Student Name: ....

2005
TRIAL HIGHER SCHOOL CERTIFICATE

MATHEMATICS
Extension 1

CARINGBAH HIGH

General Instructions

Reading Time: 5 minutes
Working Time: 2 hours

- Attempt all questions
- Start each question on a new page
- Each question is of equal value
- Show all necessary working.
- Marks may be deducted for careless work or incomplete solutions
- Standard integrals are printed on the last page
- Board-approved calculators may be used

Question One

Marks

a) Find the remainder when $P(x) = x^3 + 2x - 5$ is divided by $x - 1$. 1

b) Use the table of standard integrals to find $\int \frac{1}{\sqrt{x^2+4}}\,dx$ 1

c) Solve the inequality $\frac{2}{x-5} > 1$ 2

d) For the points $A(-3, 8)$ and $B(5, -6)$, find the coordinates of the point P which divides the interval AB **internally** in the ratio 1:3. 2

e) Use the substitution $u = x - 1$ to evaluate $\int_2^4 \frac{x}{(x-1)^2}\,dx$. 3

f) Solve the equation $\sin 2\theta = \cos\theta$ for $0 \le \theta \le 2\pi$. 3

Question Two (Start a new page)

Marks

a) Find $\frac{d}{dx}\left(x^3 \tan^{-1} 2x\right)$ 2

b) The function $f(x) = \ln x + 5x$ has a zero near $x = 0.2$

Use one application of Newton's Method to find a better approximation, giving your answer correct to 2 decimal places. 2

c) For the function $f(x) = \cos^{-1}(3x)$

i) Find $f(-\frac{1}{6})$, expressing the answer in **radians in exact form.** 1

ii) State the domain and range of this function. 1

iii) Neatly sketch $y = f(x)$. 2

d) i) Show that $\frac{x+1}{x+3} = 1 - \frac{2}{x+3}$ 1

ii) You are now given that $y = \frac{x+1}{x+3}$, find the equation of the vertical asymptote. 1

iii) Without the use of calculus **neatly** sketch the graph of $y = \frac{x+1}{x+3}$ showing all the main features. 2

Question Three (Start a new page)

Marks

a) Given that $\int_0^2 f(t)\,dt = 5$, evaluate $\int_0^1 f(t)\,dt + \int_1^2 f(t) - 1\,dt$. 2

b) If the line $y = mx + b$ is 2 units from the origin, prove that $m^2 + 1 = \frac{b^2}{4}$. 2

c) Prove, by Mathematical Induction, that for all integers $n \geq 1$ 4

$$1 + 6 + 15 + ... + n(2n-1) = \frac{1}{6}n(4n-1)(n+1).$$

d) The population N, of Keystown first reached 25 000 on 1 January 2000. The population of Keystown is set to increase according to the equation

$$\frac{dN}{dt} = k(N - 8000)$$

where t represents time in years after the population first reached 25 000. On 1 January 2005, the population of Keystown was 29 250.

i) Verify that $N = 8000 + Ae^{kt}$ is a solution to the above equation where A is a constant. 1

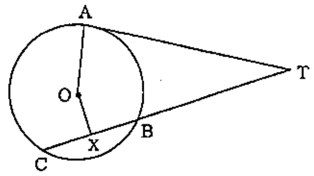
ii) Find the value of A and the value of k correct to **4 decimal places.** 2

iii) In which year will the population of Keystown reach 50 000? 1

Question Four (Start a new page)

Marks

a)



A, B and C are 3 points on a circle centre O. The tangent at A meets CB produced at T. X is the midpoint of BC.

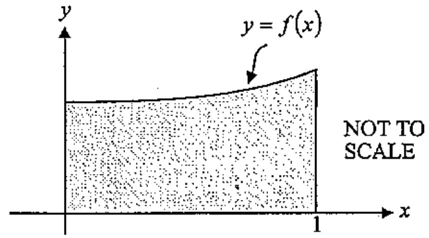
Neatly copy the diagram onto your answer sheet.

