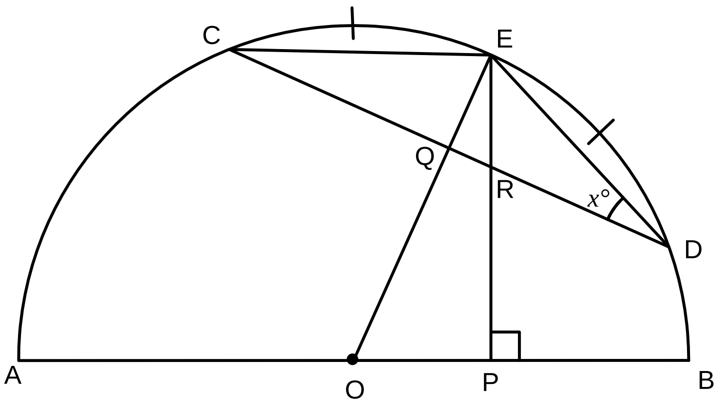
It just clears a wall $10m$ high $100m$ away. Let acceleration due to gravity be g=10ms^{-2}.

(i) Show that the equations of motion are $x = \frac{4Vt}{5}$ and $y = -5t^2 + \frac{3Vt}{5}$. **2**

(ii) Find the initial velocity, V of the projectile. **2**

(iii) At what speed is the projectile travelling the instant it clears the wall? **2**

b) Copy or trace the diagram below in your exam booklet.



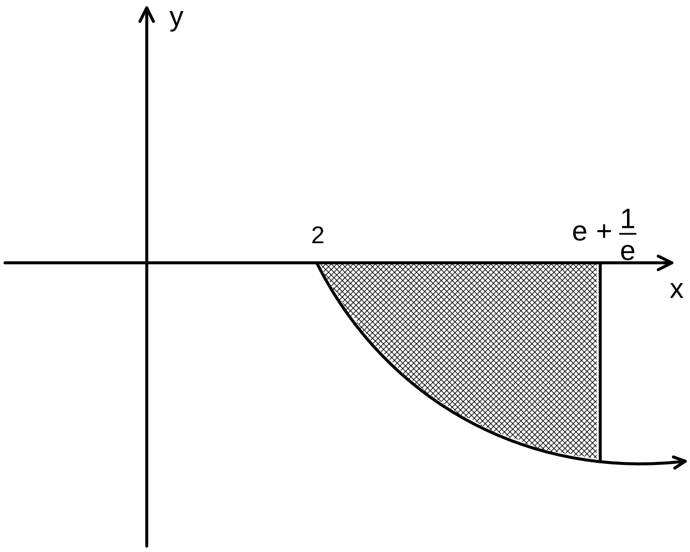
E is the midpoint of the arc CD. The radius OE meets CD at Q. EP is perpendicular to the diameter AB and meets CD at R.

(i) If $\angle CDE = x°$, show $\angle EOD = 2x°$. **2**

(ii) Prove $OPRQ$ is a cyclic quadrilateral. **2**

Question 14 continues on the following page

c) Below is the graph of $y = \ln\left(\frac{x-\sqrt{x^2-4}}{2}\right)$



(i) Show that the equation of the inverse function is given by $y = e^x + e^{-x}$. **2**

(ii) Hence find the area of the shaded region above. **3**

End of Paper.

2015 EXT.1. TRIAL SOLUTIONS:-

M. choice.

1) ~~B~~ B 2) A 3) B 4) D 5) D
6) D 7) D 8) A 9) B 10) C

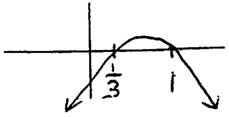
Question 11.

a) $\frac{2}{3x-1} \leq 1 \quad x \neq \frac{1}{2}$

$\frac{2(3x-1)^2}{\cancel{(3x-1)}} - 1(3x-1)^2 \leq 0$ ✓

$(3x-1)(2-(3x-1)) \leq 0$

$(3x-1)(3-3x) \leq 0$ ✓



$\therefore$ Sol'n $x < \frac{1}{3}$ or $x \geq 1$ ✓

b) $\int \frac{2x}{(2x+1)^2}\,dx \qquad u = 2x+1$

$\frac{du}{dx} = 2$

$\therefore \int \frac{2x}{(2x+1)^2}\,dx \qquad dx = \frac{du}{2}$ ✓

$2x = u-1$

$= \int \frac{u-1}{u^2} \cdot \frac{du}{2}$

$= \frac{1}{2}\int \left(\frac{u}{u^2} - \frac{1}{u^2}\right) du$

$= \frac{1}{2}\int \left(\frac{1}{u} - u^{-2}\right) du$ ✓

$= \frac{1}{2}\left(\ln u + \frac{1}{u}\right) + c$

$= \frac{1}{2}\left(\ln(2x+1) + \frac{1}{2x+1}\right) + c$ ✓

c) $\lim_{x \to 0} \frac{\sin \frac{1}{2}x}{3x}$

$= \lim_{x \to 0} \frac{\sin \frac{1}{2}x}{\frac{1}{2}x} \cdot \frac{\frac{1}{2}x}{3x}$ ✓

