BAULKHAM HILLS HS PERSEVERE

2020 HIGHER SCHOOL CERTIFICATE TRIAL EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- In Questions 11 – 14, show relevant mathematical reasoning and/or calculations

Total marks: 70

Section I – 10 marks (pages 2 – 5)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 6 – 12)

- Attempt Questions 11 – 14
- Allow about 1 hour and 45 minutes for this section

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the multiple-choice answer page in the writing booklet for Questions 1 – 10.

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1 If $\underset{\sim}{u} = 2\underset{\sim}{i} + 3\underset{\sim}{j}$, the unit vector in the direction of $\underset{\sim}{u}$ is:

A. $\frac{1}{13}\left(2\underset{\sim}{i} + 3\underset{\sim}{j}\right)$

B. $\frac{1}{5}\left(2\underset{\sim}{i} + 3\underset{\sim}{j}\right)$

C. $\frac{1}{\sqrt{13}}\left(2\underset{\sim}{i} + 3\underset{\sim}{j}\right)$

D. $\sqrt{13}\left(2\underset{\sim}{i} + 3\underset{\sim}{j}\right)$

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2 What is the domain and range of $y = \cos^{-1}\left(\frac{3x}{2}\right)$?

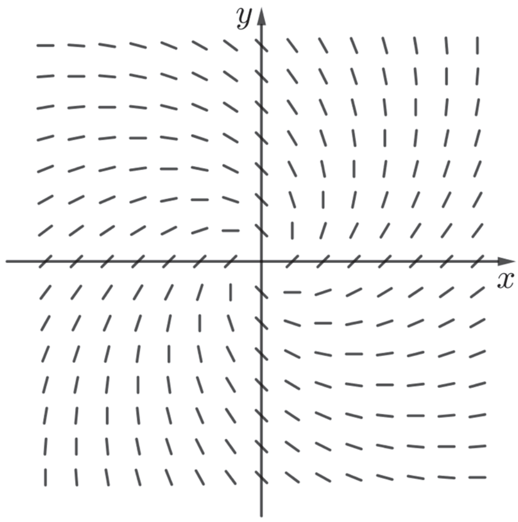
A. Domain: $x \in \left[-\frac{2}{3}, \frac{2}{3}\right]$, range: $y \in [0, \pi]$

B. Domain: $x \in \left[-\frac{3}{2}, \frac{3}{2}\right]$, range: $y \in [0, \pi]$

C. Domain: $x \in \left[-\frac{2}{3}, \frac{2}{3}\right]$, range: $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

D. Domain: $x \in \left[-\frac{3}{2}, \frac{3}{2}\right]$, range: $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

3 The slope field for a differential equation is shown below.



Which of the following could be the differential equation represented?

A. $\frac{dy}{dx} = \frac{x+y}{x-y}$

B. $\frac{dy}{dx} = \frac{y+x}{y-x}$

C. $\frac{dy}{dx} = \frac{x-y}{x+y}$

D. $\frac{dy}{dx} = \frac{y-x}{y+x}$

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4 In a large population, the probability that a person is left-handed is 0.3. Let $\hat{p}$ be the random variable that represents the sample proportion of left-handed people from a sample size of n, drawn from the population.

What is the smallest value of n such that the standard deviation of $\hat{p}$ is less than or equal to 0.07?

A. $n = 42$

B. $n = 43$

C. $n = 61$

D. $n = 62$

5 Given that n students are to be placed into 4 classrooms, what is the minimum number of students to guarantee that there are at least two students born in the same month in each of the 4 classrooms?

A. 48

B. 49

C. 52

D. 53

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6 Given $y = \sin^{-1}\left(\frac{2}{x}\right)$, which of the following is the expression for $\frac{dy}{dx}$?

A. $\frac{-2}{x\sqrt{x^2-4}}$

B. $\frac{2}{x\sqrt{x^2-4}}$

C. $\frac{-x}{\sqrt{x^2-4}}$

D. $\frac{x}{\sqrt{x^2-4}}$

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7 A solution to the differential equation $\frac{dy}{dx} = \frac{2}{\cos(x+y)+\cos(x-y)}$ can be obtained from:

A. $\int \cos y\, dy = \int \sec x\, dx$

B. $\int \sec y\, dy = \int \cos x\, dx$

C. $\int \sin y\, dy = \int \operatorname{cosec} x\, dx$

D. $\int \operatorname{cosec} y\, dy = \int \sin x\, dx$

8 In how many ways can the letters of the word **EXTENSION** be arranged if the letter E's are to be together?

A. $\dfrac{9!}{2!2!}$

B. $\dfrac{9!}{2!}$

C. $\dfrac{8!}{2!2!}$

D. $\dfrac{8!}{2!}$

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9 If $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are two non-zero vectors on the Cartesian plane, which of the following statements is **<u>NOT ALWAYS CORRECT</u>** ?

A. $\left|\text{proj}_{\underset{\sim}{v}}\underset{\sim}{u}\right| \le \left|\underset{\sim}{u}\right|$

B. $\left|\text{proj}_{\underset{\sim}{v}}\underset{\sim}{u}\right| \le \left|\underset{\sim}{v}\right|$

C. $\left|\text{proj}_{\underset{\sim}{v}}\underset{\sim}{u}\right| = \dfrac{\left|\underset{\sim}{u}\cdot\underset{\sim}{v}\right|}{\left|\underset{\sim}{v}\right|}$

D. $\left|\text{proj}_{\underset{\sim}{v}}\underset{\sim}{u}\right| = \left|\underset{\sim}{u}\cdot\hat{\underset{\sim}{v}}\right|$

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10 The graphs of $y = ax$ and $y = \tan^{-1}(bx)$ intersect at three distinct points if:

A. $0 < b < a$

B. $a < b < 0$

C. $a = b$

D. $b < a < 0$

End of Section I

Section II

60 marks
Attempt Questions 11 – 14
Allow about 1 hour and 45 minutes for this section

Answer each question in the appropriate section of the writing booklet. Extra writing paper is available.