i) Without adding any constructions to the diagram prove that AOXT is a cyclic quadrilateral. 3

ii) Hence state why $\angle AOT = \angle AXT$ 1

b) i) Find the natural domain of the function $f(x) = \frac{1}{\sqrt{4 - x^2}}$. 1

ii) The sketch below shows part of the graph of $y = f(x)$. The area under the curve for $0 \leq x \leq 1$ is shaded. Find the area of the shaded region. 2



c) If α, β, γ are the roots of $2x^3 + x^2 - x - 2 = 0$, find the value of:

i) $\alpha + \beta + \gamma$ 1

ii) $\alpha\beta\gamma$ 1

iii) $\alpha\beta + \alpha\gamma + \beta\gamma$ 1

iv) Hence or otherwise, find the value of $(\alpha - 1)(\beta - 1)(\gamma - 1)$. 2

Question Five (Start a new page)

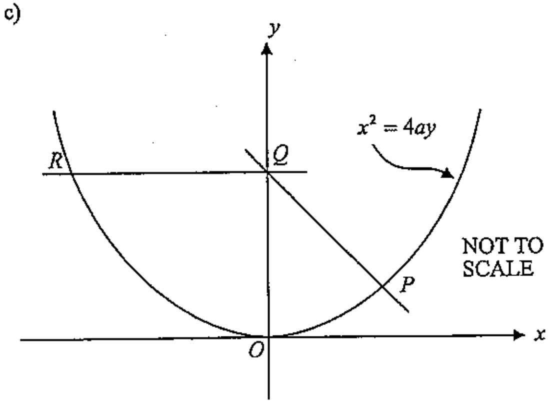
Marks

a) Air is being pumped into a spherical balloon at the rate of $450\,cm^3/s$.

Calculate in **exact form** the rate at which the radius of the balloon is increasing at the instant when the radius reaches 15cm. $\left[V=\frac{4}{3}\pi r^3\right]$ 3

b) The area between the curve $y=\sin x$, the x-axis, the lines $x=0$ and $x=\frac{3\pi}{4}$ is rotated about the x-axis. Find the volume of the solid formed. Express your answer in **exact form**. 3

c)



The diagram above shows the graph of the parabola $x^2=4ay$.
The normal to the parabola at the variable point $P(2at,at^2)$, $t>0$, cuts the y-axis at Q. The point R lies on the parabola in the 2nd quadrant.

i) Show that the equation of the normal to the parabola at P is $x+ty=at^3+2at$. 2

ii) Find the coordinates of R given that QR is parallel to the x-axis. 2

iii) Let M be the midpoint of RQ. Find the Cartesian equation of the locus of M. 2

Question Six(Start a new page)

Marks

a) A particle moves in a straight line with an acceleration given by

$$\frac{d^2x}{dt^2}=9(x-2)$$

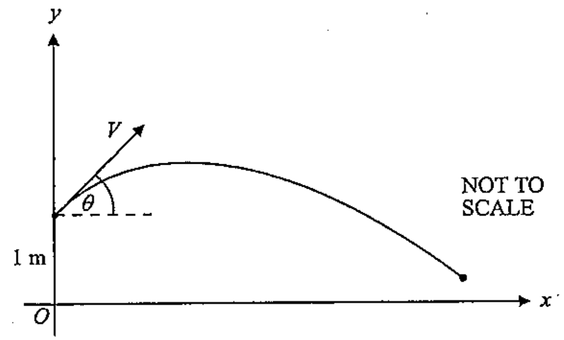
where x is the displacement in metres from an origin O after t seconds.

Initially, the particle is 4 metres to the right of O with a velocity $v=-6$.

i) Show that $v^2=9(x-2)^2$. 3

ii) Find an expression for v and hence find x as a function of t. 3

b) A boy throws a ball and projects it with a speed of V m s^{-1} from a point 1 metre above the ground. The ball lands on top of a flowerpot in a neighbour's yard.



The angle of projection is θ as indicated in the diagram. The equations of motion of the ball are $\ddot{x}=0$ and $\ddot{y}=-10$.

It has been found that $y=Vt\sin\theta-5t^2+1$.

i) Show that $x=Vt\cos\theta$. 1

ii) When the ball is at its maximum height above the ground, it passes directly above a 1.5 metre high fence and **clears** the fence by 0.5 metres. 3

Show that $V=\frac{\sqrt{20}}{\sin\theta}$

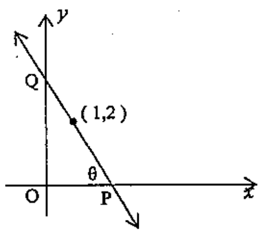
iii) Find the value of V given that $\theta=\tan^{-1}\frac{9}{40}$. 2

Give your answer in m/s, correct to 2 decimal places.