$= 1 \times \frac{\frac{1}{2}}{3}$

$= \frac{1}{6}$ ✓

d) $\left(2x + \frac{3}{x^3}\right)^8$

General term $= {}^8C_k (2x)^{8-k} (3x^{-3})^k$

$= {}^8C_k\, 2^{8-k} x^{8-k} 3^k x^{-3k}$

$= {}^8C_k\, 2^{8-k} 3^k x^{8-4k}$ ✓

$\therefore 8 - 4k = 0$

$k = 2$

$\therefore$ Constant term $= {}^8C_2\, 2^6 3^2$

$= 16128$ ✓

e) (i) $f(0.4) = -0.0267\ldots$

$f(0.5) = 0.135\ldots$

since $f(0.4) < 0$ & $f(0.5) > 0$ ✓

a root exists between 0.4 & 0.5

(ii) $f'(x) = 1 + \cos 2x$

$\therefore a_1 = 0.4 - \left(\frac{-0.026\ldots}{1.696\ldots}\right)$ ✓

$= 0.415\ldots$

$a_1 = 0.42$ ✓

f) $\sin 2\theta = \cos\theta$

$2\sin\theta\cos\theta - \cos\theta = 0$

$\cos\theta(2\sin\theta - 1) = 0$

$\sin\theta = \frac{1}{2} \qquad \cos\theta = 0$

$\therefore \theta = \underbrace{\frac{\pi}{6}, \frac{5\pi}{6}}_{✓}, \underbrace{\frac{\pi}{2}, \frac{3\pi}{2}}_{✓}$

Question 12

a) $\int_0^{\frac{\pi}{8}} \sin^2 2x\,dx$

$= \frac{1}{2}\int_0^{\frac{\pi}{8}} (1 - \cos 4x)\,dx$ ✓

$= \frac{1}{2}\left[x - \frac{\sin 4x}{4}\right]_0^{\frac{\pi}{8}}$ ✓

$= \frac{1}{2}\left[\left(\frac{\pi}{8} - \frac{\sin\frac{\pi}{2}}{4}\right) - (0-0)\right]$

$= \frac{1}{2}\left[\frac{\pi}{8} - \frac{1}{4}\right]$

$= \frac{\pi}{16} - \frac{1}{8}$ or $\frac{\pi - 2}{16}$ ✓

$= (0.0713\ldots)$

b (i) ${}^{12}C_8 = 495$ ✓

(ii) ${}^6C_4 \times {}^6C_4 = 225$ ✓

(iii) $\frac{\text{Alternating}}{\text{Total}} = \frac{4! \times 3!}{7!}$ ✓ ✓

$= \frac{1}{35}$ (or equivalent)

c) (i) $T = C + Ae^{kt} \rightarrow Ae^{kt} = T - C$ ①

$\frac{dT}{dt} = kAe^{kt}$ & from ① ✓

$= k(T - C)$

$\therefore T = C + Ae^{kt}$ is a sol'n

(ii) when $t = 10 \quad T = 36 \qquad C = 20$

$\therefore 36 = 20 + Ae^{10k}$

$\therefore Ae^{10k} = 16$ --①

when $t = 20 \quad T = 30$ ✓ (either or both)

$\therefore 30 = 20 + Ae^{20k}$

$\therefore 10 = Ae^{20k}$ --②

② ÷ ① $\frac{10}{16} = \frac{\cancel{A}e^{20k}}{\cancel{A}e^{10k}}$

$\frac{5}{8} = e^{10k}$

$\therefore k = \frac{\ln\left(\frac{5}{8}\right)}{10} \; (= -0.047)$ ✓

Sub into ② $10 = Ae^{-0.047.(20)}$

$A = 25.6$ ✓

$\therefore T = 20 + 25.6e^{kt}$

when $t = 0$

$T = 20 + 25.6e^0$

$= 45.6.$ ✓

d) $(1)(3)(5)\ldots(2n+1) = \frac{(2n+1)!}{2^n n!} \quad n \geq 0$

Step 1.

