Student Name: ....

2003
TRIAL HIGHER SCHOOL CERTIFICATE

MATHEMATICS
Extension 1

CARINGBAH HIGH

OMNIA VINCIT DILIGENTIA

General Instructions

Reading Time: 5 minutes
Working Time: 2 hours

- Attempt all questions
- Start each question on a new page
- Each question is of equal value
- Show all necessary working.
- Marks may be deducted for careless work or incomplete solutions
- Standard integrals are printed on the last page
- Board-approved calculators may be used
- This examination paper must not be removed from the examination room

QUESTION 1 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find $\frac{d}{dx}(x^2 \sin^2 x)$. 2

(b) Write down the Cartesian equation of the locus of a point $P(x, y)$ where $x = 2\cos\theta$ and $y = \frac{1}{2}\sin\theta$. 2

(c) Find the general solution, in terms of π, to $2\sin x + 1 = 0$. 2

(d) The interval AB has end points $A(2, 4)$ and $B(x, y)$. The point $P(-1, 1)$ divides AB internally in the ratio 3 : 4. Find the coordinates of B. 2

(e) If $P(x) = 5x^3 - 3x + k$ has a remainder of 7 when $P(x)$ is divided by $(x + 2)$, find the value of k. 2

(f) Use the table of standard integrals to find the exact value of $\int_0^{\frac{\pi}{6}} \sec 4x \tan 4x \, dx$. 2

QUESTION 2. (12 marks) Use a SEPARATE writing booklet. | Marks

(a) Sketch $y = \frac{\pi}{2} + \cos^{-1}\frac{x}{2}$. 2

(b) Solve $\frac{x^2 - 2}{x} \leq 1$. 4

(c) Find, correct to the nearest degree, the acute angle between the lines $x + y - 3 = 0$ and $2x - y + 2 = 0$. 2

(d) Use the substitution $u = x - 2$ to find the exact value of $\int_1^3 x(x-2)^5 dx$. 4

QUESTION 3. (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Write down the expansion of $\tan(A+B)$. 1

(ii) Hence, find the value of $\tan\left(\frac{7\pi}{12}\right)$ as a simplified surd with a rational denominator. 2

(b) Use one application of Newton's method to find an approximate root to the equation $x - \tan^{-1}2x = 0$ that lies close to $x = 1$. Write your answer correct to two significant figures. 3

(c) (i) Show that the equation of the normal to the parabola $x^2 = 4ay$ at the point $(2ap, ap^2)$ is $x + py = 2ap + ap^3$. 2

(ii) Derive the equation of the line that passes through the focus $S(0, a)$ and is perpendicular to the normal. 1

(iii) If the line in (c) (ii) meets the normal at N, find the coordinates of N. 2

(iv) Find a Cartesian equation for the locus of N. 1

Marks

QUESTION 4. (12 marks) Use a SEPARATE writing booklet.

(a) Determine the exact value of $\cos\left(\sin^{-1}\left(-\frac{12}{13}\right)\right)$. 2

(b) A golf ball is hit towards a tree 60 metres away and standing in the same horizontal plane as the ball. The tree is 20 metres high. The initial velocity of the ball is 30 *m/s* and the angle of projection θ.

(i) Show that the vertical distance travelled by the ball is $y = 30t\sin\theta - 5t^2$. (take $g = 10m/s^2$). 1

(ii) Show that the horizontal distance travelled by the ball is $x = 30t\cos\theta$. 1

(iii) Find the range of angles (θ) in which the ball must be hit to clear the tree. 3

(c) Newton's law of cooling states that the rate of cooling of a body is proportional to the excess of the temperature of a body above the surrounding temperature.

The temperature of a cup of chocolate drink satisfies an equation of the form $T = B + Ae^{-kt}$ where T is the temperature of the drink, t is time in minutes, A and k are constants and B is the temperature of the surroundings.

The drink cools from 90°C to 80°C in two minutes in a room of temperature 25°C.

