Student Name: ....

2004
TRIAL HIGHER SCHOOL CERTIFICATE

MATHEMATICS
Extension 1

CARINGBAH HIGH

OMNIA DILIGENTIA VINCIT

General Instructions

Reading Time: 5 minutes
Working Time: 2 hours

- Attempt all questions
- Start each question on a new page
- Each question is of equal value
- Show all necessary working
- Marks may be deducted for careless work or incomplete solutions
- Standard integrals are printed on the last page
- Board-approved calculators may be used

Question One

Marks

a) Find the remainder when $P(x) = x^3 - 4x + 2$ is divided by $x - 1$. — 1

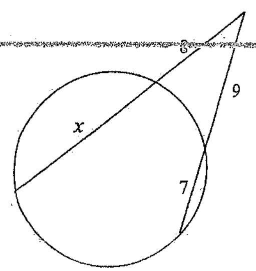
b) Find $\int xe^{x^2}\,dx$ — 1

c) Solve the inequality $\dfrac{2x-3}{x} \le 4$ — 3

d) For the points $A(3,-5)$ and $B(-4,2)$, find the coordinates of the point P which divides the interval AB **externally** in the ratio 2:1. — 2

e) Solve the equation $x - 1 = \sqrt{x+1}$. — 3

f)



Not to scale

Calculate the value of x, giving a reason for your answer. — 2

Question Two (Start a new page)

Marks

a) Find $\int \dfrac{1}{x\log_e x}\,dx$ using the substitution $u = \log_e x$. — 2

b) The cubic equation $x^3 - 4x^2 + x + 1 = 0$ has a root near $x = 0.7$

Use one application of Newton's Method to find a better approximation, giving your answer to 2 decimal places. — 2

c) For the function $f(x) = 2\sin^{-1}\left(\dfrac{x}{2}\right)$

i) Find $f(2)$ — 1

ii) State the domain and range of this function. — 1

iii) Neatly sketch $y = f(x)$. — 2

d) A parabola is defined by the parametric equations

$$x = 12t$$
$$y = 6t^2 + 3$$

i) Find $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$. — 1

ii) If Q is the point where $t = -1$, find

α) the coordinates of Q. — 1

β) the slope of the tangent to the curve at Q. — 1

γ) the equation of the tangent at Q. — 1

Question Three (Start a new page) Marks

a) Evaluate $\lim_{x \to 1} \frac{x^3 - 1}{x^2 - 1}$ 2

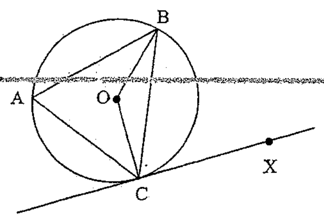
b) Evaluate $\int_{-3}^{0} \frac{x}{\sqrt{x+4}}\,dx$ using the substitution $u^2 = x + 4$, where $u > 0$. 3

c) Prove, by Mathematical Induction, that for $n \geq 1$ 4

$$\frac{1}{1\times 4} + \frac{1}{4\times 7} + \frac{1}{7\times 10} + \dots\dots + \frac{1}{(3n-2)\times(3n+1)} = \frac{n}{3n+1}$$

d)

CX is tangent to the circle centre O. Let $\angle CAB = \alpha$.



i) Copy the diagram to your answer sheet.

ii) Find with reasons $\angle COB$ in terms of α. 1

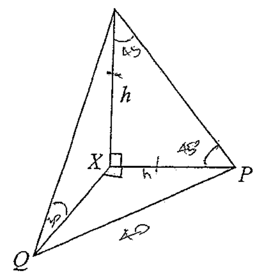
iii) Find with reasons $\angle OCB$ in terms of α. 1

iv) Hence show that $\angle BCX = \angle BAC$. 1

Question Four (Start a new page)

