Question 1 (12 marks) Start a NEW booklet. **Marks**

a) Find the remainder when the polynomial $P(x)=2x^3-4x+3$ is divided by $(x+1)$. 1

b) Using the points $A(-4,1)$ and $B(2,-2)$ find the coordinates of the point $P(x,y)$ that divides the interval AB **externally** in the ratio $5:2$. 2

c) Solve $\dfrac{2}{x-1} \le 3$. 3

d) i) Show that $f(x) = \dfrac{x^3}{1+x^2}$ is an odd function. 1

.

ii) Hence or otherwise evaluate $\int_{-1}^{1} \dfrac{x^3}{1+x^2}\, dx$. 1

e) Neatly sketch $y=3\cos^{-1}2x$ clearly showing the domain and range. 2

f) Show that $\dfrac{d}{dx}(\sec x) = \sec x \tan x$. 2

Question 2 (12 marks) Start a NEW booklet. **Marks**

a) Find $\int \frac{1}{\sqrt{3-x^2}}\,dx$. 1

b) The parametric equations of a curve a given by $x = 6r,\ y = \frac{3}{r}$.

Find the cartesian equation of the curve. 1

c) Given the function $f:\ y = \frac{x-1}{x+2}$, find its inverse function $f^{-1}(x)$ in terms of x and **state** the range of the inverse function. 3

d) Solve $\sin 2\theta = \sin\theta$ for $0 \le \theta \le 2\pi$. 3

e) If α, β and γ are the roots of $2x^3 + 12x^2 - 6x + 1 = 0$, find

i) $\alpha + \beta + \gamma$ 1

ii) $\alpha\beta\gamma$ 1

iii) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ 2

Question 3 (12 marks) Start a NEW booklet. **Marks**

a) Find $\lim_{\theta \to 0} \frac{\sin 2\theta}{\tan 3\theta}$ 2

.

b) Use the substitution $u = 5 - x$ to evaluate $\int_1^5 x\sqrt{5-x}\ dx$ 3

c) i) Express $\cos\theta - \sqrt{3}\sin\theta$ in the form $R\cos(\theta + \alpha)$, 2

where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.

ii) Hence solve $\cos\theta - \sqrt{3}\sin\theta = 2$ for $0 \le \theta \le 2\pi$. 2

d) A particle is moving in a straight line x centimetres from the origin O.

After t minutes its displacement is given by $x = 3 - 5\cos 2t$.

i) Show that it acceleration is given by $\ddot{x} = -4(x-3)$. 2

ii) Assuming it is moving in simple harmonic motion find its centre of motion. 1

Question 4 (12 marks) Start a NEW booklet. **Marks**

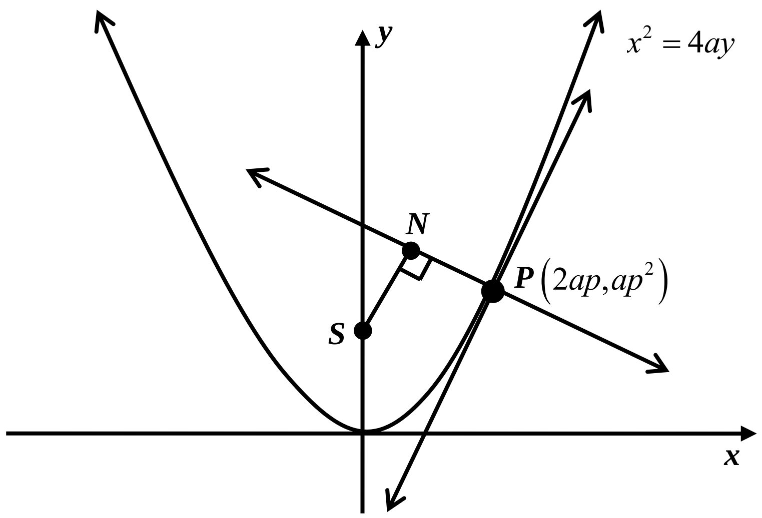
a) The acceleration of a body P is given by $\frac{d^2x}{dt^2} = 18x^3 + 18x$, where x is the displacement of P from O at time t. The velocity is v.

