Student Name: ....

CARINGBAH H.S.

2006

TRIAL HIGHER SCHOOL CERTIFICATE

MATHEMATICS

Extension 1

Question 1 (12 Marks)

	Marks
(a) Find $\int \frac{1}{\sqrt{25-x^2}}\,dx$	1
(b) Find the exact value of $\int_0^1 \left(\frac{1}{1+x} + e^{-x}\right) dx$	2
(c) State the domain and range of $y = \sin^{-1}(\frac{x}{3})$	2
(d) Using the substitution $u = 2 + x^2$, or otherwise, find $\int x\sqrt{2+x^2}\;dx$	3
(e) Find the coordinates of the point which divides the interval AB with A(1, 4) and B(5,2) externally in the ratio 1 : 3.	2
(f) The acute angle between the lines $y = 2x + 7$ and $y = mx - 3$ is 45°. Find the two possible values of m.	2

13

Question 7 (continued)

	Marks
(iii) Find the rate at which the water is overflowing from the container when the vertex of the cone is 2 metres below the surface.	2
(c) (i) What is the domain of $h(x) = \sin^{-1}\sqrt{1-x^2} + \sin^{-1} x$?	1
(ii) Find $h'(x)$	2
(iii) Hence determine the interval over which $h(x)$ is constant and find this constant.	2

End of paper

Question 2 (12 Marks) Start a new page | **Marks**

(a) Find $\frac{d}{dx}\left(\cos^{-1}3x\right)$ **1**

(b) A particle is moving along the x-axis. Its velocity V at position x is given by $V=\sqrt{8x-x^2}$. Find the acceleration when $x=3$. **2**

(c) (i) Differentiate $x-x\ln x$ **2**

(ii) Hence, or otherwise, evaluate $\int_1^{e^2}\ln x\,dx$ **2**

(d) A bowl of hot soup at temperature T° C, when placed in cooler surrounding air, loses heat according to the law

$$\frac{dT}{dt}=-k(T-S)$$

where t is the time elapsed in minutes and S is the temperature of the surrounding air in degrees Celsius.

(i) Show that $T=S+Ae^{-kt}$ satisfies this equation. **1**

(ii) A bowl of soup at 96° is left to stand in a room at a temperature of 18° C. After 3 minutes the soup cools to 75°. Calculate the value of k to 4 decimal places. **2**

(iii) Melissa wishes to enjoy her soup at a temperature of 60°. How long should she wait? **2**

Question 3 (12 Marks) Start a new page. | **Marks**

(a) (i) Let $g(x)=x^3+5x^2+17x-10$. Show that $g(x)=0$ has a root between 0 and 2. [This is the only root] **1**

(ii) Use one application of the "halving the interval" method to find a smaller interval containing the root. **1**

(iii) Which end of the smaller interval found in part (ii) is closer to the root? Briefly justify your answer. **2**

(b) Show that $\frac{d}{dx}(\sec x)=\sec x\tan x$ **2**

(c) Using the substitution $x=u-1$ find the integral $\int\frac{1}{x^2+2x+4}\,dx$ **2**

(d)

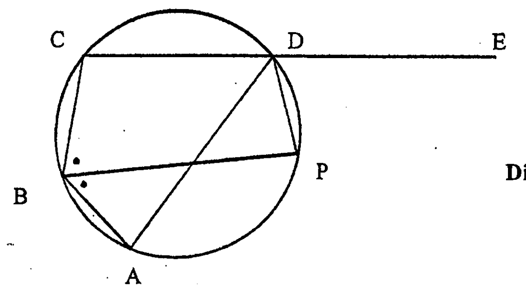


Diagram not to scale

In the diagram above ABCD is a cyclic quadrilateral. CD is produced to E. P is a point on the circle through A,B,C,D such that $\angle ABP=\angle PBC$.

(i) Copy the diagram showing the above information.

(ii) Explain why $\angle ABP=\angle ADP$. **1**

(iii) Show that PD bisects $\angle ADE$. **2**

(iv) If, in addition, $\angle BAP=90^\circ$ and $\angle APD=90^\circ$, explain where the centre of the circle is located. **1**

Question 4 (12 Marks) Start a new page. **Marks**

(a) Find $\displaystyle\int_0^{\frac{\pi}{4}} \frac{e^{\tan x}}{\cos^2 x}\,dx$ **2**

(b) By using the "t-results", where $t = \tan\dfrac{\theta}{2}$, or otherwise, show that **2**

$$\frac{1-\cos\theta}{\sin\theta} = \tan\frac{\theta}{2}$$

(c) $P(2at, at^2)$ is a point on the parabola $x^2 = 4ay$ with focus $S(0, a)$ and directrix $y = -a$. $D(2at, -a)$ is a point on the directrix.