In Questions 11 – 14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 section of the writing booklet.

(a) Using the substitution $u = 1 + x$, find $\int \frac{x}{\sqrt{1+x}}\,dx$. **2**

(b) Solve $\frac{dy}{dx} = \frac{y^2}{1+x^2}$ given $y(0) = 1$. **3**

(c) By letting $t = \tan\frac{x}{2}$, solve $\sin x + \cos x = -1$ in the domain $0 \le x \le 2\pi$. **3**

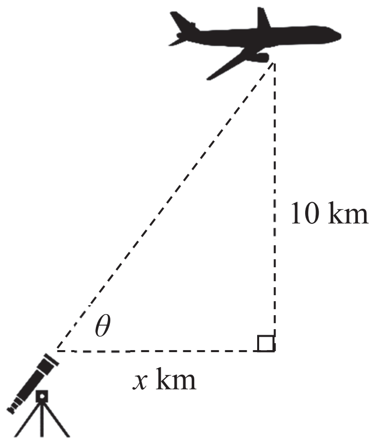
(d) (i) Show that $\int_0^{\frac{3\pi}{2}} \sin^2 x\,dx = \frac{3\pi}{4}$. **2**

(ii) The graph of $y = 1 + \sin x$ for $0 \le x \le \frac{3\pi}{2}$ is rotated about the x-axis. Find the exact volume of the solid generated. **2**

Question 11 continues on the next page

Question 11 (continued)

(e) A plane flies horizontally at an altitude of 10 km. It passes directly over a tracking telescope on the ground. When the angle of elevation from the telescope to the plane is $\frac{\pi}{3}$ radians, the angle of elevation is decreasing at a rate of $\frac{\pi}{6}$ rad/min . How fast is the plane travelling at the time of measurement? **3**



End of Question 11

Question 12 (15 marks) Use the Question 12 section of the writing booklet.

(a) The following table is the distribution of a random variable X where $X \sim \text{Bin}(4, p)$.

x	0	1	2	3	4
$P(X = x)$	0.2401	a	b	0.0756	0.0081

(i) Use the known entries in the probability distribution table to calculate the value of p. **1**

(ii) Find the values of a and b **2**

(iii) Calculate $\text{Var}(X)$. **1**

(iv) Find $P(X \geq 3)$. **1**

(b) A team of biologists released 500 fish into a lake with a maximum carrying capacity of 10000 fish. It was found that the number of fish tripled during the first year. The fish population, N, in the lake after t years is modelled by the logistic equation:

$$\frac{dN}{dt} = kN(10000 - N)$$

where k is a constant.

(i) Given $\dfrac{10000}{N(10000-N)} = \dfrac{1}{N} + \dfrac{1}{10000-N}$, solve the differential equation to show that the fish population after t years is $N = \dfrac{10000}{1+19e^{-10000kt}}$. **3**

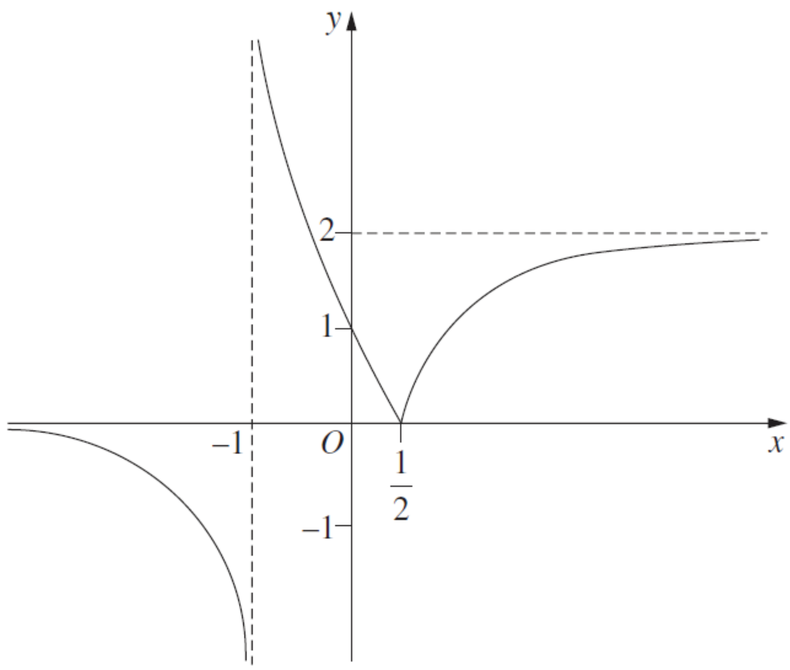
(ii) Show that $k = \dfrac{1}{10000}\ln\left(\dfrac{57}{17}\right)$. **2**

(iii) Correct to the nearest month, how long will it take for the fish population in the lake to reach 7000 fish? **2**

Question 12 continues on the next page

Question 12 (continued)

(c) The diagram shows the graph of $y = f(x)$.



Draw sketches of the graphs of the following on separate number planes:

(i) $y = f(|x|)$ **1**

(ii) $y^2 = f(x)$ **2**

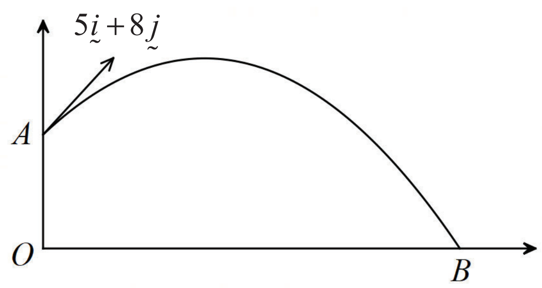
End of Question 12

Question 13 (16 marks) Use the Question 13 section of the writing booklet.