Marks

a) A vertical tower with base X and height h metres stands on horizontal ground. From a point P on the ground due east of the tower the angle of elevation of the top of the tower is $45°$ and from a point Q on the ground due south of the tower the angle of elevation of the top of the tower is $30°$. If distance PQ is 40 metres:



i) Find the length of QX in terms of h. 1

ii) Hence find the height of the tower. 2

b) Find $\int \frac{1}{4+x^2}\,dx$ 1

c) A sphere is being heated so that its surface area is increasing at a constant rate of $15\,mm^2/s$. Find the rate of increase of the volume of the sphere when the radius is $5\,mm$. (You are given that $V = \frac{4}{3}\pi r^3$ and $S.A. = 4\pi r^2$). 3

d) Corn cobs are cooked by immersing them in boiling water. On being removed, a corn cob cools in the air according to the equation $\frac{dT}{dt} = -k(T - B)$, where t is time in minutes, T is temperature in degrees celsius (°C), B is the temperature of the surrounding air and k is a positive constant.

i) Show that $T = B + Ae^{-kt}$ is a solution of the above equation where A is a constant. 1

ii) If the temperature of the boiling water is 100°C and the surrounding air is a constant 25°C, find the value of A and the value of k (correct to 4 decimal places) if a corn cob cools to 70°C in 3 minutes. 2

iii) How long (to the nearest minute) will it take for the corn to cool to 50°C ? 2

Question Five (Start a new page)

Marks

a) If $y = \sin^{-1}(3x-2)$, find $\frac{dy}{dx}$. 2

b) If $p, q,$ and r are the roots of the cubic equation $x^3 + 2x^2 + 3x + 5 = 0$, find the value of:

i) $p + q + r$. 1

ii) $p^{-1} + q^{-1} + r^{-1}$. 3

c) $A\left(t, e^t\right)$ and $B\left(-t, e^{-t}\right)$ are points on the curve $y = e^x$, where $t > 0$.

The tangents at A and B form an angle of 45°.

i) Prove that $e^t - \frac{1}{e^t} = 2$. 3

ii) Hence by solving the equation in part (i) find the coordinates of A in ***exact form***. 3

Question Six(Start a new page)

Marks

a) Consider the function $f(x) = \frac{x+1}{x^2+3}$

i) Find the points where the curve crosses the x-axis and the y-axis. 1

ii) Find the coordinates of any stationary points on the curve $y = f(x)$ and, without finding the second derivative, determine their nature. 3

iii) Describe the behaviour of $y = f(x)$ for large positive and large negative values of x. 1

iv) Using an **appropriate scale** neatly sketch $y = f(x)$ showing all important information. 2

b) The points $P\left(2ap, ap^2\right)$ and $Q\left(2aq, aq^2\right)$ lie on the parabola $x^2 = 4ay$.

[You are given that the equation of the chord PQ is $2y = (p+q)x - 2apq$]

i) If the chord PQ passes through $(2a, 0)$ show that $pq = p + q$. 1

ii) Hence, find the locus of M, the midpoint of PQ. 4

Question Seven(Start a new page)

Marks

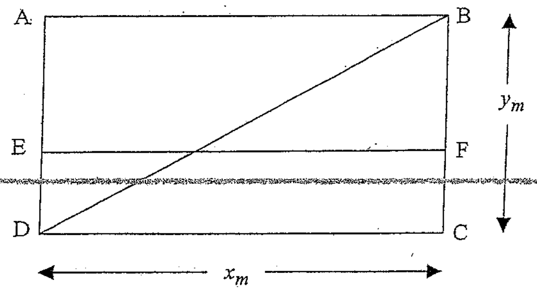
a) An urn contains W white balls and B black balls.
If the probability of selecting 2 white balls in succession at random is $\frac{1}{3}$ and the probability of selecting 3 white balls in succession at random is $\frac{1}{6}$, find the number of white balls in the urn. **3**
[You must justify your answer to gain full marks]

b) A particle is travelling in a straight line executing Simple Harmonic Motion about O according to the equation $x = a\cos nt$.