(i) Find the values of A and k. 3

(ii) Find the temperature of the cup of chocolate, to the nearest degree, after a further five minutes have passed. 2

QUESTION 5. (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Given the polynomial $P(x) = 2x^3 - 9x^2 + kx + 6$,

(i) find the value of k if $(x - 3)$ is a factor of $P(x)$. 1

(ii) Hence, or otherwise, determine all the roots of the equation $P(x) = 0$. 3

(b) A particle is moving in simple harmonic motion. Its velocity v m s^{-1} is given by $v^2 = 15 + 4x - 4x^2$.

(i) Find an expression for the acceleration, $\ddot{x}$, of the particle in terms of x. 1

(ii) Find the centre, amplitude and period of the motion. 3

(c) Use mathematical induction to show that $\cos(x + n\pi) = (-1)^n \cos x$ for all positive integers $n \geq 1$ 4

Marks

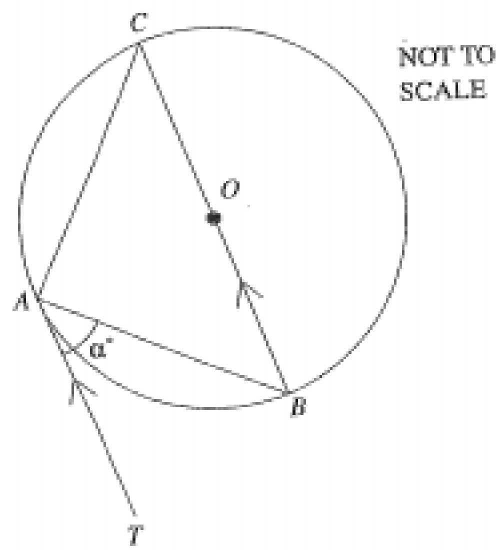
QUESTION 6. (12 marks) Use a SEPARATE writing booklet.

(a) Consider the function $f(x) = \dfrac{1}{1+x^2}$.

(i) What is the largest domain containing $x = 1$ for which $f(x)$ has an inverse function? 1

(ii) Find an expression for $f^{-1}(x)$. 2

(b) 4



In the diagram, A, B and C are points on the circle with centre O. The line AT is a tangent to the circle at A and is parallel to the diameter CB. Angle $TAB = \alpha°$.

Find the value of $\alpha°$ giving reasons.

(c) A surveyor observes two towers, one due north of height $80m$ and the other on a bearing of $\theta\ (<90°)$ of height $120m$. The angles of elevation of the two towers are $40°$ and $36°$ respectively. The towers are $150m$ apart on a horizontal plane.

(i) Find an expression in terms of $\tan 50°$ for the distance of the surveyor from the base of the first tower. 1

(ii) Find an expression in terms of $\tan 54°$ for the distance of the surveyor from the base of the second tower. 1

(iii) Calculate the value of θ to the nearest minute. 3

QUESTION 7. (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Solve $3\cos\theta - \sqrt{3}\sin\theta + 3 = 0$ for $0 \le \theta \le 2\pi$. 4

(b) A car leasing company provides finance to customers. Clients can borrow $\$P$ at $r\%$ per month reducible interest, calculated monthly. The loan is to be repaid in equal monthly payments of $\$M$.

Let $R = \left(1 + \frac{r}{100}\right)$ and let $\$A_n$ be the amount owing after n monthly repayments have been made.