Given that $t = 0, x = 0, v = 3$ and that $v > 0$ throughout the motion:

i) Find v in terms of x . 2

ii) Show that $x = \tan 3t$. 2

b) $P\left(2ap, ap^2\right)$ is a point on the parabola $x^2 = 4ay$. SN is perpendicular to the normal at P, where S is the focus of the parabola and N the foot of the perpendicular from S to the normal.



i) Show that the equation of the normal at P is $x + py = 2ap + ap^3$ 2

ii) Find the equation of SN. 2

iii) Show that the coordinates of the point N are $\left(ap, ap^2 + a\right)$. 2

iv) Find the locus of N as P moves on the parabola. 2

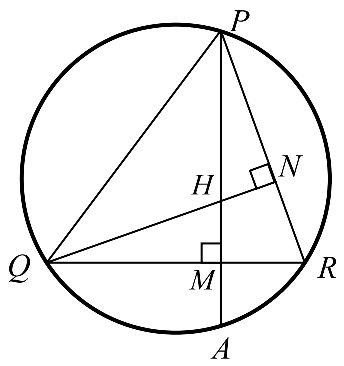
Question 5 (12 marks) Start a NEW booklet. **Marks**

a) i) Find the values of a and b such that $x^2 - 4x + 7 = (x-a)^2 + b$. 1

ii) Hence, state the largest positive domain for which $y = x^2 - 4x + 7$ has an inverse function. 1

b) Evaluate $\sin\left(\tan^{-1}\frac{1}{2}\right)$ in exact form. 2

c) The altitudes PM and QN of an acute angled triangle PQR meet at H. PM produced cuts the circle PQR at A. [A **larger diagram** is included at the end of the exam paper, **remove** it, use it and **submit** it with your **solutions**]



i) Explain why $PQMN$ is a cyclic quadrilateral. 1

ii) With the aid of congruent triangles prove that $HM = MA$. 3

d) The rate of growth of the number of bacteria in a colony is proportional to the excess of the colony's population over 5000 and is given by $\frac{dN}{dt} = k(N - 5000)$.

i) Show that $N = 5000 + Ae^{kt}$ is a solution to the differential equation. 1

ii) If the initial population is 15 000 and reaches 20 000 after 2 days find the values of A and k. 2

iii) Hence, calculate the expected population after 7 days. 1

Question 6 (12 marks) Start a NEW booklet. **Marks**

a) Using $t = \tan\frac{\theta}{2}$, prove that $\frac{1+\cos\theta}{1-\cos\theta} = \cot^2\frac{\theta}{2}$. 2

b) Air is being pumped into a spherical balloon at the rate of $20\ cm^3/s$. 4

Find the rate of increase of the balloon's surface area when the radius is 5cm.

$$\left[V = \frac{4}{3}\pi r^3, SA = 4\pi r^2\right]$$

c) For the function $y = \frac{x^2-4}{x^2-1}$

i) Write down the equations of any horizontal and vertical asymptotes. 2

ii) Find any stationary points and determine their nature. 2

iii) Neatly sketch the graph showing the above features. 2

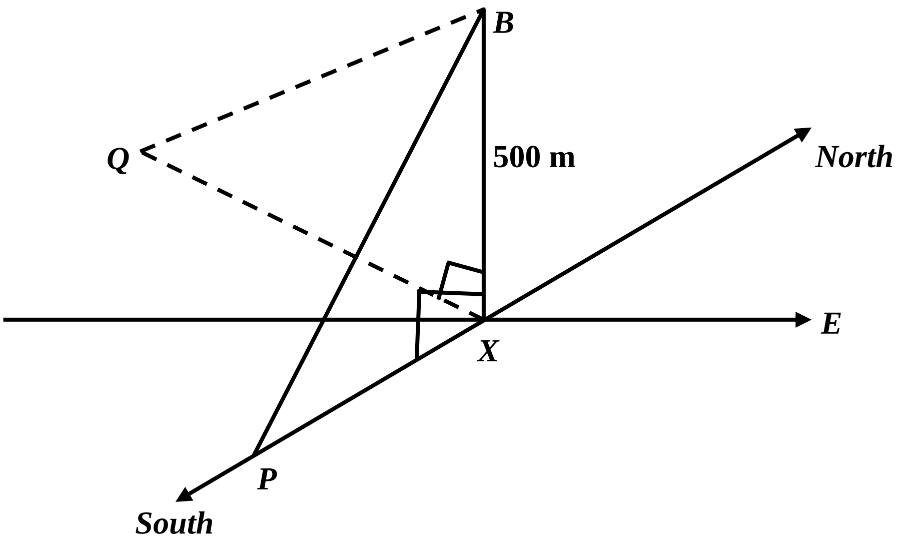
Question 7 (12 marks) Start a NEW booklet. **Marks**

a) Prove by mathematical induction that for any positive integer $n \geq 1$. 4

$$\frac{1}{1\times 5}+\frac{1}{5\times 9}+\frac{1}{9\times 13}+...........+\frac{1}{(4n-3)(4n+1)}=\frac{n}{4n+1}$$

b) From a balloon 500 metres above a road junction *X*, the angle of depression to a point *P*, which lies due south of the junction *X* is 42º.