i) Show that the velocity v and the displacement x of the particle at any time t are related by the equation $v^2 = n^2\left(a^2 - x^2\right)$. **2**

ii) Hence using part (i) show that the acceleration of the particle can be written as $\ddot{x} = -n^2 x$. **1**

c) A rectangle ABCD with sides of length x metres and y metres has an area of $9\,m^2$. Two metal construction strips, one a diagonal BD and the other EF parallel to the sides AB and CD are required to strengthen the rectangle.



i) Show that the total length L of both strips is given by **2**

$$L = x + \frac{\sqrt{x^4 + 81}}{x} \text{ metres.}$$

ii) Find the value of x that will minimise the total length L of the strips, writing your answer in the form $x = a^{b/c}$, where a, b and c are integers. **4**

[You do not need to justify that your value of x gives a minimum L]

END OF EXAM

STANDARD INTEGRALS

$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$

$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$

$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$

$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$

$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$

$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$

$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$

$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$

$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$

$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$

$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$

NOTE : $\ln x = \log_e x, \quad x > 0$

Ext 1 2004 Trial

SOLUTIONS

Question One

a) Using the remainder theorem find $P(1) = -1$.

b) $\frac{1}{2}e^{x^2} + c$

c) $x^2\left(\frac{2x-3}{x}\right) \le 4x^2$

$x(2x-3) - 4x^2 \le 0$

$-x(2x+3) \le 0$

$\therefore x \le -\frac{3}{2}$ and $x > 0$ as $x \ne 0$.

d) $m : n = 2 : -1$

$\therefore x = \frac{2\times -4 + -1\times 3}{2+(-1)} = -11,\ y = \frac{2\times 2 + -1\times 3}{2+(-1)} = 9$

$\therefore P$ has coordinates $(-11, 9)$

e) $(x-1)^2 = x+1$

$x^2 - 2x + 1 = x + 1$

$x^2 - 3x = 0 \Rightarrow x(x-3) = 0$

$\therefore x = 0$ or $x = 3$

but as x must ≥ 1, then $x = 3$ only.

f) $8\times(x+8) = 9\times 16$ [Product of intersecting secants]

$8x + 64 = 144 \Rightarrow x = 10$.

Question Two

a) $u = \ln x \Rightarrow du = \frac{1}{x}dx$

$\therefore I = \int \frac{1}{u}du = \ln(u)$

$= \ln(\ln x) + c$

b) Let $P(x) = x^3 - 4x^2 + x + 1$

$\therefore P'(x) = 3x^2 - 8x + 1$

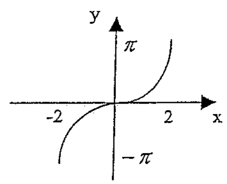
Now with $x_1 = 0.7$, $x_2 = 0.7 - \frac{P(0.7)}{P'(0.7)} =$ 0.73 (2dp)

c) i) π

ii) Domain : $-2 \le x \le 2$

Range : $-\pi \le y \le \pi$.

iii)



d) i) $\frac{dx}{dt} = 12$ and $\frac{dy}{dt} = 12t$

ii) α) $Q(-12, 9)$.

β) $\frac{dy}{dx} = \frac{dy}{dt}\times\frac{dt}{dx}$ using the chain rule

$= 12t \times \frac{1}{12} = -1$ when $t = -1$.

γ) $y - 9 = -1(x+12)$

$\therefore y = -x - 3$.

Question Three

a) $\lim_{x\to 1} \frac{(x-1)(x^2+x+1)}{(x-1)(x+1)}$

$= \lim_{x\to 1} \frac{x^2+x+1}{x+1} = \frac{1+1+1}{1+1} = \frac{3}{2}$

b) $u^2 = x+4 \Rightarrow 2u\,du = dx$ and $u = \sqrt{x+4}$

Also, when $x = -3$, $u = 1$ and when $x = 0$, $u = 2$.