(i) Write an expression for the amount owing after two repayments, A_2, in terms of P, R and M. 1

(ii) Show that the amount owing after the nth repayment is given by 2

$$A_n = PR^n - \frac{M(R^n - 1)}{R - 1}.$$

(iii) If the amount owing after the nth repayment is $K\%$ of the amount borrowed, show that 3

$$R^n = \frac{PK(R-1) - 100M}{100[P(R-1) - M]}.$$

(iv) Hence, find the minimum number of years required for the amount owing to fall to 20% of the amount borrowed, if a client borrows \$40 000 and undertakes to make equal monthly payments of \$800. Interest is charged at 9% per annum compounding monthly. 2

End of paper

Standard integrals

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \; x \neq 0, \; \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right).$$

Note: $\ln x = \log_e x, \quad x > 0$

T. 1 TRIAL HSC 2003 SOLNS

$2x^2\sin x\cos x + 2x\sin^2 x$ [PRODUCT RULE]

$\cos^2\theta + \sin^2\theta = 1$

$\left(\frac{x}{2}\right)^2 + (2y)^2 = 1$

$\frac{x^2}{4} + 4y^2 = 1$

$\sin x = -\frac{1}{2}$

$\sin x = \sin\frac{7\pi}{6}$

$x = n\pi + (-1)^n\left(\frac{-\pi}{6}\right)$

$1 = \frac{3x+4\times2}{7}$ $\quad 1 = \frac{3y+4\times4}{7}$

$x = -5$ $\quad y = -3$

i.e. $B(-5,-3)$

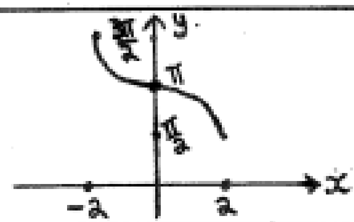
$P(-2) = 7$ [REMAINDER THEOREM]

$\therefore K = 41$

$\frac{1}{4}\left[\sec 4x\right]_0^{\frac{\pi}{6}} = \frac{1}{4}\left[\sec\frac{2\pi}{3} - \sec 0\right]$

$= \frac{1}{4}\left[-2 - 1\right]$

$= -\frac{3}{4}$



(b) $\frac{x-2}{x} \times \frac{x}{1} \leqslant 1 \times x^2$

$x(x^2-2) - x^2 \leqslant 0$

$x\left[(x^2-2) - x\right] \leqslant 0$

$x(x-2)(x+1) \leqslant 0$

$x \leqslant -1$ OR $0 < x \leqslant 2$

(c) $m_1 = -1$, $m_2 = 2$

$\tan\theta = \left|\frac{-1-2}{1+-1\times2}\right|$

$= 3$

$\theta = 72^\circ$ [NEAREST DEGREE]

(d) $u = x-2 \Rightarrow x = 1 : u = -1$

$x = 3 : u = 1$

$\frac{du}{dx} = 1$

$du = dx$

$\int_{-1}^{1}(u+2)u^5\,du = \int_{-1}^{1} u^6 + 2u^5\,du$

$= \left[\frac{1}{7}u^7 + \frac{1}{3}u^6\right]_{-1}^{1}$

$= \frac{2}{7}$

3. (a) (i) $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$

(ii) $\tan\left(\frac{\pi}{3} + \frac{\pi}{4}\right) = \frac{\tan\frac{\pi}{3} + \tan\frac{\pi}{4}}{1 - \tan\frac{\pi}{3}\tan\frac{\pi}{4}}$

$= \frac{\sqrt{3}+1}{1-\sqrt{3}\times1}$

$= -2-\sqrt{3}$

(b) $P(x) = x - \tan^{-1}2x \therefore P(1) = 1 - \tan^{-1}2$

$P'(x) = 1 - \frac{2}{1+4x^2} \therefore P'(1) = \frac{3}{5}$

$z_2 = z_1 - \frac{P(1)}{P'(1)}$

$= 1 - \frac{1-\tan^{-1}(2)}{3/5}$

$\doteqdot 1\cdot2$

(c) (i) $y' = \frac{1}{2a}y$

At $(2ap, ap^2)$: $m_{TAN} = p$

$\therefore m_{NORM} = -\frac{1}{p}$

EQUN. NORMAL: $y - ap^2 = -\frac{1}{p}(x - 2ap)$

$x + py = 2ap + ap^3$

(ii) $m = p$

EQUN: $y - a = p(x - 0)$

$y = px + a$

(iii) Solve $x + py = 2ap + ap^3$

$y = px + a$

(iii) cont. $N(ap, a(p^2+1))$

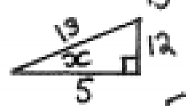
(iv) sub $p = \frac{x}{a}$: $y = a\left[\left(\frac{x}{a}\right)^2 + 1\right]$