Question Seven(Start a new page) Marks

a) Find $\lim_{x \to 0} \frac{1-\cos 2x}{3x^2}$ 2

b)



A line passes through the point (1, 2) and meets the x and y axes at P and Q respectively as shown in the above diagram. $\angle OPQ = \theta$.

i) Show that the equation of the line PQ is given by $y = \tan\theta - x\tan\theta + 2$. 2

ii) Show that the area (A) of ΔOPQ is given by 3

$$A = \frac{\tan\theta}{2} + 2 + \frac{2}{\tan\theta}.$$

iii) Prove that the area is a minimum when $\tan\theta = 2$. 4

iv) Hence, find the minimum area. 1

END OF EXAM

STANDARD INTEGRALS

$\int x^n\,dx$	$= \frac{1}{n+1}x^{n+1},\quad n \neq -1;\ x \neq 0,\ \text{if } n < 0$
$\int \frac{1}{x}\,dx$	$= \ln x,\quad x > 0$
$\int e^{ax}\,dx$	$= \frac{1}{a}e^{ax},\ a \neq 0$
$\int \cos ax\,dx$	$= \frac{1}{a}\sin ax,\ a \neq 0$
$\int \sin ax\,dx$	$= -\frac{1}{a}\cos ax,\ a \neq 0$
$\int \sec^2 ax\,dx$	$= \frac{1}{a}\tan ax,\ a \neq 0$
$\int \sec ax \tan ax\,dx$	$= \frac{1}{a}\sec ax,\ a \neq 0$
$\int \frac{1}{a^2+x^2}\,dx$	$= \frac{1}{a}\tan^{-1}\frac{x}{a},\ a \neq 0$
$\int \frac{1}{\sqrt{a^2-x^2}}\,dx$	$= \sin^{-1}\frac{x}{a}, a > 0,\ -a < x < a$
$\int \frac{1}{\sqrt{x^2-a^2}}\,dx$	$= \ln\left(x+\sqrt{x^2-a^2}\right),\ x > a > 0$
$\int \frac{1}{\sqrt{x^2+a^2}}\,dx$	$= \ln\left(x+\sqrt{x^2+a^2}\right)$

NOTE : $\ln x = \log_e x,\quad x > 0$

CARINGBAH HIGH SCHOOL 2005 EXTENSION 1 MATHEMATICS SOLUTIONS

Question 1.

a) $P(1) = -2$

b) $\ln\left(x+\sqrt{x^2+4}\right)+C$

c) $2(x-5) > (x-5)^2$

$2(x-5)-(x-5)^2 > 0$

$(x-5)(7-x) > 0 \Rightarrow \quad 5 < x < 7$

d) $x = \dfrac{3(-3)+1(5)}{1+3}, \quad y = \dfrac{3(8)+1(-6)}{1+3}$

$\therefore$ P has coordinates $\left(-1, \dfrac{9}{2}\right)$.

e) Let $u = x-1 \Rightarrow x = u+1$ and $du = dx$.
Also when $x = 4, u = 3$ and when $x = 2, u = 1$.

$\therefore I = \displaystyle\int_1^3 \frac{u+1}{u^2}\,du \quad \Rightarrow \quad I = \int_1^3 \frac{1}{u}+u^{-2}\,du$

$\therefore I = \left[\ln u - \dfrac{1}{u}\right]_1^3 = \left(\ln 3 - \dfrac{1}{3}\right) - (\ln 1 - 1)$

$= \dfrac{2}{3} + \ln 3.$

f) $2\sin\theta\cos\theta - \cos\theta = 0$

$\cos\theta(2\sin\theta - 1) = 0$

$\therefore \cos\theta = 0$ or $\sin\theta = \dfrac{1}{2}$

$\therefore \theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$ and $\theta = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$.