Prove true for $n = 0$

LHS $\therefore 2(0)+1 = 1 \qquad$ RHS $= \frac{(2(0)+1)!}{2^0.0!}$

$= \frac{1}{1} = 1$

$\therefore$ LHS = RHS (true for $n = 0$)

Step 2

Assume true for $n = k$ ie

$(1)(3)(5)\ldots(2k+1) = \frac{(2k+1)!}{2^k k!}$

Step 3

Prove true for $n = k+1$ ie

$(1)(3)\ldots(2k+1)(2k+3) = \frac{(2k+3)!}{2^{k+1}(k+1)!}$ ✓

ie $\frac{(2k+1)!}{2^k k!} \cdot (2k+3) = \frac{(2k+3)!}{2^{k+1}(k+1)!}$

$= \frac{(2k+1)!\,(2k+2)(2k+3)}{2^k k!\,(2k+2)}$ ✓

$= \frac{(2k+3)!}{2^k k!\; 2(k+1)}$ ✓

$= \frac{(2k+3)!}{2^{k+1}(k+1)!}$

$=$ RHS.

Step 4

$\therefore$ True for $n = k+1$ if true for $n = k$. Therefore true for $n = 0, n = 1, n = 2 \ldots$ & for all n by M. Induction.

a) $P(x) = x^3 - 6x^2 + kx - 8$

(i) $\alpha+\beta+\gamma = 6$ ✓

(ii) $\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2$

$= \alpha\beta\gamma(\alpha+\beta+\gamma)$ ✓

$= 8(6)$

$= 48$ ✓

(iii) If $P(x)$ has a triple root then

$\alpha^3 = 8 \rightarrow \alpha = 2$ ✓

sum of roots 2 at a time

$3\alpha^2 = k$

$\therefore 3(2)^2 = k$

$k = 12$ ✓

b) (i) $y = \frac{x^2}{4a}$

$y' = \frac{2x}{4a}$

at $x = 2ap$ $y' = \frac{2ap}{2a} = p$ ✓

$\therefore$ Normal

$y - ap^2 = -\frac{1}{p}(x - 2ap)$ ✓

$py - ap^3 = -x + 2ap$

$x + py = 2ap + ap^3$ — ①

(ii) grad. of SP $= p$

$\therefore$ SN $\rightarrow y = px + a$ ✓ — ②

(iii) $y = px + a$ sub into ①

$x + p(px+a) = 2ap + ap^3$

$x + p^2x + ap = 2ap + ap^3$

$x(1+p^2) = ap^3 + ap$

$x = \frac{ap\cancel{(p^2+1)}}{\cancel{(1+p^2)}}$

$x = ap$ sub into ② ✓

$y = ap^2 + a$ ✓

$\therefore N(ap, ap^2+a)$

(iv) $x = ap$ $y = ap^2 + a$

✓ $p = \frac{x}{a}$ ↗ $y = a\left(\frac{x^2}{a^2}\right) + a$

$y = \frac{x^2}{a} + a$ ✓

$x^2 = ay - a^2$

$x^2 = a(y - a)$

c) $V = \frac{1}{3}\pi r^2 h$ ① $\tan 30 = \frac{r}{h}$

$r = h \tan 30°$

$r = \frac{h}{\sqrt{3}}$ sub in ①

$\therefore V = \frac{1}{3}\pi\left(\frac{h}{\sqrt{3}}\right)^2 . h$

$= \frac{1}{3}\pi\frac{h^3}{3}$

$V = \frac{\pi h^2}{3}$ ✓

(ii) find $\frac{dh}{dt}$ when $h = 2$

$\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$

$\frac{dV}{dh} = \frac{2\pi h}{3}$ ✓

$\frac{\pi}{100} = \frac{2\pi(2)}{3} \cdot \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{\pi}{100} \times \frac{3}{4\pi}$ ✓

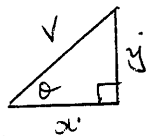
$= \frac{3}{400}$ m/s

14a) $\tan\theta = \frac{3}{4}$ $\therefore \sin\theta = \frac{3}{5}$

$\cos\theta = \frac{4}{5}$

$\ddot{x} = 0$

$\dot{x} = c_1$



when $t = 0$ $\dot{y} = V\sin\theta$

$\dot{x} = V\cos\theta$

$\therefore \dot{x} = V\cos\theta$

$= \frac{4V}{5}$

$x = \frac{4tV}{5}$ ✓

$\ddot{y} = -10$

$\dot{y} = -10t + c_2$

when $t = 0$ $\dot{y} = V.\frac{3}{5}$

$\therefore \dot{y} = -10t + \frac{3V}{5}$

$y = -5t^2 + \frac{3tV}{5} + c_3$ ✓

when $t = 0$ $y = 0 \Rightarrow c_3 = 0$

$\therefore y = -5t^2 + \frac{3tV}{5}$

when $x = 100$ $y = 10$

$\therefore 100 = \frac{4tV}{5}$ $10 = -5t^2 + \frac{3tV}{5}$ — ①

$tV = 125$ — ②

sub ② into ①

$10 = -5t^2 + \frac{3(125)}{5}$

$t = \sqrt{13}$ ✓

sub into ② (Must find V in part ii not part iii)