$= \frac{1}{a}x^2 + a$

4. (a) $\cos\left(-\sin^{-1}\left(\frac{12}{13}\right)\right) = \cos\left(\sin^{-1}\left(\frac{12}{13}\right)\right)$

let $x = \sin^{-1}\left(\frac{12}{13}\right)$

$\sin x = \frac{12}{13}$

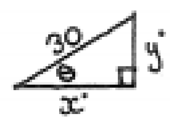


$\therefore \cos x = \frac{5}{13}$

$= \cos x$

$= \frac{5}{13}$

(b) (i)



INITIALLY
$\dot{y} = 30\sin\theta$
$\dot{x} = 30\cos\theta$

$\ddot{y} = -10$

$\dot{y} = -10t + C$

sub $t=0, \dot{y} = 30\sin\theta \Rightarrow C = 30\sin\theta$

$\dot{y} = 30\sin\theta - 10t$

$y = 30t\sin\theta - 5t^2 + K$

sub $t=0, y=0 \Rightarrow K=0$

$y = 30t\sin\theta - 5t^2$

(ii) $\ddot{x} = 0$

$\dot{x} = C = 30\cos\theta$

$x = 30t\cos\theta + C$

sub $t=0, x=0 \Rightarrow C=0$

$x = 30t\cos\theta$

(iii) We want $y > 20$ when $x = 60$

$x = 60 \Rightarrow 2 = t\cos\theta \Rightarrow t = \frac{2}{\cos\theta}$

$30t\sin\theta - 5t^2 > 0$

$30 \cdot \frac{2}{\cos\theta} \cdot \sin\theta - 5 \cdot \left(\frac{2}{\cos\theta}\right)^2 > 20$

$60\tan\theta - 20\sec^2\theta > 20$

$60\tan\theta - 20 - 20\tan^2\theta > 20$

$\tan^2\theta - 3\tan\theta + 1 > -1$

$(\tan\theta - 2)(\tan\theta - 1) > 0$

$1 < \tan\theta < 2$

$\frac{\pi}{4} < \theta < 1{\cdot}11^c$

or $45° < \theta < 63° 26'$

(c) (i) sub $t=0, T=90 \Rightarrow 90 = 25 + Ae^0$

$A = 65$

sub $t=2, T=80 \Rightarrow 80 = 25 + 65e^{-2k}$

$\frac{11}{13} = e^{-2k}$

$k = \ln\left(\frac{11}{13}\right) \div -2$

$= 0{\cdot}0835$

(ii) sub $t=7$: $T = 25 + 65e^{-0{\cdot}0835 \times 7}$

$= 61$ (nearest degree)

5. (a) (i) $P(3) = 0 \Rightarrow k = 7$

(ii) $P(x) = (x-3)(2x+1)(x-2)$

roots of $P(x) = 0$

$x = 3, -\frac{1}{2}, 2$

(b) (i) $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$\frac{1}{2}v^2 = \frac{15}{2} + 2x - 2x^2$

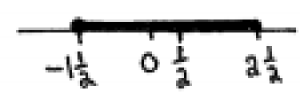
$\ddot{x} = 2 - 4x$

(ii) $V = \sqrt{15 + 4x - 4x^2}$

$15 + 4x - 4x^2 \geq 0$

$4x^2 - 4x - 15 \leq 0$

$(2x+3)(2x-5) \leq 0$



i.e. $-1\frac{1}{2} \leq x \leq 2\frac{1}{2}$

$\therefore$ centre of motion is $\frac{1}{2}$

amplitude $= 2$ m

period $= \frac{2\pi}{n} = \frac{2\pi}{2} = \pi$ s

(c) STEP 1 : Prove true for $n=1$

$n=1$: LHS $= \cos(x + \pi)$

$= -\cos x$

RHS $= -\cos x$

5.(c) cont. = LHS

$\therefore$ True for $n=1$

STEP 2: Assume true for $n=k$.