Question 2.

a)
$\dfrac{d}{dx}\left(x^3\tan^{-1}x\right) = \tan^{-1}2x \times 3x^2 + x^3 \times \dfrac{1}{1+(2x)^2} \times 2$

$= 3x^2\tan^{-1}2x + \dfrac{2x^3}{1+4x^2}.$

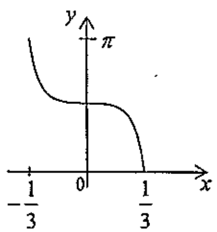
b) $f(x) = \ln x + 5x \quad \Rightarrow \quad f'(x) = \dfrac{1}{x}+5$

$\therefore x = 0.2 - \dfrac{\ln(0.2)+1}{10} \quad \Rightarrow \quad x = 0.26$

c) i) $\cos^{-1}\left(-\dfrac{1}{2}\right) = \dfrac{2\pi}{3}$

ii) D: $-\dfrac{1}{3} \le x \le \dfrac{1}{3}$, R: $0 \le y \le \pi$

iii)

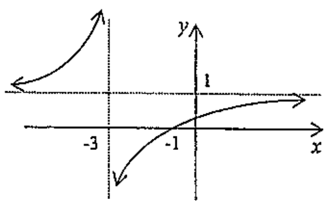


d) $LHS = \dfrac{x+3-2}{x+3} = \dfrac{x+3}{x+3} - \dfrac{2}{x+3}$

$= 1 - \dfrac{2}{x+3} = RHS.$

ii) $x = -3$.

iii)



Question 3.

a) $I = \displaystyle\int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt - \int_1^2 1\,dt$

$= \displaystyle\int_0^2 f(t)\,dt - [t]_1^2 = 5 - [2-1] = 4$

b) $mx - y + b = 0$ and using the perpendicular distance formula with (x_1, y_1) as $(0, 0)$ and $a = m$, $b = -1$, $c = b$, $d = 2$ we obtain:

$2 = \dfrac{|m(0)+-1(0)+b|}{\sqrt{m^2+1}} \quad \Rightarrow \quad 2 = \dfrac{b}{\sqrt{m^2+1}}$

$\therefore 4 = \dfrac{b^2}{m^2+1} \quad \Rightarrow \quad m^2+1 = \dfrac{b^2}{4}$

c) When $n = 1$, $LHS = 1$, $RHS = \dfrac{1}{6}(1)(3)(2) = 1$

$\therefore$ true for $n = 1$.

Assume true for $n = k$

i.e. $S_k = \dfrac{1}{6}k(4k-1)(k+1)$

Prove true for $n = k+1$

i.e. Prove that $S_{k+1} = S_k + T_{k+1}$

$RHS = \dfrac{1}{6}k(4k-1)(k+1) + (k+1)(2k+1)$

$= \dfrac{k(4k-1)(k+1)+6(k+1)(2k+1)}{6}$

$= \dfrac{(k+1)[k(4k-1)+6(2k+1)]}{6}$

$= \dfrac{(k+1)(4k^2+11k+6)}{6}$

$= \dfrac{1}{6}(k+1)(k+2)(4k+3) = S_{k+1}.$

Thus proved true for $n = 1$, assumed true for $n = k$ and proved true for $n = k+1$, so by Mathematical Induction it is true for $n = 1+1 = 2$, $n = 2+1 = 3$, and so on for all positive integers n.

d) i) $N = 8000 + Ae^{kt} \quad \Rightarrow \quad Ae^{kt} = N - 8000$

$\therefore \dfrac{dN}{dt} = kAe^{kt} = k(N-8000)$

ii) When $t = 0, N = 25000$

$\therefore 25000 = 8000 + Ae^0 \quad \Rightarrow \quad A = 17000$

When $t = 5, N = 29250$

$\therefore 29250 = 8000 + 17000e^{5k}$

$e^{5k} = \dfrac{21250}{17000}$

$5k = \ln\left(\dfrac{21250}{17000}\right) \quad \Rightarrow \quad k = 0.0446$

iii) $50000 = 8000 + 17000e^{0.0446t}$

$e^{0.0446t} = \dfrac{42000}{17000} \quad \Rightarrow \quad t = 20$

$\therefore$ In 2020 the population will reach 50 000.

Question 4.

a) i) The tangent AT meets radius OA at 90°.
$\therefore \angle OAT = 90°$.