$\therefore I = \int_1^2 \frac{u^2-4}{u}\times 2u\,du$

$= 2\int_1^2 u^2 - 4\,du$

$= 2\left[\left(\frac{u^3}{3} - 4u\right)\right]_1^2$

$= 2\left[\left(\frac{8}{3} - 8\right) - \left(\frac{1}{3} - 4\right)\right] = -\frac{10}{3}$

c) When $n = 1$, LHS $= \frac{1}{4}$, RHS $= \frac{1}{3\times 1+1} = \frac{1}{4}$

$\therefore$ true for $n = 1$.

Assume true for $n = k$, i.e.

$\frac{1}{1\times 4} + \frac{1}{4\times 7} + \frac{1}{7\times 10} + \ldots + \frac{1}{(3k-2)\times(3k+1)} = \frac{k}{3k+1}$

Now prove true for $n = k+1$, i.e. prove that

$S_k + T_{k+1} = S_{k+1}$.

$S_k = \frac{k}{3k+1}$, $S_{k+1} = \frac{k+1}{3k+4}$, $T_{k+1} = \frac{1}{(3k+1)(3k+4)}$

$\therefore S_k + T_{k+1} = \frac{k}{3k+1} + \frac{1}{(3k+1)(3k+4)}$

$= \frac{k(3k+4)+1}{(3k+1)(3k+4)}$

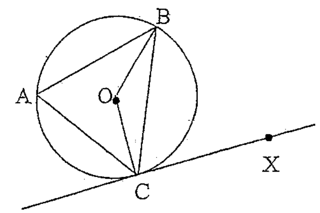
$= \frac{3k^2+4k+1}{(3k+1)(3k+4)}$

$= \frac{(3k+1)(k+1)}{(3k+1)(3k+4)} = \frac{k+1}{3k+4}$

$= S_{k+1}$

$\therefore$ If true for $n = k$ then true for $n = k+1$ and since true for $n = 1$, then true for $n = 2$ and so on for all integers n.

d)



i) $\angle COB = 2\alpha$ ($\angle$ at centre = twice $\angle$ at circumference)

ii) $\angle OCB = 90 - \alpha$ (base $\angle$'s of isos Δ and $\angle$ sum of Δ)

iii) $\angle BCX = \angle OCX - \angle OCB$

$= 90 - (90 - \alpha)$ (rt.$\angle$, tangent $\perp$ to radius)

$= \alpha$

$= \angle BAC$.

Question Four

a) i) $\tan 30° = \frac{h}{QX} \Rightarrow QX = \sqrt{3}h$.

ii) Similarly $\tan 45° = \frac{h}{PX} \Rightarrow PX = h$

$\therefore$ Using ΔPXQ: $40^2 = (\sqrt{3}h)^2 + h^2$

$1600 = 4h^2 \Rightarrow h = 20m$

b) $\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$

Question Four (cont)

c) $\frac{dA}{dt}=15$, $\frac{dA}{dr}=8\pi r$, $\frac{dV}{dr}=4\pi r^2$, $\frac{dV}{dt}=?$

Need to use the chain rule twice:

$\frac{dA}{dt}=\frac{dA}{dr}\times\frac{dr}{dt} \Rightarrow 15=8\pi r \times \frac{dr}{dt} \Rightarrow \frac{dr}{dt}=\frac{15}{8\pi r}$

Also $\frac{dV}{dt}=\frac{dV}{dr}\times\frac{dr}{dt} = 4\pi r^2 \times \frac{15}{8\pi r}$

$\therefore \frac{dV}{dt}=\frac{15r}{2} \Rightarrow 37.5mm^3/s$ when $r=5$.

d) i) $LHS=\frac{dT}{dt}=-kAe^{-kt}$

$RHS=-k(T-B)=-k\left(B+Ae^{-kt}-B\right)$

$=-kAe^{-kt}=LHS$

ii) When $t=0, T=100°$ and when $t=3, T=70°$.