ie $\cos(x+k\pi) = (-1)^k \cos x$

Hence prove true for $n=k+1$

ie $\cos(x+(k+1)\pi) = (-1)^{k+1}\cos x$

Now $\cos(x+(k+1)\pi) = \cos(x+k\pi+\pi)$

$= \cos(x+k\pi)\cos\pi - \sin(x+k\pi).\sin\pi$

$= -\cos(x+k\pi) - 0$

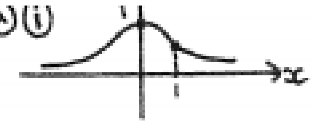
$= -1.(-1)^k \cos x$ by our assumption

$= (-1)^{k+1}\cos x.$

ie. if true for $n=k$ then true for $n=k+1$.

STEP 3: We assumed true for $n=k$ and hence proved true for $n=k+1$. Since true for $n=1$ then true for $n=2$. Since true for $n=2$ then true for $n=3$ and so on for all positive integers $n \geq 1$.

6. (a)(i)



largest domain: $x \geq 0$

(ii) SWAP x AND y: $x = \frac{1}{1+y^2}$

$f^{-1}(x) = \sqrt{\frac{1-x}{x}}$

(b) $\angle BAC = 90°$ ($\angle$ in semicircle)

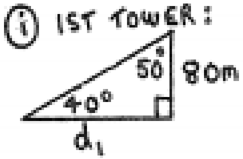
$\angle ACB = \alpha°$ (alternate segment theorem)

$\angle ABC = \alpha°$ (alternate $\angle$s, $AT \parallel BC$)

$\therefore 2\alpha° + 90° = 180°$ ($\angle$ sum of ΔABC)

$\alpha = 45°$

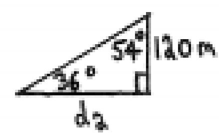
(c)(i) 1ST TOWER:



$\tan 50° = \frac{d_1}{80}$

$d_1 = 80\tan 50°$

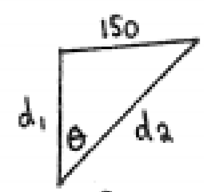
(ii) 2ND TOWER



$\tan 54° = \frac{d_2}{120}$

$d_2 = 120\tan 54°$

(iii) On ground:



$\cos\theta = \frac{(80\tan 50°)^2 + (120\tan 54°)^2 - 150^2}{2.80\tan 50°.120\tan 54°}$

$\theta = 63°52'$

7. (a) $2\sqrt{3}\left(\frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta\right) = -3$

$\cos\alpha\cos\theta - \sin\alpha\sin\theta = -\frac{\sqrt{3}}{2}$

where $\left.\begin{array}{l}\cos\alpha = \frac{\sqrt{3}}{2}\\ \sin\alpha = \frac{1}{2}\end{array}\right\} \alpha = \frac{\pi}{6}$

$\cos\left(\theta + \frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$

$\theta + \frac{\pi}{6} = \frac{5\pi}{6}$ or $\frac{7\pi}{6}$

$\theta = \frac{2\pi}{3}$ or π

(b)(i) $A_1 = PR - M$

$A_2 = A_1R - M$

$= (PR-M)R - M$

$= PR^2 - MR - M$

$= PR^2 - M(1+R)$

(ii) $A_n = PR^n - M(1+R+R^2+\ldots+R^{n-1})$

$= PR^n - \frac{M(R^n-1)}{R-1}$ USING $S_n = \frac{a(r^n-1)}{r-1}$

(iii) $\frac{PK}{100} = PR^n - \frac{M(R^n-1)}{R-1}$

$\frac{PK(R-1)}{100} = PR^n(R-1) - M(R^n-1)$

7(b) cont.