(a) Prove using mathematical induction that $2^{n+2}+3^{2n+1}$ is divisible by 7 for all positive integers n. **3**

(b) A ball is struck by a racquet from a point A which has position vector $21\underset{\sim}{j}$ metres relative to a fixed origin O. Initially, the velocity vector of the ball is $\left(5\underset{\sim}{i}+8\underset{\sim}{j}\right)$ m/s, where $\underset{\sim}{i}$ and $\underset{\sim}{j}$ are the unit vectors in the horizontal and vertical directions respectively.

After being struck, the ball undergoes projectile motion under the influence of gravity only, that is, the acceleration vector of the ball $\underset{\sim}{a}=-10\underset{\sim}{j}$. The ball will hit the ground at point B, a point that is on the same horizontal level as the origin.



(i) Show that the velocity vector of the ball t seconds after being struck is $\underset{\sim}{v}=5\underset{\sim}{i}+\left(8-10t\right)\underset{\sim}{j}$. **2**

(ii) Find an expression for the position vector $\underset{\sim}{r}$ of the ball relative to O at a time t seconds after it was struck. **2**

(iii) Find the maximum height that the ball reaches. **2**

(iv) Find the speed of the ball when it lands at B and the angle at which the ball strikes the ground. **3**

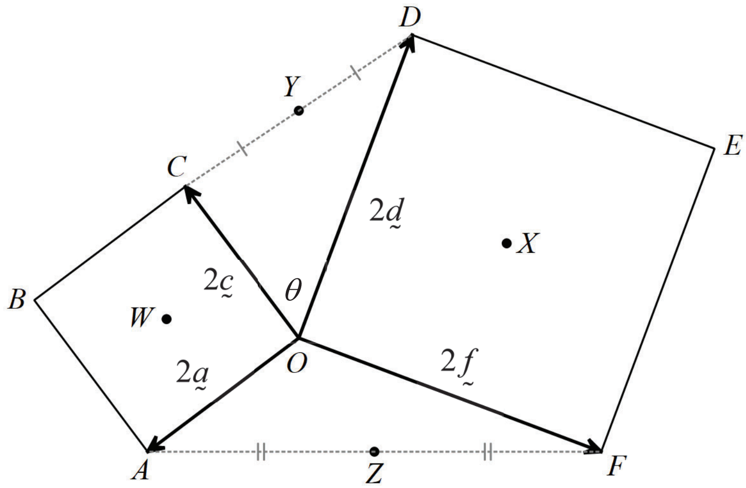
(c) By first expressing $\cos x-\sin x$ in the form $R\cos\left(x+\alpha\right)$, determine the maximum value of $2+\cos x-\sin x$ and the smallest positive angle, in radians, at which this maximum value occurs. **4**

End of Question 13

Question 14 (14 marks) Use the Question 14 section of the writing booklet.

(a) In the diagram, *OABC* and *ODEF* are two non-overlapping squares. *W* and *X* are the centres of these two squares respectively.

Points *Y* and *Z* are the midpoints on the intervals *CD* and *AF* respectively, as shown in the diagram, making *WYXZ* a parallelogram (*W*, *Y*, *X* and *Z* are the midpoints of the sides of quadrilateral *ACDF*).



Let *A*, *C*, *D* and *F* have position vectors $2\underset{\sim}{a}$, $2\underset{\sim}{c}$, $2\underset{\sim}{d}$ and $2\underset{\sim}{f}$ respectively.

It can be shown that the interval joining the midpoint of two sides of a triangle is parallel to the third side and is half the length of this third side. You may use this result without proving it.

(i) Consider ΔACF, express $\overrightarrow{WZ}$ in terms of $\underset{\sim}{c}$ and $\underset{\sim}{f}$. Similarly, find an expression for $\overrightarrow{WY}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{d}$ using ΔACD. **2**

(ii) ΔADO and ΔCFO are congruent triangles. By considering the lengths of *AD* and *CF*, show that $\left|\overrightarrow{WY}\right| = \left|\overrightarrow{WZ}\right|$. **1**

(iii) Let $\angle COD = \theta$ as shown above, show that $\underset{\sim}{a} \cdot \underset{\sim}{f} = -\underset{\sim}{c} \cdot \underset{\sim}{d}$. **2**

(iv) By considering the dot product of vectors $\overrightarrow{WY}$ and $\overrightarrow{WZ}$, as well as the results from parts (ii) and (iii), show that *WYXZ* is a square. **2**

Question 14 continues on the next page

Question 14 (continued)

(b) If α and β are the roots of the equation $x^2+mx+n=0$, find the roots of $nx^2-mx+1=0$ in terms of α and β. **3**

(c) Suppose there are *n* different books, and there are *p* identical copies of each of these *n* books. (You may assume *n* and *p* are non-zero integers by definition.)

(i) If all the books are to be placed in a line on one shelf, in how many ways can this be done? **2**

Consider now a library with *n* different books, where there are *p* identical copies of each of these *n* books on one shelf. The library barcodes each book differently to keep track of each book in its catalogue.

(ii) Suppose you want to borrow at least one book from the shelf, but you do not want to borrow multiple copies of the same book (despite the different barcodes). In how many ways can this be done? **2**

(To clarify: At the borrowing desk, the librarian will scan out copy 1 of Book A differently to copy 2 of Book A because they have different barcodes.)