Now $T=25+Ae^{-kt}$

$\therefore 100=25+Ae^{0} \Rightarrow A=75.$

Also $70=25+75e^{-3k}$

$\therefore e^{-3k}=\frac{45}{75} \Rightarrow -3k=\ln\left(\frac{3}{5}\right) \Rightarrow k\approx 0.1703$

iii) $50=25+75e^{-0.1703t} \Rightarrow \frac{1}{3}=e^{-0.1703t}$

$\therefore -0.1703t=\ln\left(\frac{1}{3}\right) \Rightarrow t\approx 6$ minutes.

Question Five

a) $\frac{dy}{dx}=\frac{3}{\sqrt{1-(3x-2)^2}}$

b) i) $p+q+r=-\frac{b}{a}=-2$

ii) $pq+pr+qr=\frac{c}{a}=3$ and $pqr=-\frac{d}{a}=-5$

$\therefore p^{-1}+q^{-1}+r^{-1}=\frac{1}{p}+\frac{1}{q}+\frac{1}{r}$

$=\frac{pq+pr+qr}{pqr}=-\frac{3}{5}.$

c) $y=e^x \Rightarrow \frac{dy}{dx}=e^x$

$\therefore$ at A: $x=t \Rightarrow m_1=e^t$

and at B: $x=-t \Rightarrow m_2=e^{-t}$

$\therefore \tan 45°=\left|\frac{e^t-e^{-t}}{1+e^t\times e^{-t}}\right|$

$\therefore 1=\frac{e^t-e^{-t}}{1+e^0} \Rightarrow 2=e^t-e^{-t}$

Hence $2=e^t-\frac{1}{e^t}$ ----------**

ii) $\left(e^t\right)^2-2e^t-1=0$ using **

$\therefore e^t=\frac{2\pm\sqrt{(-2)^2-4\times1\times-1}}{2\times1}$

$=\frac{2\pm\sqrt{8}}{2}=\frac{2\pm2\sqrt{2}}{2}=1\pm\sqrt{2}$

but as $e^t>0$, then $e^t=1+\sqrt{2} \Rightarrow t=\ln\left(1+\sqrt{2}\right)$

$\therefore$ A has coordinates $\left[\ln\left(1+\sqrt{2}\right),\left(1+\sqrt{2}\right)\right]$

Question Six

a) i) At $x=0 \Rightarrow y=\frac{1}{3}$ $\therefore$ y-intercept is $\left(0,\frac{1}{3}\right)$

At $y=0 \Rightarrow x=-1$ $\therefore$ x-intercept is $(-1,0)$.

ii) $\frac{dy}{dx}=\frac{\left(x^2+3\right)\times1-(x+1)\times2x}{\left(x^2+3\right)^2}=\frac{3-2x-x^2}{\left(x^2+3\right)^2}$

For stationary points $\frac{dy}{dx}=0$

$\therefore x^2+2x-3=0 \Rightarrow (x+3)(x-1)=0$

$\therefore x=-3$ or $x=1$

When $x=-3$, $y=-\frac{1}{6}$ and

x	-3^-	-3	-3^+
y'	$-ve$	0	$+ve$

$\therefore \left(-3,-\frac{1}{6}\right)$ is a minimum .

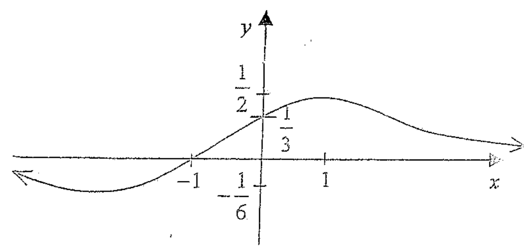
When $x=1$, $y=\frac{1}{2}$ and

x	1^-	1	1^+
y'	$+ve$	0	$-ve$

$\therefore \left(1,\frac{1}{2}\right)$ is a maximum.

iii) As $x\to\infty$, $f(x)\to 0$ from above.

and as $x\to-\infty$, $f(x)\to 0$ from below.