$$\frac{PK(R-1)}{100} = PR^{n+1} - PR^n - MR^n + M$$

$$\frac{PK(R-1)}{100} - M = R^n(PR-P-M)$$

$$\frac{PK(R-1)-100M}{100} = R^n(PR-P-M)$$

$$R^n = \frac{PK(R-1)-100M}{100[PR-P-M]}$$

$$= \frac{PK(R-1)-100M}{100\,[P(R-1)-M]}$$

(iv) $$1\cdot0075^n = \frac{40000\times20(0\cdot0075)-80000}{100[40000\times0\cdot0075-800]}$$

$$= 1\cdot48$$

$$\ln 1\cdot0075^n = \ln 1\cdot48$$

$$n = \frac{\ln 1\cdot48}{\ln 1\cdot0075}$$

$\doteqdot$ 52·5 months

$\doteqdot$ 5 years

Q1 (a) – Poorly set out, many differentiated $(\sin x)^2 \neq \sin^2 x$ incorrectly by not using chain rule.

b) – Usually well done, but many could not link $\sin\theta$, $\cos\theta$ using $\sin^2\theta + \cos^2\theta = 1$.

c) Well done, but a number forgot general form.

d) Very straight forward, caused no problem

e) Easy for people who used remainder theorem.

f) This was also done well by most students.

Q2 (a) Some candidates lost marks for not showing y-intercept or "end points" of the curve.

(b) N.B. Here $x \neq 0$!

(c) Some need to learn formula

(d) Generally well done.

Q3 a) Usually done well, some found it easier to convert to degrees. Some careless errors with exact values cost some students a mark.

b) This caused many problems! Very few could correctly differentiate $\tan^{-1}2x$!! Poor setting out meant that marks could not be awarded for showing what you could do. Far too many did not use rads on calculator!

c) i), ii) Usually well attempted.

iii) poor algebra skills cost marks – check working.

iv) poorly attempted because of algebra probs in part iii). Many did not realise they simply had to eliminate ρ using $p = \frac{3}{2}$.

Q4 comments not available.

Q5 (a) Many factorised correctly but failed to continue and write down the roots.

(b) $\ddot{x} = -4(x-\frac{1}{2})$ of form $\ddot{x} = -n^2(x-\frac{1}{2})$

... Period $= \frac{2\pi}{n} = \frac{2\pi}{2} = \pi$

(c) N.B. $\cos[(x+k\pi)+\pi] = \cos(x+k\pi)\cos\pi - \sin(x+k\pi)\sin\pi$

Q6 Inverse trig question not well done from an algebraic point of view (ie making y the subject when swapping x & y).

Circle geometry well done, most students got it out. 3D trig was OK but not knowing the Cosine Rule let some students down.

Q7. a) Half of the students who used the "t" results neglected to check $\theta = \pi$. Of those that used $R\cos(\theta \pm \alpha)$ or $R\sin(\theta \pm \alpha)$ most found R correctly but a lot of students had difficulty with α.

b) i) Well done.

ii) Show means to insert every line of working — too many students assumed that $S_n = \frac{a(r^n-1)}{r-1}$ and that $a=1$, $r=R$ and $n=n$. A lot of students extrapolated from $A_2 = \ldots$ to $A_n = PR^n - \frac{M(R^n-1)}{R-1}$ without writing $A_n = PR^n - M(1+R+R^2+\ldots+R^{n-1})$.

iii) Half of all candidates did not realise that K% of 100n meant $\frac{PK}{100}$. Very few students completed this section correctly.

iv) Very few students gained full marks. A lot could not solve $1\cdot0075^n = 1\cdot48$.