To a point *Q*, which lies at a bearing of 280º from the road junction, the angle of depression to the balloon is 32º .



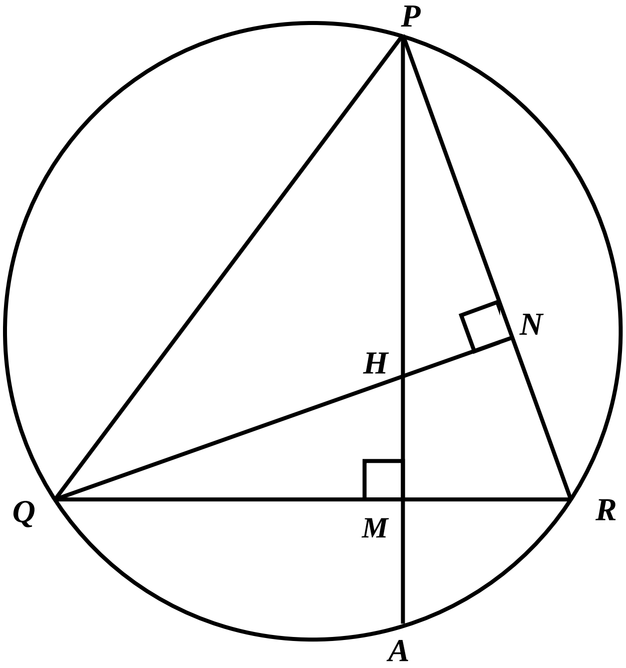
i) Clearly explain why $\angle QXP = 100^\circ$. 1

ii) Calculate the distance *PQ* (correct to the nearest metre). 3

c) A particle is moving in a straight line with simple harmonic motion. The velocity of the particle is respectively $\sqrt{20}\,ms^{-1}$ and $4\,ms^{-1}$ at distances of 1 metre and 2 metres from the centre of motion. Find the period and amplitude of the motion. 4

END OF EXAM

Question 5c (additional diagram)   Candidate Number: ____



1a) $P\ -1\ = -2+4+3=5$ ☑

b) $x=\dfrac{5\times 2\ +\ -2\times -4}{5-2}$ $\quad y=\dfrac{5\times -2\ +\ -2\times 1}{5-2}$ ☑

$=6$ $\qquad\qquad =-4$ ☑

$\therefore P\ x,y\ =\ 6,-4$

c) CV1: $2=3\ x-1 \quad\rightarrow\quad x=\dfrac{5}{3}$ ☑

CV2: $x=1$ ☑

On testing $x<1$ and $x\geq \dfrac{5}{3}$, since $x\neq 1$ ☑

d) i) An odd function exists if $f\ x\ =-f\ -x$

$f\ -x\ =\dfrac{-x^{\ 3}}{1+\ -x^{\ 2}}$

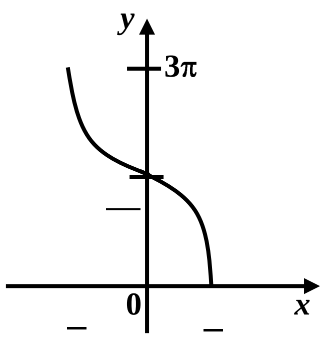
$=\dfrac{-x^3}{1+x^2}=-f\ x$ ☑

$\therefore$ an odd function.

ii) $\therefore \displaystyle\int_{-1}^{1}\frac{x^3}{1+x^2}\,dx\ =\ 0$ ☑

an odd function with symmetrical limits.

e)



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f) $\dfrac{d}{dx}\sec x\ =\dfrac{d}{dx}\cos x^{\ -1}$