End of Paper

YEAR 12 TRIAL EXAMINATION 2020
MATHEMATICS EXTENSION 1
MARKING GUIDELINES

Section I

Multiple-choice Answer Key

Question	Answer
1	C
2	A
3	A
4	B
5	C

Question	Answer
6	A
7	A
8	D
9	B
10	D

Questions 1 – 10

Sample solution		
1.	$\hat{\underset{\sim}{u}} = \dfrac{\underset{\sim}{u}}{\lvert \underset{\sim}{u} \rvert}$ $= \dfrac{2\underset{\sim}{i} + 3\underset{\sim}{j}}{\sqrt{2^2+3^2}}$ $= \dfrac{1}{\sqrt{13}}\left(2\underset{\sim}{i} + 3\underset{\sim}{j}\right)$	
2.	Domain: $-1 \le \dfrac{3x}{2} \le -1$ $-2 \le 3x \le 2$ $-\dfrac{2}{3} \le x \le \dfrac{2}{3}$	Range: $0 \le \cos^{-1}\left(\dfrac{3x}{2}\right) \le \pi$ $0 \le y \le \pi$
3.	From the slope field, $\dfrac{dy}{dx} = 0$ when $y = -x$ and $\dfrac{dy}{dx}$ is undefined when $y = x$.	
4.	$\sigma \le 0.07$ $\sqrt{\dfrac{pq}{n}} \le 0.07$ $\sqrt{\dfrac{0.3 \times 0.7}{n}} \le 0.07$ $0.21 \le n \times 0.07^2$ $n \ge \dfrac{0.21}{0.07^2}$ $n \ge 42.85$ The smallest value of n would be $n = 43$.	
5.	Worst case scenario, in each of the 4 classrooms, there are 12 students all born in different months. By the pigeonhole principle, one more student in each classroom would guarantee that there are at least two students in each classroom who are born in the same month. Therefore, a minimum of 52 students is required to guarantee that at least two students in each classroom are born in the same month.	

Sample solution	
6.	$y=\sin^{-1}\left(\frac{2}{x}\right)$ $\frac{dy}{dx}=\frac{\frac{d}{dx}\left(\frac{2}{x}\right)}{\sqrt{1-\left(\frac{2}{x}\right)^2}}$ $=\frac{-\frac{2}{x^2}}{\sqrt{1-\frac{4}{x^2}}}$ $=\frac{-\frac{2}{x^2}}{\frac{1}{x}\sqrt{x^2-4}}$ $=\frac{-2}{x\sqrt{x^2-4}}$
7.	$\frac{dy}{dx}=\frac{2}{\cos(x+y)+\cos(x-y)}$ $\frac{dy}{dx}=\frac{2}{\cos x\cos y-\sin x\sin y+\cos x\cos y+\sin x\sin y}$ $=\frac{2}{2\cos x\cos y}$ $=\frac{1}{\cos x\cos y}$ $\int\cos y\,dy=\int\frac{1}{\cos x}dx$ $\int\cos y\,dy=\int\sec x\,dx$
8.	The number of ways to arrange **(EE), X, T, N, S, I, O, N** is $\frac{8!}{2!}$ (8 “letters” to arrange, treating EE as a single “letter”, dividing through by 2! for the repeated N’s)
9.	By the geometry of projection of $\underset{\sim}{u}$ onto $\underset{\sim}{v}$, its length must be shorter than or equal to the length of $\underset{\sim}{u}$. The “signed” length of $\text{proj}_{\underset{\sim}{v}}\underset{\sim}{u}$ is $\underset{\sim}{u}\cdot\hat{\underset{\sim}{v}}$, therefore the length of $\text{proj}_{\underset{\sim}{v}}\underset{\sim}{u}$ is $\lvert\underset{\sim}{u}\cdot\hat{\underset{\sim}{v}}\rvert$. Similarly, the “signed” length of $\text{proj}_{\underset{\sim}{v}}\underset{\sim}{u}$ is $\underset{\sim}{u}\cdot\hat{\underset{\sim}{v}}=\frac{\underset{\sim}{u}\cdot\underset{\sim}{v}}{\lvert\underset{\sim}{v}\rvert}$, therefore the length of $\text{proj}_{\underset{\sim}{v}}\underset{\sim}{u}$ is $\left\lvert\frac{\underset{\sim}{u}\cdot\underset{\sim}{v}}{\lvert\underset{\sim}{v}\rvert}\right\rvert=\frac{\lvert\underset{\sim}{u}\cdot\underset{\sim}{v}\rvert}{\lvert\underset{\sim}{v}\rvert}$. A counter example of option B could be $\underset{\sim}{u}=2\underset{\sim}{i}+2\underset{\sim}{j}$ and $\underset{\sim}{v}=\underset{\sim}{i}$.
10.	$y=\tan^{-1}(bx)$ $\frac{dy}{dx}=\frac{b}{1+(bx)^2}$ At the origin, $\frac{dy}{dx}=\frac{b}{1+0^2}$ $=b$ $y=ax$ is a straight line through the origin with gradient a. These two curves will intersect at three distinct points if the tangent of the inverse tangent function is steeper than the gradient of the line, i.e. $b>a>0$ or $b<a<0$.

Section II

Question 11

	Sample solution	**Suggested marking criteria**
(a)	$\int \frac{x}{\sqrt{1+x}}\,dx = \int \frac{u-1}{\sqrt{u}}\,du$ $u = x+1 \Rightarrow x = u-1$, $du = dx$ $= \int u^{\frac{1}{2}} - u^{-\frac{1}{2}}\,du$ $= \frac{2u^{\frac{3}{2}}}{3} - 2u^{\frac{1}{2}} + c$ $= \frac{2(x+1)^{\frac{3}{2}}}{3} - 2(x+1)^{\frac{1}{2}} + c$ $= \frac{2}{3}\sqrt{x+1}(x-2) + c$	• 2 – correct solution (answer in terms of x suffices, simplification not required) • 1 – correctly converts the integral into an integral in terms of u
(b)	$\frac{dy}{dx} = \frac{y^2}{1+x^2}$ $\int_1^y \frac{dy}{y^2} = \int_0^x \frac{dx}{1+x^2}$ $\int_1^y y^{-2}\,dy = \int_0^x \frac{dx}{1+x^2}$ $\left[\frac{y^{-1}}{-1}\right]_1^y = \left[\tan^{-1} x\right]_0^x$ $-\frac{1}{y} - (-1) = \tan^{-1} x$ $-\frac{1}{y} + 1 = \tan^{-1} x$ $\frac{1}{y} = 1 - \tan^{-1} x$ $y = \frac{1}{1 - \tan^{-1} x}$	• 3 – correct solution • 2 – uses the initial condition • 1 – correctly separates the variables and finding correct integrals with respect to each variable
(c)	$\sin x + \cos x = -1$ $\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2} = -1$ $2t + 1 - t^2 = -1 - t^2$ $2t = -2$ $t = -1$ $\tan\frac{x}{2} = -1$ $\frac{x}{2} = \frac{3\pi}{4}$ $x = \frac{3\pi}{2}$ Check $x = \pi$: $\sin\pi + \cos\pi = 0 + (-1)$ $= -1$ Therefore, the answers to the equation are $x = \frac{3\pi}{2}, \pi$	• 3 – correct solution • 2 – finds $x = \frac{3\pi}{2}$ as a solution • 1 – finds the solution for t – finds $x = \pi$ as a solution