(i) Display this information on a diagram.

(ii) Show that the equation of the tangent to the parabola at P is **2**

$$y = tx - at^2$$

(iii) Write down the coordinates of Q, the x-intercept of this tangent. **1**

(iv) Show that the tangent is the perpendicular bisector of the interval DS. **2**

(d) Use the principle of mathematical induction to show that if x is a positive integer then $(1+x)^n - 1$ is divisible by x for all positive integers $n \geq 1$. **3**

Question 5 (12 Marks) Start a new page. **Marks**

(a) The area between the curve $y = \sin^2 x$ and the x-axis between $x = 0$ and $x = \dfrac{\pi}{2}$ is rotated through one complete revolution about the x-axis.

(i) Find the exact value of the <u>area</u> of the region. **3**

(ii) Use Simpson's Rule with three function values to find an approximation to the <u>volume</u> of the solid of revolution, leaving the answer in terms of π. **2**

(b) A particle moving in a straight line is performing simple harmonic motion. At time t seconds its displacement x metres from a fixed point O on the line is given by $x = 2\cos^2 t$.

(i) Show that its velocity v ms^{-1} and its acceleration $\ddot{x}$ ms^{-2} are given by $v^2 = 4(2x - x^2)$ and $\ddot{x} = -4(x-1)$ respectively. **4**

(ii) Find the centre, amplitude and period of the motion. **3**

15

Question 6 (12 Marks) Start a new page. **Marks**

(a) Consider the function $f(x) = \dfrac{x}{4-x^2}$.

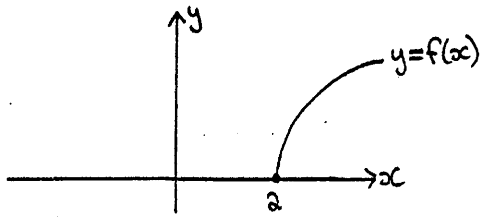
(i) Find the domain of the function. **1**

(ii) Show that the function is increasing throughout its domain. **2**

(iii) Sketch the graph of the function showing clearly the coordinates of any points of intersection with the x-axis and the y-axis and the equations of any asymptotes. **3**

(b) Take $x = 2.5$ as a first approximation for a root of $x^3 - 3x - 20 = 0$. Use one application of Newton's Method to find a second approximation correct to 2 decimal places. **2**

(c) The diagram below shows a sketch of the graph of $y = f(x)$ where where $f(x) = \sqrt{x^2 - 4}$ for $x \geq 2$.



(i) Copy this diagram onto your answer sheet. On the same set of axes, sketch the graph of the inverse function $y = f^{-1}(x)$. **1**

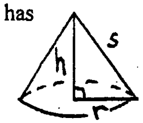
(ii) State the domain of $f^{-1}(x)$ **1**

(iii) Find an expression for $y = f^{-1}(x)$ in terms of x. **2**

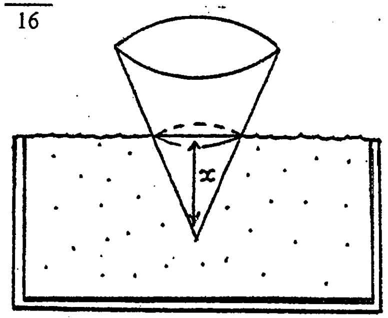
Question 7 (12 Marks) Start a new page. **Marks**

(a) Evaluate $\lim\limits_{x \to 0} \dfrac{\sin\left(\frac{x}{2}\right)}{3x}$ **1**

(b) A cone with base radius r, vertical height h and slant height s has curved surface area $A = \pi r s$ and volume $V = \frac{1}{3}\pi r^2 h$.