Also, a line from the centre that bisects a chord is perpendicular to the chord
$\therefore \angle OXT = 90°$

Hence $AOXT$ is a cyclic quad (opp. ∠'s supp.)

ii) Since $AOXT$ is cyclic $\angle AOT = \angle AXT$
[Angles at circumference standing on same arc AT].

b) i) $4 - x^2 > 0 \quad \Rightarrow \quad -2 < x < 2$.

ii) $A = \displaystyle\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx = \left[\sin^{-1}\frac{x}{2}\right]_0^1$

$= \sin^{-1}\left(\dfrac{1}{2}\right) - \sin^{-1}(0) = \dfrac{\pi}{6}u^2$

Question 4(continued).

c) i) $\alpha+\beta+\gamma=-\frac{b}{a}=-\frac{1}{2}$

ii) $\alpha\beta\gamma=-\frac{d}{a}=1$

iii) $\alpha\beta+\alpha\gamma+\beta\gamma=\frac{c}{a}=-\frac{1}{2}$

iv) On Expansion

$$(\alpha-1)(\beta-1)(\gamma-1)=\alpha\beta\gamma-(\alpha\beta+\alpha\gamma+\beta\gamma)+(\alpha+\beta+\gamma)-1$$
$$=1-\left(-\frac{1}{2}\right)+\left(-\frac{1}{2}\right)-1=0$$

Question 5.

a) $\frac{dV}{dt}=450$, $V=\frac{4}{3}\pi r^3$, $\frac{dV}{dr}=4\pi r^2$, $r=15\,cm$

$\therefore$ using the chain rule $\frac{dV}{dt}=\frac{dV}{dr}\cdot\frac{dr}{dt}$

$\therefore 450=4\pi r^2\cdot\frac{dr}{dt} \Rightarrow \frac{dr}{dt}=\frac{450}{4\pi\times 15^2}=\frac{1}{2\pi}cm/s.$

b) $V=\pi\int_0^{\frac{3\pi}{4}}\sin^2 x\,dx$ $\quad\left[\cos 2\theta=1-2\sin^2\theta\right]$

$\therefore\left[\sin^2\theta=\frac{1}{2}(1-\cos 2\theta)\right]$

$$\therefore V=\pi\int_0^{\frac{3\pi}{4}}\frac{1}{2}(1-\cos 2x)\,dx=\frac{\pi}{2}\left[x-\frac{1}{2}\sin 2x\right]_0^{\frac{3\pi}{4}}$$
$$=\frac{\pi}{2}\left[\left(\frac{3\pi}{4}-\frac{1}{2}\sin\frac{3\pi}{2}\right)-(0-0)\right]$$
$$=\frac{\pi}{2}\left[\frac{3\pi}{4}+\frac{1}{2}\right]=\frac{3\pi^2+2\pi}{8}\,u^3$$

c) i) $y=\frac{x^2}{4a} \Rightarrow \frac{dy}{dx}=\frac{x}{2a} \Rightarrow m_T=t$ when $x=2at$

$\therefore m_N=-\frac{1}{t} \Rightarrow$ Eq^n of Normal: $y-at^2=-\frac{1}{t}(x-2at)$

$\therefore ty-at^3=2at-x \Rightarrow x+ty=2at+at^3$

ii) Using (i) and putting $x=0 \Rightarrow y=2a+at^2$

$\therefore Q$ has coordinates $\left(0,2a+at^2\right)$

and since R lies on the parabola then when $y=2a+at^2$, $x^2=4a\left(2a+at^2\right)=4a^2\left(2+t^2\right)$

which gives $x=-2a\sqrt{2+t^2}$ since R is in the second quadrant.

$\therefore R$ has coordinates $\left(-2a\sqrt{2+t^2},2a+at^2\right)$.

iii) M has coordinates $\left(-a\sqrt{2+t^2},2a+at^2\right)$

For the locus of M : $x=-a\sqrt{2+t^2}$

$\therefore x^2=a^2\left(2+t^2\right)=a\left(2a+at^2\right)$
$=ay$

Hence $x=-\sqrt{ay}$ since R is in the 2^{nd} Quad. is the locus of M.

Question 6.

a) i) $\ddot{x}=\frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$\therefore\frac{d}{dx}\left(\frac{1}{2}v^2\right)=9x-18$

$\therefore\frac{1}{2}v^2=\frac{9x^2}{2}-18x+C$

$\frac{1}{2}(-6)^2=\frac{9(4)^2}{2}-18(4)+C \Rightarrow C=18$

$\therefore\frac{1}{2}v^2=\frac{9x^2}{2}-18x+18$

$v^2=9x^2-36x+36$
$=9\left(x^2-4x+4\right)=9(x-2)^2$

ii) $v=-3(x-2)$ since $v<0$ when $t=0$

$\frac{dx}{dt}=-3(x-2) \Rightarrow \frac{dt}{dx}=\frac{-1}{3(x-2)}$

Question 6 a) ii) continued.