$=-\ \cos x^{\ -2}\times -\sin x$ ☑

$=\dfrac{\sin x}{\cos^2 x}$

$=\dfrac{\sin x}{\cos x}\times\dfrac{1}{\cos x}\ =\ \tan x\sec x$ ☑

2a) $\displaystyle\int\frac{1}{\sqrt{3-x^2}}\,dx=\sin^{-1}\left(\frac{x}{\sqrt{3}}\right)+c$ ☑

b) $x=6r \quad\rightarrow\quad r=\dfrac{x}{6}$

$\therefore\ y=\dfrac{3}{r}\quad\rightarrow\quad y=\dfrac{3}{x/6}$ ☑

Hence $xy\ =18$

c) $f^{-1}\ x\ : x=\dfrac{y-1}{y+2}$ ☑

$xy+2x=y-1$

$2x\ +1\ =y-xy$

$2x\ +1\ =y\ 1-x$

$\therefore\ y=\dfrac{2x\ +1}{1-x}$ ☑

Range is all real y except $y=-2$ ☑

d) $\sin 2\theta=\sin\theta$ for $0\leq\theta\leq 2\pi$

$2\sin\theta\cos\theta-\sin\theta=0$ ☑

$2\sin\theta\ 2\cos\theta-1\ =0$

$\therefore\ \sin\theta=0$ or $\cos\theta=\dfrac{1}{2}$ ☑

$\therefore\ \theta=0,\pi,2\pi$ or $\theta=\dfrac{\pi}{3},\dfrac{5\pi}{3}$ ☑

e) i) $\alpha+\beta+\gamma=\dfrac{-b}{a}=-6$ ☑

ii) $\alpha\beta\gamma=\dfrac{d}{a}=\dfrac{-1}{2}$ ☑

iii) $\dfrac{1}{\alpha}+\dfrac{1}{\beta}+\dfrac{1}{\gamma} = \dfrac{\beta\gamma+\alpha\gamma+\alpha\beta}{\alpha\beta\gamma}$ ☑

$= \dfrac{-6/2}{-1/2} = 6$ ☑

3a) $\lim\limits_{\theta\to 0}\dfrac{\sin 2\theta}{\tan 3\theta} = \lim\limits_{\theta\to 0}\dfrac{2}{3}\left[\dfrac{\sin 2\theta}{2\theta}\times\dfrac{3\theta}{\tan 3\theta}\right]$ ☑

$= \dfrac{2}{3}$ ☑

b) When $u=5-x \rightarrow x=5-u$
When $x=1, u=4;\ \ x=5, u=0.$
$du = -dx$

$I = -\int_4^0 5-u\ \sqrt{u}\ du$ ☑

$= \int_0^4 5u^{\frac{1}{2}} - u^{\frac{3}{2}}\ du$

$= \left[\dfrac{10u^{\frac{3}{2}}}{3} - \dfrac{2u^{\frac{5}{2}}}{5}\right]_0^4$ ☑

$= \left[\dfrac{10\ 4^{\frac{3}{2}}}{3} - \dfrac{2\ 4^{\frac{5}{2}}}{5}\right] - 0$

$= \dfrac{208}{15}$ ☑

c) i) $\cos\theta-\sqrt{3}\sin\theta = R\cos\ \theta+\alpha$
$= R\cos\alpha\cos\theta - R\sin\alpha\sin\theta$

$\therefore\ R\cos\alpha = 1$ and $R\sin\alpha = \sqrt{3}$ ☑

$\therefore\ R^2\cos^2\alpha + R^2\sin^2\alpha = 1+3$

$\therefore\ R^2 = 4 \rightarrow R=2$

and hence $\alpha = \dfrac{\pi}{3}$ ☑

$\therefore\ \cos\theta-\sqrt{3}\sin\theta = 2\cos\left(\theta+\dfrac{\pi}{3}\right)$

ii) $\therefore\ 2\cos\left(\theta+\dfrac{\pi}{3}\right)=2$

$\therefore\ \cos\left(\theta+\dfrac{\pi}{3}\right)=1$ ☑

$\therefore\ \theta+\dfrac{\pi}{3}=0,\ 2\pi \rightarrow \theta=\dfrac{5\pi}{3}$ only. ☑

d) i) $x = 3-5\cos 2t$

$\therefore\ \dot{x} = 10\sin 2t$

$\ddot{x} = 20\cos 2t$ ☑

$\ddot{x} = 20\left(\dfrac{3-x}{5}\right) = 4\ 3-x$ ☑

$\therefore\ \ddot{x} = -4\ x-3$

ii) Centre of motion is at $x=3.$ ☑

4a) i) $\dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right) = 18x^3+18x$