(i) Show that a cone with vertical height x and base radius $\dfrac{3x}{4}$ has $A = \dfrac{15\pi x^2}{16}$ and $V = \dfrac{3\pi x^3}{16}$ **2**



(ii) A similar cone whose base radius is $\frac{3}{4}$ of its vertical height, with its axis vertical and vertex downwards, is being lowered into a container which is overflowing with water. When the vertex is 2 metres below surface of the water (with the cone not fully submerged) the surface area of the cone is being covered with water at a rate of $0.5\, m^2 s^{-1}$. Show that the cone is being lowered into the water at a rate of $\dfrac{2}{15\pi} ms^{-1}$. **2**

Question 7 continued ~~**over page.**~~ on first page of

Question 7 (continued) **Marks**

(iii) *Find the rate at which the water is overflowing* from the container when the vertex of the cone is 2 metres below the surface. **2**

(c) (i) What is the domain of $h(x)=\sin^{-1}\sqrt{1-x^2}+\sin^{-1}x$? **1**

(ii) Find $h'(x)$ **2**

(iii) Hence determine the interval over which $h(x)$ is constant and find this constant. **2**

End of paper

EXT. 1 TRIAL HSC 2006 SOLNS

1. (a) $\sin^{-1}\frac{x}{5} + c$

(b) $\left[\ln(1+x) - e^{-x}\right]_0^1 = \ln 2 - \frac{1}{e} + 1$

(c) DOMAIN: $-3 \leq x \leq 3$

RANGE: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

(d) $\frac{du}{dx} = 2x$

$\frac{1}{2}\,du = x\,dx$

$\frac{1}{2}\int \sqrt{u}\,du$

$= \frac{1}{2}\cdot\frac{2}{3}u^{3/2} + c$

$= \frac{1}{3}(2+x^2)^{3/2} + c$

(e) $x = \frac{-1\times5+3\times1}{2}$, $y = \frac{-1\times2+3\times4}{2}$

$= -1$, $= 5$

i.e. $(-1, 5)$

(f) $m_1 = 2$, $m_2 = m$ $\quad \tan 45^\circ = \left|\frac{m-2}{1+2m}\right|$

i.e. $\frac{m-2}{1+2m} = -1$ or $\frac{m-2}{1+2m} = 1$

$m = \frac{1}{3}, -3$

2. (a) $\frac{-3}{\sqrt{1-9x^2}}$

(b) $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$= \frac{d}{dx}\left[\frac{1}{2}(8x-x^2)\right]$

$= 4 - x$

when $x=3$: $a = 1$

(c) (i) $u = x$, $u' = 1$; $v = \ln x$, $v' = \frac{1}{x}$ $\quad y' = 1 - \left[\ln x \cdot 1 + x\frac{1}{x}\right]$

$= 1 - \ln x - 1$

$= -\ln x$

(ii) $\int_1^{e^2} \ln x\,dx = -\left[x - x\ln x\right]_1^{e^2}$

$= -\left(e^2 - e^2\ln e^2\right) - (1 - 1\ln 1)$

$= e^2 + 1$

(d) (i) $T = S + Ae^{-kt}$

$\frac{dT}{dt} = -k.Ae^{-kt} = -k(T-S)$

(ii) sub. $T=96, S=18, t=0 \Rightarrow A = 78$

sub. $T=75, t=3$: $75 = 18 + 78e^{-3k}$

$78e^{-3k} = 57$

$e^{-3k} = \frac{57}{78}$

$k = \frac{\ln\frac{57}{78}}{-3} = 0.1046$

(iii) $60 = 18 + 78e^{-0.1046t}$

$t = 5.9$ min

3. (a) (i) $g(0) = -10$, $g(2) = 52$ — opposite signs ∴ root between 0 and 2.

(ii) $g(1) = 13$ ∴ root in 0 to 1.

(iii) $g(\frac{1}{2}) = -\frac{1}{8}$ ∴ root in $\frac{1}{2}$ to 1

∴ closer to 1.

(b) $\frac{d}{dx}(\cos x)^{-1} = -(\cos x)^{-2}\cdot -\sin x$

$= \frac{1}{\cos x}\cdot\frac{\sin x}{\cos x}$

$= \sec x \cdot \tan x$

(c) $dx = du$

$\int \frac{du}{(u-1)^2 + 2(u-1) + 4} = \int \frac{du}{u^2+3}$

$= \frac{1}{\sqrt{3}}\tan^{-1}\frac{u}{\sqrt{3}} + c$

$= \frac{1}{\sqrt{3}}\tan^{-1}\frac{x+1}{\sqrt{3}} + c$

… continued.

(ii) ∠s in same segment equal.