6b) i) Substitute the point $(2a,0)$ into the equation

$2y=(p+q)x-2apq$ to obtain

$0=(p+q)\times2a-2apq$

$\therefore 0=2ap+2aq-2apq$

$0=p+q-pq \Rightarrow pq=p+q$

ii) The midpoint M of the chord PQ has coordinates $M\left(a(p+q),a\left(\frac{p^2+q^2}{2}\right)\right)$

$\therefore x^2=a^2(p+q)^2$

$=a^2\left(p^2+q^2\right)+2a^2pq$

$=2a\times a\left(\frac{p^2+q^2}{2}\right)+2a^2pq$

$=2ay+2a\times a(p+q)$ -------- using (i)

$=2ay+2ax$

Hence the locus of M is given by the equation

$x^2=2a(y+x)$

Question Seven

a) $\dfrac{W}{W+B}\times\dfrac{W-1}{(W-1)+B}=\dfrac{1}{3}$ --------------------(1)

$\dfrac{W}{W+B}\times\dfrac{W-1}{(W-1)+B}\times\dfrac{W-2}{(W-2)+B}=\dfrac{1}{6}$

$\therefore\ \dfrac{1}{3}\times\dfrac{W-2}{(W-2)+B}=\dfrac{1}{6} \quad\Rightarrow\quad \dfrac{W-2}{(W-2)+B}=\dfrac{1}{2}$

$\therefore\ 2W-4=W-2+B \quad\Rightarrow\quad B=W-2$ --(2)

Substitute (2) into (1) to obtain

$\dfrac{W}{2W-2}\times\dfrac{W-1}{2W-3}=\dfrac{1}{3}$

$\dfrac{W^2-W}{4W^2-10W+6}=\dfrac{1}{3}$

$3W^2-3W=4W^2-10W+6$
$W^2-7W+6=0$
$(W-1)(W-6)=0$
$\therefore\ W=1$ or $W=6$

But $W=1$ is trivial hence there are six white balls.

b) i) $x=a\cos nt$
$\therefore\ v=-an\sin nt$

Hence $v^2=a^2n^2\sin^2 nt$
$=a^2n^2(1-\cos^2 nt)$
$=a^2n^2-n^2(a\cos nt)^2$
$=a^2n^2-n^2x^2$
$=n^2(a^2-x^2)$

ii) $\ddot{x}=\dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$

$=\dfrac{1}{2}n^2\dfrac{d}{dx}\left[(a^2-x^2)\right]$

$=\dfrac{1}{2}n^2\times -2x$
$=-n^2x$

c) i) $xy=9 \quad\Rightarrow\quad y=\dfrac{9}{x}$ -------------------- (1)

Also $L=DC+DB=x+\sqrt{x^2+y^2}$

$\therefore\ L=x+\sqrt{x^2+\left(\dfrac{9}{x}\right)^2}$ using (1)

$=x+\sqrt{x^2+\dfrac{81}{x^2}} = x+\sqrt{\dfrac{x^4+81}{x^2}}$

$\therefore\ L=x+\dfrac{\sqrt{x^4+81}}{x}$

ii) $\dfrac{dL}{dx}=1+\dfrac{x\times\frac{1}{2}(x^4+81)^{-\frac{1}{2}}\times 4x^3-\sqrt{x^4+81}\times 1}{x^2}$

$=1+\dfrac{\dfrac{2x^4}{\sqrt{x^4+81}}-\sqrt{x^4+81}}{x^2}$

$=1+\dfrac{\dfrac{2x^4-(x^4+81)}{\sqrt{x^4+81}}}{x^2}$

$=1+\dfrac{x^4-81}{x^2\sqrt{x^4+81}}$

For a minimum $\dfrac{dL}{dx}=0 \Rightarrow 1=\dfrac{81-x^4}{x^2\sqrt{x^4+81}}$

$\therefore\ x^2\sqrt{x^4+81}=81-x^4$ and square both sides

$x^4(x^4+81)=6561-162x^4+x^8$
$x^8+81x^4=6561-162x^4+x^8$
$6561=243x^4$

$\therefore\ x^4=27$

Hence $x=27^{\frac{1}{4}}$ or $x=3^{\frac{3}{4}}$