Question 11 (continued)

Sample solution		Suggested marking criteria
(d)	(i) $\displaystyle\int_0^{\frac{3\pi}{2}} \sin^2 x\,dx = \frac{1}{2}\int_0^{\frac{3\pi}{2}} 1-\cos 2x\,dx$ $\displaystyle = \frac{1}{2}\times\left[x-\frac{1}{2}\sin 2x\right]_0^{\frac{3\pi}{2}}$ $\displaystyle = \frac{1}{2}\times\left(\frac{3\pi}{2}-\frac{1}{2}\sin 3\pi\right)$ $\displaystyle = \frac{1}{2}\times\frac{3\pi}{2}$ $\displaystyle = \frac{3\pi}{4}$	• 2 – correct solution • 1 – correctly rewrites the integral using the double angle formula
	(ii) $\displaystyle V = \pi\int_0^{\frac{3\pi}{2}} (1+\sin x)^2\,dx$ $\displaystyle = \pi\int_0^{\frac{3\pi}{2}} 1+2\sin x+\sin^2 x\,dx$ $\displaystyle = \pi\left[\int_0^{\frac{3\pi}{2}} 1+2\sin x\,dx + \int_0^{\frac{3\pi}{2}} \sin^2 x\,dx\right]$ $\displaystyle = \pi\times\left[x-2\cos x\right]_0^{\frac{3\pi}{2}} + \pi\times\frac{3\pi}{4}$ $\displaystyle = \pi\times\left[\left(\frac{3\pi}{2}-2\cos\frac{3\pi}{2}\right)-(0-2)\right]+\frac{3\pi^2}{4}$ $\displaystyle = \pi\times\left(\frac{3\pi}{2}+2\right)+\frac{3\pi^2}{4}$ $\displaystyle = \frac{3\pi^2}{2}+2\pi+\frac{3\pi^2}{4}$ $\displaystyle = \left(\frac{9\pi^2}{4}+2\pi\right)$ cubic units	• 2 – correct solution • 1 – establishes the correct integral for the volume and splits the integral appropriately
(e)	$\displaystyle \tan\theta = \frac{10}{x}$ $\displaystyle x = \frac{10}{\tan\theta}$ $x = 10\cot\theta$ $\displaystyle \frac{dx}{d\theta} = -10\,\mathrm{cosec}^2\,\theta$ $\displaystyle \frac{dx}{dt} = \frac{dx}{d\theta}\times\frac{d\theta}{dt}$ At the time of measurement: $\displaystyle \frac{dx}{dt} = -10\,\mathrm{cosec}^2\left(\frac{\pi}{3}\right)\times\left(-\frac{\pi}{6}\right)$ $\displaystyle = 10\times\left(\frac{2}{\sqrt{3}}\right)^2\times\frac{\pi}{6}$ = 6.98 km/min	• 3 – correct solution • 2 – correct application of chain rule • 1 – finds $\dfrac{dx}{d\theta}$, or equivalent merit

Question 12

Sample solution		Suggested marking criteria
(a)	(i) $P(X=4)=0.0081$ ${}^4C_4 p^4(1-p)^0=0.0081$ $p^4=0.0081$ $p=0.3$	• 1 – correct value of p
	(ii) $a=P(X=1)$ $={}^4C_1\times 0.3^1\times 0.7^3$ $=0.4116$ $b=P(X=2)$ $={}^4C_2\times 0.3^2\times 0.7^2$ $=0.2646$ Alternatively: $b=1-0.2401-0.4116-0.0756-0.0081$ $=0.2646$	• 2 – correct values for both a and b • 1 – one correct value
	(iii) $\mathrm{Var}(X)=npq$ $=4\times 0.3\times 0.7$ $=0.84$	• 1 – correct solution
	(iv) $P(X\geq 3)=P(X=3)+P(X=4)$ $=0.0756+0.0081$ $=0.0837$	• 1 – correct solution
(b)	(i) $\frac{dN}{dt}=kN(10000-N)$ $\frac{10000\,dN}{N(10000-N)}=10000k\,dt$ $\int_{500}^{N}\frac{10000\,dN}{N(10000-N)}=10000k\int_0^t dt$ $\int_{500}^{N}\frac{1}{N}+\frac{1}{10000-N}\,dN=10000k\int_0^t dt$ $\left[\ln N-\ln(10000-N)\right]_{500}^{N}=10000k\left[t\right]_0^t$ $\left[\ln\left(\frac{N}{10000-N}\right)\right]_{500}^{N}=10000kt$ $\ln\left(\frac{N}{10000-N}\right)-\ln\left(\frac{500}{10000-500}\right)=10000kt$ $\ln\left(\frac{19N}{10000-N}\right)=10000kt$ $\ln\left(\frac{10000-N}{19N}\right)=-10000kt$ $\frac{10000-N}{19N}=e^{-10000kt}$ $10000-N=19Ne^{-10000kt}$ $10000=N+19Ne^{-10000kt}$ $N=\frac{10000}{1+19e^{-10000kt}}$	• 3 – correct solution • 2 – uses the initial condition • 1 – correctly rewrite the integral using the given partial fraction and finds the integral