(iii) ∠PDE = ∠PBC (ext. ∠ cyclic quad. DCBP equals opp. interior ∠)

∠ADP = ∠PBC (both equal ∠ABP)

∴ ∠PDE = ∠ADP

i.e. PD bisects ∠ADE

) intersection of AD and BP

∠ in semi-circle 90° means both are diameters.

(a) $\left[e^{\tan x}\right]_0^{\frac{\pi}{4}} = e^1 - e^0 = e - 1$

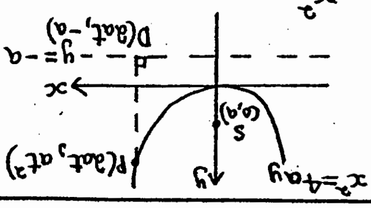
) $\frac{1 - \frac{1-t^2}{1+t^2}}{\frac{2t}{1+t^2}} = \frac{(1+t^2)-(1-t^2)}{2t}$

$= \frac{2t^2}{2t}$

$= t$

$= \tan\frac{\theta}{2}$

(c) (i)



(ii) $y = \frac{x^2}{4a}$

$y' = \frac{x}{2a}$

At $P(2at, at^2)$ $\quad m_{TAN} = \frac{2at}{2a} = t$

Equn. tang. $y - at^2 = t(x - 2at)$

$y = tx - at^2$

(iii) $Q(at, 0)$

(iv) $m_{DS} = -\frac{1}{t} \Rightarrow m_{TAN} \times m_{DS} = -1$

∴ TAN. ⊥ DS.

Also midpoint DS: $(at, 0)$ which is Q (on tangent)

∴ tangent perp. bisector of DS

(d) STEP 1: Prove true for $n=1$.

$n=1$: $(1+x)^1 - 1 = x$ which is divisible by x.

STEP 2: Assume true for $n=k$.

i.e. $(1+x)^k - 1 = Mx$ [M a polynom]

i.e. $(1+x)^k = Mx + 1$

Hence prove true for $n = k+1$

i.e. $(1+x)^{k+1} - 1$ also divisible by x.

Now $(1+x)^{k+1} - 1 = (1+x)(1+x)^k - 1$

$= (1+x)(Mx+1) - 1$ by our assumpt.

$= x(Mx + M + 1)$ which is divis. by x.

∴ if true for $n=k$ then true for $n=k+1$

STEP 3: we assumed true for $n=k$ and hence proved true for $n=k+1$. Since true for $n=1$ then true for $n=2$ and so on for all tve integral values of n.

5. (a) (i) $\int_0^{\frac{\pi}{2}} \sin^2 x\,dx$

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} 1 - \cos 2x\,dx$

$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{2}}$

$= \frac{1}{2}\left[\left(\frac{\pi}{2} - \frac{1}{2}\sin\pi\right) - (0-0)\right]$

$= \frac{\pi}{4}$ units²

(ii) $V = \pi\int_0^{\frac{\pi}{2}} (\sin^2 x)^2\,dx$

: continued

i) $\doteqdot \pi \times \dfrac{\frac{\pi}{4}}{3}\left[0^4 + 1^4 + 4\times\left(\frac{1}{\sqrt{2}}\right)^4\right]$

$= \dfrac{\pi^2}{6}$ units^3

) (i) $x = 2(\cos t)^2$

$\dot{x} = 4\cos t . -\sin t = -4\cos t.\sin t$

$\therefore v^2 = 16\cos^2 t \sin^2 t$

$= 16\cos^2 t(1-\cos^2 t)$

$= 16 \times \frac{x}{2}\left(1 - \frac{x}{2}\right)$

$= 4(2x - x^2)$

$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$= \frac{d}{dx}\left(4x - 2x^2\right)$

$= 4 - 4x$

$= -4(x-1)$

) CENTRE : $\ddot{x} = 0$ $\therefore$ CENTRE $x = 1$

XTREMITIES : $v = 0$ $\therefore 2x - x^2 = 0$

$x(2-x) = 0$

$x = 0, 2$

$\therefore$ AMPLITUDE $= 1$

) PERIOD $= \dfrac{2\pi}{n} = \dfrac{2\pi}{2} = \pi$

0 ⊢—1—⊣ 2

6. (a) (i) all real x, $x \neq \pm 2$

(ii) Find $\frac{dy}{dx}$: $u = x$ $\quad v = 4 - x^2$

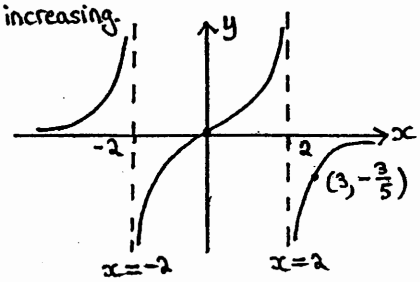
$u' = 1$ $\quad v' = -2x$

$\dfrac{dy}{dx} = \dfrac{(4-x^2).1 - x.-2x}{(4-x^2)^2}$

$= \dfrac{4+x^2}{(4-x^2)^2} > 0$ for all x in domain

$\therefore$ increasing.