$\therefore\ \dfrac{1}{2}v^2 = \dfrac{9x^4}{2}+9x^2 + c$

$v^2 = 9x^4+18x^2 + d$

when $x=0,\ v=3$

$\therefore\ 9 = 0+0+d \rightarrow d=9$ ☑

$v^2 = 9x^4+18x^2+9$

$= 9\ x^4 + 2x^2 + 1$

$= 9\ x^2 + 1\ ^2$

$\therefore\ v = 3\ x^2 + 1$, since $v>0.$ ☑

ii) $\dfrac{dx}{dt} = 3\ x^2 + 1$

$\therefore\ \dfrac{dt}{dx} = \dfrac{1}{3\ x^2 + 1}$

$\therefore \quad t = \frac{1}{3}\tan^{-1}x + c$ ☑

$\therefore \quad 0 = \frac{1}{3}\tan^{-1}0 + c \quad \rightarrow \quad c = 0$

$\therefore \quad t = \frac{1}{3}\tan^{-1}x$

$\therefore \quad 3t = \tan^{-1}x \quad \rightarrow \quad x = \tan 3t$ ☑

b) i) $\quad x^2 = 4ay \quad \rightarrow \quad y = \frac{x^2}{4a}$

$\therefore \frac{dy}{dx} = \frac{x}{2a} \quad \rightarrow \quad m_1 = \frac{2ap}{2a} = p$

$\therefore$ gradient of the normal at P is $m_2 = \frac{-1}{p}$ ☑

$\therefore$ Equation of the normal:

$y - ap^2 = \frac{-1}{p} \; x - 2ap$ ☑

$py - ap^3 = -x + 2ap$

$\therefore \quad x + py = 2ap + ap^3$

ii) gradient of SN is p and S has coordinates $(0,a)$ ☑

$\therefore$ eq^n of SN is $y - a = p(x-0) \quad \rightarrow \quad y = px + a$ ☑

iii) $\quad x + py = 2ap + ap^3 ----\boxed{1}$

$y = px + a -----\boxed{2}$

$\therefore \boxed{2}$ in $\boxed{1} \quad \rightarrow \quad x + p \;\; px + a \;\; = 2ap + ap^3$

$\therefore \quad x + p^2x + ap = 2ap + ap^3$

$\therefore \quad x \;\; 1 + p^2 \;\; = ap \;\; 1 + p^2$ ☑

$\therefore \quad x = ap$

and $\quad y = ap^2 + a$ ☑

$\therefore \quad N \;\; ap, ap^2 + a$

iv) $\quad x = ap \quad \rightarrow \quad p = \frac{x}{a}$ ☑

$\therefore \quad y = a\left(\frac{x}{a}\right)^2 + a \quad \rightarrow \quad y = \frac{x^2}{a} + a$ ☑

$x^2 = a \;\; y - a$

5a) i) $\quad x^2 - 4x + 7 = x - a^{\;2} + b$

$x^2 - 4x + 4 \;\; + 3 = x - a^{\;2} + b$

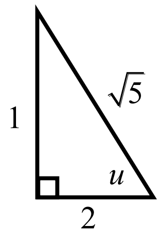
$x - 2^{\;2} + 3 = x - a^{\;2} + b$

$\therefore \quad a = 2, b = 3$ ☑

ii) all $\; x \geq 2$ ☑

b) Let $u = \tan^{-1}\frac{1}{2}$ and evaluate $\sin u$

$u = \tan^{-1}\frac{1}{2} \quad \rightarrow \quad \tan u = \frac{1}{2}$ ☑



$\sin u = \frac{1}{\sqrt{5}}$ ☑

c) i) $\angle QMP = \angle PNQ$ { both 90º }
$\therefore PQMN$ is a cyclic quad - $\angle$'s in same segment standing on same arc or chord PQ. ☑

ii) Join QA: $\quad \angle AQR = \angle APR$ { $= \theta$ say }
{$\angle$'s in same segment on arc AR}
Also $\angle MQN = \angle MPN$ { $= \theta$ }
{$\angle$'s in same segment on arc MN} ☑

$\therefore$ in Δ's QHM and QAM

$\angle QMH = \angle QMA$ { both 90º }
$\angle HQM = \angle AQM$ { proven above $(= \theta)$}
QM = QM { common } ☑