$t=-\frac{1}{3}\ln(x-2)+C$

when $t=0, x=4 \Rightarrow 0=\frac{-1}{3}\ln 2+C \Rightarrow C=\frac{1}{3}\ln 2$

$\therefore t=\frac{1}{3}[\ln 2-\ln(x-2)] \Rightarrow 3t=\ln\left(\frac{2}{x-2}\right)$

$\therefore e^{3t}=\frac{2}{x-2} \Rightarrow x-2=\frac{2}{e^{3t}}$

$\therefore x=2+2e^{-3t} \Rightarrow x=2\left(1+e^{-3t}\right)$

b) i) When $t=0, x=0, \dot{x}=V\cos\theta$ (a constant)

$\ddot{x}=0 \Rightarrow \dot{x}=C \Rightarrow \dot{x}=V\cos\theta$

$\therefore x=Vt\cos\theta+C$ but $C=0$ by I.C.

Hence $x=Vt\cos\theta$.

ii) $y=Vt\sin\theta-5t^2+1 \Rightarrow \dot{y}=V\sin\theta-10t$

At maximum height $\dot{y}=0 \Rightarrow t=\frac{V\sin\theta}{10}$

and when $t=\frac{V\sin\theta}{10}$, $y=2$

$$\therefore 2=V\left(\frac{V\sin\theta}{10}\right)\sin\theta-5\left(\frac{V\sin\theta}{10}\right)^2+1$$
$$2=\frac{V^2\sin^2\theta}{10}-\frac{5V^2\sin^2\theta}{100}+1$$
$$200=10V^2\sin^2\theta-5V^2\sin^2\theta+100$$

$\therefore V^2\sin^2\theta=20 \Rightarrow V^2=\frac{20}{\sin^2\theta} \Rightarrow V=\frac{\sqrt{20}}{\sin\theta}$.

iii) $\tan\theta=\frac{9}{40} \Rightarrow \sin\theta=\frac{9}{41}$ using Pythagoras

$\therefore V=\frac{\sqrt{20}}{\frac{9}{41}}=\frac{41\sqrt{20}}{9}$

$\therefore V\approx 20.37\ ms^{-1}$ correct to 2 decimal places.

Question 7.

a) $\lim_{x\to 0}\frac{1-\cos 2x}{3x^2}=\lim_{x\to 0}\frac{2\sin^2 x}{3x^2}$

$=\frac{2}{3}\lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{\sin x}{x}$

$=\frac{2}{3}\cdot 1\cdot 1=\frac{2}{3}$

b) i) Gradient of $PQ=\tan(180-\theta)=-\tan\theta$

$\therefore y-2=-\tan\theta(x-1)$

$y-2=-x\tan\theta+\tan\theta$

$y=\tan\theta-x\tan\theta+2$

ii) Using i) when $x=0, y=2+\tan\theta$

and when $y=0, x=\frac{2+\tan\theta}{\tan\theta}$

$\therefore$ Area $\Delta OPQ=\frac{1}{2}OP\times OQ$

$$=\frac{1}{2}\left(\frac{2+\tan\theta}{\tan\theta}\right)(2+\tan\theta)$$
$$=\frac{1}{2}\left(\frac{4+4\tan\theta+\tan^2\theta}{\tan\theta}\right)$$
$$=\frac{1}{2}\left(\frac{4}{\tan\theta}+4+\frac{\tan^2\theta}{\tan\theta}\right)$$
$$=\frac{2}{\tan\theta}+2+\frac{\tan\theta}{2}$$

iii) In part ii) let $t=\tan\theta$

$\therefore A=\frac{t}{2}+2+\frac{2}{t}=\frac{t}{2}+2+2t^{-1}$

For a minimum area $\frac{dA}{dt}=0$ and $\frac{d^2A}{dt^2}>0$

$\frac{dA}{dt}=\frac{1}{2}-2t^{-2}=\frac{1}{2}-\frac{2}{t^2}$ and $\frac{d^2A}{dt^2}=\frac{4}{t^3}$

$\therefore\frac{2}{t^2}=\frac{1}{2} \Rightarrow t^2=4 \Rightarrow t=2$, also $\frac{d^2A}{dt^2}=\frac{1}{2}$

$\therefore$ minimum area when $\tan\theta=2$

iv) $\therefore$ Minimum Area $=1+2+1=4\ u^2$.