(iii)



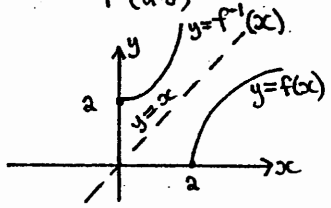
(b) $P(2.5) = -11.875$

$P'(x) = 3x^2 - 3$

$P'(2.5) = 15.75$

$z_2 = 2.5 - \dfrac{P(2.5)}{P'(2.5)} = 3.25$

(c) (i)



(ii) domain: $x \geqslant 0$

(iii) $x = \sqrt{y^2 - 4}$

$x^2 = y^2 - 4$

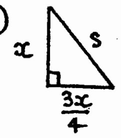
$y^2 = x^2 + 4$

$y = \pm\sqrt{x^2+4}$ [TAKE +VE ONLY]

$y = \sqrt{x^2+4}$

7. (a) $\displaystyle\lim_{x \to 0} \frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}} \times \frac{1}{6} = 1 \times \frac{1}{6} = \frac{1}{6}$

(b) (i)



$s = \sqrt{x^2 + \left(\frac{3x}{4}\right)^2}$

$= \frac{5x}{4}$

$A = \pi \times \frac{3x}{4} \times \frac{5x}{4} = \frac{15\pi x^2}{16}$

$V = \frac{1}{3} \times \pi \times \left(\frac{3x}{4}\right)^2 \times x = \frac{3\pi x^3}{16}$

(ii) we want $\frac{dx}{dt}$. we know $\frac{dA}{dt} = 0.5$

$\frac{dA}{dt} = \frac{dA}{dx} \times \frac{dx}{dt}$

$0.5 = \frac{15\pi x}{8} \times \frac{dx}{dt}$

$0.5 = \frac{15\pi}{4} \times \frac{dx}{dt}$ when $x = 2$.

7. continued.

(ii) $\dfrac{dx}{dt} = \dfrac{2}{15\pi}$ i.e. lowered at rate of $\frac{2}{15\pi}$ ms^{-1}

(iii) we want $\frac{dV}{dt}$.

$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$

$= \frac{9\pi x^2}{16} \times \frac{2}{15\pi}$

$= \frac{36\pi}{16} \times \frac{2}{15\pi}$

$= 0.3$

i.e. overflowing at rate of 0.3 m^3/s

(c) (i) $-1 \leqslant x \leqslant 1$

(ii) $h'(x) = \dfrac{1}{\sqrt{1-\left(\sqrt{1-x^2}\right)^2}} \times \dfrac{-x}{\sqrt{1-x^2}} + \dfrac{1}{\sqrt{1-x^2}}$

$= \dfrac{-x}{\sqrt{x^2}.\sqrt{1-x^2}} + \dfrac{1}{\sqrt{1-x^2}}$

(iii) Now $\sqrt{x^2} = x$ for $x \geqslant 0$

$= -x$ for $x < 0$

so for $x > 0$:

$h'(x) = \dfrac{-x}{x.\sqrt{1-x^2}} + \dfrac{1}{\sqrt{1-x^2}}$

$= \dfrac{-1}{\sqrt{1-x^2}} + \dfrac{1}{\sqrt{1-x^2}}$

$= 0.$

$\therefore h(x)$ is constant for $x > 0$

But domain is $-1 \leqslant x \leqslant 1$

and $h(0) = h(1) = \frac{\pi}{2}$

$\therefore$ constant for $0 \leqslant x \leqslant 1$ and this constant is $\frac{\pi}{2}$.

N.B. For $x < 0$:

$h'(x) = \dfrac{-x}{-x\sqrt{1-x^2}} + \dfrac{1}{\sqrt{1-x^2}}$

$= \dfrac{2}{\sqrt{1-x^2}}$

$\therefore h(x)$ is not constant for $x < 0$.