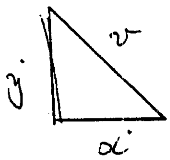
$V\sqrt{13} = 125$

$= \frac{125}{\sqrt{13}}$

$= 34.67$ ✓

at $(100, 10)$

need $\dot{y}$ & $\dot{x}$



at $t = \sqrt{13}$

$\dot{x} = \frac{4(34.6...)}{5}$

$= 27.73....$ ✓

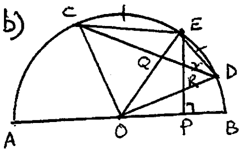
$\dot{y} = -10\sqrt{13} + \frac{3}{5}(34.67)$

$= -15.25...$

$v = \sqrt{(27.73)^2 + (-15.25)^2}$ ✓

$v = 31.64....$

b)



Construct OD & OC

(i) $\angle EDC = x°$ (Given)

✓ $\angle ECD = x°$ (Equal chord subtend = ∠'s at the Circumference)

✓ $\therefore \angle EOD = 2x°$ (∠ at centre = 2 times ∠'s at circumference on same arc)

(ii) $OE = OD$ (Radii)

$\angle OED = \angle ODE$ (opp. = sides in a Δ)

$\therefore \angle ODE = \frac{180 - 2x}{2} = 90 - x$ (Angle sum Δ)

$\angle EDC + \angle ODQ = 90 - x$

$x + \angle ODQ = 90 - x$

$\angle ODQ = 90 - 2x$

$\angle OQD + 90 - 2x + 2x = 180°$ ✓

(∠ sum Δ OQD)

$\therefore \angle OQD = 90°$

$\angle OPE = 90°$ (EP ⊥ OB)

$\therefore \angle OQR = \angle OPE = 90°$ ✓

$\therefore$ QRPO is a cyclic quad

(opposite angles of a cyclic quad. are supplementary)

14c) $y = \ln\left(\frac{x-\sqrt{x^2-4}}{2}\right)$

inverse

✓ $x = \ln\left(\frac{y-\sqrt{y^2-4}}{2}\right)$

$e^x = \frac{y-\sqrt{y^2-4}}{2}$

$2e^x = y - \sqrt{y^2-4}$

$\sqrt{y^2-4} = y - 2e^x$ ✓

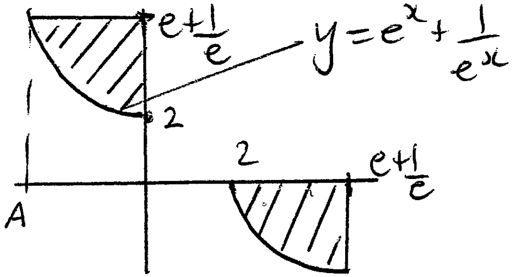
$\cancel{y^2} - 4 = \cancel{y^2} - 4ye^x + 4e^{2x}$

$4ye^x = 4e^{2x} + 4$

$y = \frac{4e^{2x}}{4e^x} + \frac{4}{4e^x}$

$y = e^x + \frac{1}{e^x}$

$y = e^x + e^{-x}$ ✓



2 areas are equal because of symmetry.

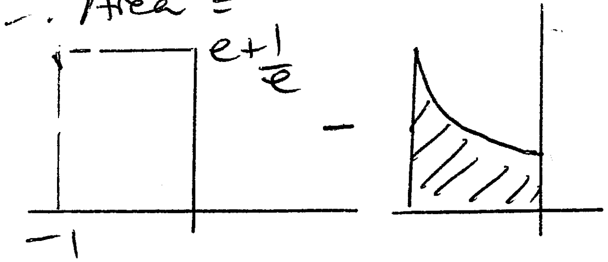
At A $y = e + \frac{1}{e}$

$\therefore e + \frac{1}{e} = e^x + \frac{1}{e^x}$

✓ $x = \pm 1$ by inspection.

here $x = -1$

∴ Area =



$= \left(e+\frac{1}{e}\right)\times 1 - \int_{-1}^{0} e^x + e^{-x}$ ✓

$= e + \frac{1}{e} - \left(e^x - e^{-x}\right)_{-1}^{0}$

$= e + \frac{1}{e} - \left(e^0 - e^0 - (e^{-1} - e^1)\right)$

$= e + \frac{1}{e} - \left(\cancel{1-1} - \frac{1}{e} + e\right)$

$= e + \frac{1}{e} + \frac{1}{e} - e$

$= \frac{2}{e}$ ✓

NB AWARD 2 for

$\int_{-1}^{0} e^x + e^{-x} = e - \frac{1}{e}$

AWARD 2 $\int_{0}^{1} e^x + e^{-x}$

$= e - \frac{1}{e}$.