Question 12 (continued)

<table>
<tr><th colspan="2">Sample solution</th><th>Suggested marking criteria</th></tr>
<tr><td rowspan="2">(b)</td><td>(ii)

$$N = \frac{10000}{1+19e^{-10000kt}}$$

$$1500 = \frac{10000}{1+19e^{-10000k}}$$

$$1+19e^{-10000k} = \frac{20}{3}$$

$$19e^{-10000k} = \frac{17}{3}$$

$$e^{-10000k} = \frac{17}{57}$$

$$-10000k = \ln\left(\frac{17}{57}\right)$$

$$k = \frac{1}{10000}\ln\left(\frac{57}{17}\right)$$
</td><td>

- 2 – correct solution
- 1 – substitutes $N = 1500$ and attempts to solve for k
</td></tr>
<tr><td>(iii)

$$N = \frac{10000}{1+19e^{-10000kt}}$$

$$7000 = \frac{10000}{1+19e^{\ln\left(\frac{17}{57}\right)t}}$$

$$7000 = \frac{10000}{1+19\times\left(\frac{17}{57}\right)^t}$$

$$1+19\times\left(\frac{17}{57}\right)^t = \frac{10}{7}$$

$$19\times\left(\frac{17}{57}\right)^t = \frac{3}{7}$$

$$\left(\frac{17}{57}\right)^t = \frac{3}{133}$$

$$t\ln\left(\frac{17}{57}\right) = \ln\left(\frac{3}{133}\right)$$

$$t = \ln\left(\frac{3}{133}\right) \div \ln\left(\frac{17}{57}\right)$$

$$= 3.134 \text{ years } (3 \text{ d.p.})$$

$$= 38 \text{ months } (\text{nearest month})$$

or 3 years and 2 months (nearest month)
</td><td>

- 2 – correct solution
- 1 – substitutes $N = 7000$ and attempts to solve for t
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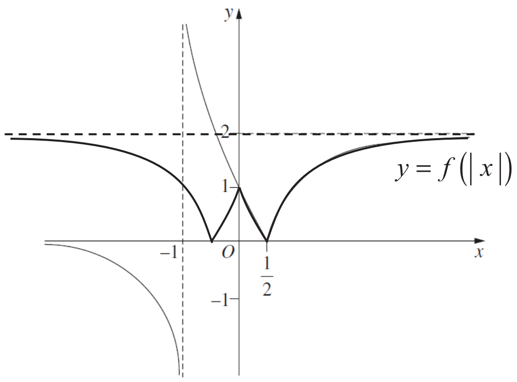
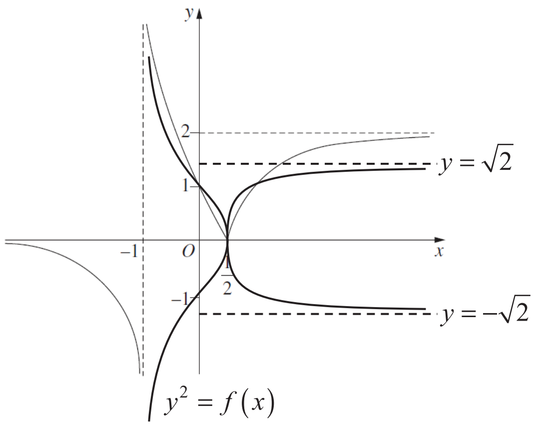
</table>

Question 12 (continued)

Sample solution		**Suggested marking criteria**
(c)	(i) 	• 1 – correct graph
	(ii) 	• 2 – correct graph • 1 – recognises invariant points $(x,0)$ and $(x,1)$, as well as the position of the horizontal asymptote

Question 13

Sample solution		Suggested marking criteria
(a)	Let $S(n)$ be the statement that $2^{n+2}+3^{2n+1}$ is divisible by 7. Show $S(1)$ is true: $2^{n+2}+3^{2n+1} = 2^{1+2}+3^{2\times1+1}$ $= 2^3+3^3$ $= 35$ $= 7\times5$, clearly divisible by 7. $\therefore S(1)$ is true. Assume $S(k)$ is true: i.e. $2^{n+2}+3^{2n+1}=7M$, where M is an integer. Prove that $S(k+1)$ is true: i.e. $2^{n+3}+3^{2n+3}=7N$, where N is an integer. $\text{LHS} = 2^{n+3}+3^{2n+3}$ $= 2\times2^{n+2}+3^2\times3^{2n+1}$ $= 2\times\left(7M-3^{2n+1}\right)+9\times3^{2n+1}$ $= 2\times7M-2\times3^{2n+1}+9\times3^{2n+1}$ $= 2\times7M+7\times3^{2n+1}$ $= 7\left(2M+3^{2n+1}\right)$ $= 7N$, where $N=2M+3^{2n+1}$ is an integer. $= \text{RHS}$ $\therefore S(k+1)$ is true if $S(k)$ is assumed true. Since $S(1)$ is proven true, then by the principle of mathematical induction, $S(n)$ is true for all integers $n\ge1$.	• 3 – correct solution • 2 – uses the inductive hypothesis to attempt to prove the result inductively • 1 – showing the result to be true for $n=1$

Question 13 (continued)

<table>
<tr><th colspan="2">Sample solution</th><th>Suggested marking criteria</th></tr>
<tr><td>(b)</td><td>(i) $\underset{\sim}{a} = -g\underset{\sim}{j}$
$= -10\underset{\sim}{j}$
$\underset{\sim}{v} = 5\underset{\sim}{i} - (10t + c_1)\underset{\sim}{j}$ (since horizontal velocity is a constant 5 m/s)

Given $\underset{\sim}{v}(0) = 5\underset{\sim}{i} + 8\underset{\sim}{j}$:
$-(10 \times 0 + c_1) = 8$
$c_1 = -8$

$\underset{\sim}{v} = 5\underset{\sim}{i} - (10t - 8)\underset{\sim}{j}$
$= 5\underset{\sim}{i} + (8 - 10t)\underset{\sim}{j}$</td><td>• 2 – correct solution
• 1 – correctly integrates the vertical component of acceleration

(Note: Using established results such as $v = u + at$ or $\underset{\sim}{v} = \underset{\sim}{u} + \underset{\sim}{a}t$ does not constitute as a "show", and will not receive full marks)</td></tr>
<tr><td></td><td>(ii) $\underset{\sim}{v} = 5\underset{\sim}{i} + (8 - 10t)\underset{\sim}{j}$
$\underset{\sim}{r} = (5t + c_2)\underset{\sim}{i} + (8t - 5t^2 + c_3)\underset{\sim}{j}$

Given $\underset{\sim}{r}(0) = 21\underset{\sim}{j}$:
$5 \times 0 + c_2 = 0$ $8 \times 0 - 5 \times 0^2 + c_3 = 21$
$c_2 = 0$ $c_3 = 21$

$\underset{\sim}{r} = (5t)\underset{\sim}{i} + (8t - 5t^2 + 21)\underset{\sim}{j}$</td><td>• 2 – correct solution
• 1 – correctly integrates the vertical component of velocity</td></tr>
<tr><td></td><td>(iii) Maximum height occurs when $\dot{y}(t) = 0$:
$8 - 10t = 0$
$t = 0.8$

$y(0.8) = 8 \times 0.8 - 5 \times 0.8^2 + 21$
$= 24.2$ metres</td><td>• 2 – correct solution
• 1 – finds the time when the ball is at maximum height
– attempts to find the maximum height by substituting a time value into the vertical displacement expression</td></tr>
<tr><td></td><td>(iv) When the ball will lands, $y(t) = 0$:
$-5t^2 + 8t + 21 = 0$
$(3 - t)(5t + 7) = 0$
$t = 3$ (since $t \geq 0$)

$\underset{\sim}{v}(3) = 5\underset{\sim}{i} + (8 - 10 \times 3)\underset{\sim}{j}$
$= 5\underset{\sim}{i} - 22\underset{\sim}{j}$

Speed at $B = \sqrt{5^2 + 22^2}$
$= 22.56$ m/s (2 d.p.)

Angle of impact $= \tan^{-1}\left(\frac{22}{5}\right)$
$= 77°$ (nearest degree)
(Also accept the obtuse angle 103° and disregard whether they state "above" or "below the horizontal")</td><td>• 3 – correct solution
• 2 – finds the speed of the ball at B
– finds the angle at which the ball strikes the ground
• 1 – finds the time of landing</td></tr>
</table>

Question 13 (continued)

Sample solution		Suggested marking criteria
(c)	Let $\cos x - \sin x \equiv R\cos(x+\alpha)$ $= R\cos x\cos\alpha - R\sin x\sin\alpha$ $\therefore R\sin\alpha = 1$ and $R\cos\alpha = 1$ Solving simultaneously gives $R=\sqrt{2}$, $\alpha = \dfrac{\pi}{4}$. $\therefore \cos x - \sin x = \sqrt{2}\cos\left(x+\dfrac{\pi}{4}\right)$ $\cos\left(x+\dfrac{\pi}{4}\right) \le 1$ $\sqrt{2}\cos\left(x+\dfrac{\pi}{4}\right) \le \sqrt{2}$ $\cos x - \sin x \le \sqrt{2}$ $2+\cos x - \sin x \le 2+\sqrt{2}$ $\therefore$ maximum value of $2+\cos x-\sin x$ is $2+\sqrt{2}$. Maximum value occurs when $\cos\left(x+\dfrac{\pi}{4}\right)=1$: $\cos\left(x+\dfrac{\pi}{4}\right)=1$ $x+\dfrac{\pi}{4} = 2\pi$ $x = \dfrac{7\pi}{4}$ $\therefore$ smallest positive angle for this maximum value to occur is $\dfrac{7\pi}{4}$.	• 4 – correct solution • 3 – finds a maximum value of $2+\cos x-\sin x$ using appropriate methods • 2 – finds appropriate values for R and α • 1 – attempts to rewrite $\cos x - \sin x$ into the form $R\cos(x+\alpha)$

Question 14

Sample solution			Suggested marking criteria
(a)	(i)	$\overrightarrow{WZ} = \frac{1}{2}\overrightarrow{CF}$ $= \frac{1}{2}\left(\overrightarrow{OF} - \overrightarrow{OC}\right)$ $= \frac{1}{2}\left(2\underset{\sim}{f} - 2\underset{\sim}{c}\right)$ $= \underset{\sim}{f} - \underset{\sim}{c}$ $\overrightarrow{WY} = \frac{1}{2}\overrightarrow{AD}$ $= \frac{1}{2}\left(\overrightarrow{OD} - \overrightarrow{OA}\right)$ $= \frac{1}{2}\left(2\underset{\sim}{d} - 2\underset{\sim}{a}\right)$ $= \underset{\sim}{d} - \underset{\sim}{a}$	• 2 – correct solution • 1 – uses appropriate vector operations towards a correct solution
	(ii)	$\left\|\overrightarrow{AD}\right\| = \left\|\overrightarrow{CF}\right\|$ $\frac{1}{2}\left\|\overrightarrow{AD}\right\| = \frac{1}{2}\left\|\overrightarrow{CF}\right\|$ $\left\|\overrightarrow{WY}\right\| = \left\|\overrightarrow{WZ}\right\|$	• 1 – correct reasoning
	(iii)	$\angle AOF = 180° - \theta$ $\overrightarrow{OA} \cdot \overrightarrow{OF} = \left\|\overrightarrow{OA}\right\|\left\|\overrightarrow{OF}\right\|\cos(\angle AOF)$ $(2\underset{\sim}{a}) \cdot \left(2\underset{\sim}{f}\right) = \left\|2\underset{\sim}{a}\right\|\left\|2\underset{\sim}{f}\right\|\cos(180° - \theta)$ $4\underset{\sim}{a} \cdot \underset{\sim}{f} = -\left\|2\underset{\sim}{a}\right\|\left\|2\underset{\sim}{f}\right\|\cos\theta \quad (\text{since } \cos(180° - \theta) = -\cos\theta)$ $4\underset{\sim}{a} \cdot \underset{\sim}{f} = -\left\|2\underset{\sim}{c}\right\|\left\|2\underset{\sim}{d}\right\|\cos\theta \quad \left(\text{since } \left\|2\underset{\sim}{a}\right\| = \left\|2\underset{\sim}{c}\right\| \text{ and } \left\|2\underset{\sim}{f}\right\| = \left\|2\underset{\sim}{d}\right\|\right)$ $4\underset{\sim}{a} \cdot \underset{\sim}{f} = -\left(\overrightarrow{OC} \cdot \overrightarrow{OD}\right)$ $4\underset{\sim}{a} \cdot \underset{\sim}{f} = -2\underset{\sim}{c} \cdot 2\underset{\sim}{d}$ $\underset{\sim}{a} \cdot \underset{\sim}{f} = -\underset{\sim}{c} \cdot \underset{\sim}{d}$	• 2 – correct proof • 1 – correct expression for $\overrightarrow{OA} \cdot \overrightarrow{OF}$ or $\overrightarrow{OC} \cdot \overrightarrow{OD}$
	(iv)	$\overrightarrow{WY} \cdot \overrightarrow{WZ} = (\underset{\sim}{d} - \underset{\sim}{a}) \cdot \left(\underset{\sim}{f} - \underset{\sim}{c}\right)$ $= \underset{\sim}{d} \cdot \underset{\sim}{f} - \underset{\sim}{d} \cdot \underset{\sim}{c} - \underset{\sim}{a} \cdot \underset{\sim}{f} + \underset{\sim}{a} \cdot \underset{\sim}{c}$ $= \underset{\sim}{d} \cdot \underset{\sim}{f} - \underset{\sim}{d} \cdot \underset{\sim}{c} + \underset{\sim}{c} \cdot \underset{\sim}{d} + \underset{\sim}{a} \cdot \underset{\sim}{c} \quad (\text{from part (iii)})$ $= \underset{\sim}{d} \cdot \underset{\sim}{f} + \underset{\sim}{a} \cdot \underset{\sim}{c}$ $= \frac{1}{4}\left(2\underset{\sim}{d} \cdot 2\underset{\sim}{f}\right) + \frac{1}{4}(2\underset{\sim}{a} \cdot 2\underset{\sim}{c})$ $= 0 \quad \left(\text{since } 2\underset{\sim}{d} \cdot 2\underset{\sim}{f} = 0 \text{ as } OD \perp OF \text{ and } 2\underset{\sim}{a} \cdot 2\underset{\sim}{c} = 0 \text{ as } OA \perp OC\right)$ Since $\left\|\overrightarrow{WY}\right\| = \left\|\overrightarrow{WZ}\right\|$ and $\overrightarrow{WY} \cdot \overrightarrow{WZ} = 0$ (i.e. $WY \perp WZ$), parallelogram $WYXZ$ is a square (a pair of adjacent sides equal and perpendicular in a parallelogram).	• 2 – correct conclusion • 1 – shows $\overrightarrow{WY} \cdot \overrightarrow{WZ} = 0$

Question 14 (continued)

<table>
<tr><th colspan="2">Sample solution</th><th>Suggested marking criteria</th></tr>
<tr><td>(b)</td><td>$x^2+mx+n=0$ has roots α and β, therefore:
$\alpha+\beta=-m$
$\alpha\beta=n$

$nx^2-mx+1=0$
$$\begin{aligned} x &= \frac{m\pm\sqrt{(-m)^2-4n}}{2n} \\ &= \frac{m\pm\sqrt{m^2-4n}}{2n} \\ &= \frac{-(\alpha+\beta)\pm\sqrt{(\alpha+\beta)^2-4\alpha\beta}}{2\alpha\beta} \\ &= \frac{-(\alpha+\beta)\pm\sqrt{(\alpha-\beta)^2}}{2\alpha\beta} \\ &= \frac{-(\alpha+\beta)\pm(\alpha-\beta)}{2\alpha\beta} \\ &= \frac{-\alpha-\beta+\alpha-\beta}{2\alpha\beta} \text{ or } \frac{-\alpha-\beta-\alpha+\beta}{2\alpha\beta} \\ &= -\frac{2\beta}{2\alpha\beta} \text{ or } \frac{-2\alpha}{2\alpha\beta} \\ &= -\frac{1}{\alpha} \text{ or } -\frac{1}{\beta} \end{aligned}$$</td><td>• 3 – correct solution
• 2 – finds the roots of $nx^2-mx+1=0$ in terms of m and n
• 1 – finds the sum and product of roots for the original equation</td></tr>
<tr><td>(c)</td><td>(i) Number of ways $=\frac{(np)!}{(p!)^n}$</td><td>• 2 – correct solution
• 1 – consider the arrangement of the total number of books without considering the consequences of having copies</td></tr>
<tr><td></td><td>(ii) For each of the n books, there are $(p+1)$ ways of selecting the book, given they have different barcodes on them (1 ways for each of the p copies, and the option of not selecting the book at all).
Number of ways $=(p+1)^n-1$ (take 1 away for the option of selecting none of each of the n books because you were tasked with borrowing at least 1 book).</td><td>• 2 – correct solution
• 1 – considering a few valid cases and the number of ways of achieving those cases</td></tr>
</table>