$\therefore \quad \Delta QHM \equiv \Delta QAM \;\; (AAS)$

Hence $HM = MA$ { corresponding sides in $\equiv \Delta$'s } ☑

d) i) $N = 5000 + Ae^{kt} \quad \rightarrow \quad Ae^{kt} = N - 5000$

$LHS = \frac{d}{dt} \;\; 5000 + Ae^{kt}$

$= k \;\; Ae^{kt}$ ☑

$= k \;\; N - 5000 \;\; = RHS$

ii) $\quad t = 0 \quad \rightarrow \quad N = 15\,000$

$15000 = 5000 + Ae^0 \quad \rightarrow \quad A = 10\,000$ ☑

$t = 2 \quad \rightarrow \quad N = 20\,000$

$20\,000 = 5000 + 10\,000e^{2k}$

$\therefore\ e^{2k} = \frac{3}{2} \quad \rightarrow \quad 2k = \ln\frac{3}{2}$

$k \ = \frac{1}{2}\ln\frac{3}{2} \ \approx \ 0.202733$ ☑

iii) When $t = 7, \quad N = \ 5000 \ + \ 10\,000e^{7k}$

$\approx 46\,335$ ☑

6a) Using $t = \tan\frac{\theta}{2}$:

$$LHS \ = \ \frac{1+\frac{1-t^2}{1+t^2}}{1-\frac{1-t^2}{1+t^2}}$$ ☑

$$= \ \frac{1+t^2 + 1-t^2}{1+t^2 - \ 1-t^2}$$

$$= \ \frac{2}{2t^2} \ = \frac{1}{t^2}$$ ☑

$$= \frac{1}{\tan^2\frac{\theta}{2}} \ = \ \cot^2\frac{\theta}{2}$$

b) $\frac{dV}{dt} = 20, V = \frac{4}{3}\pi r^3, \ \frac{dV}{dr} = \pi r^2, A = 4\pi r^2, \frac{dA}{dr} = 8\pi r$

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt} \ \rightarrow \ 20 = 4\pi r^2 \times \frac{dr}{dt}$$ ☑

$$\therefore \ \frac{dr}{dt} = \frac{5}{\pi r^2}$$ ☑

Also $\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$

$$= 8\pi r \ \times \frac{5}{\pi r^2} = \frac{40}{r}$$ ☑

and when $r = 5, \frac{dA}{dt} = 8cm^2 / s.$ ☑

c) i) Vertical asymptotes: $x = \pm 1$ ☑

Horizontal asymptote: $y = \ 1$ ☑

ii) $$\frac{dy}{dx} \ = \ \frac{(x^2 - 1) \bullet 2x \ - \ (x^2 - 4) \bullet 2x}{(x^2 - 1)^2}$$

$$= \ \frac{6x}{(x^2 - 1)^2}$$ ☑

Stationary points when $\frac{dy}{dx} \ = \ 0$

$\therefore$ when $6x = 0 \ \rightarrow \ x = 0$

at $x = 0, y = 4$

x	−½	0	½
y′	− ve	0	+ ve

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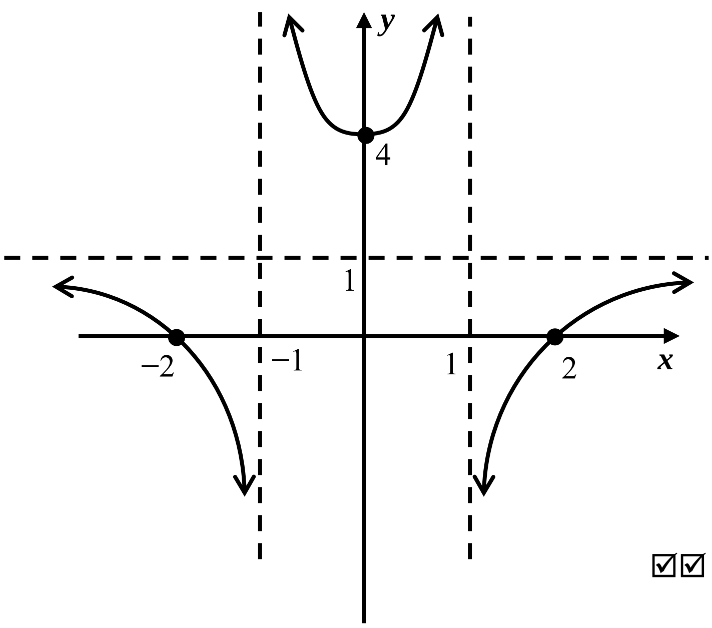
$\therefore$ local minimum turning point at (0, 4)

iii)



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7a) $\frac{1}{1\times 5}+\frac{1}{5\times 9}+\frac{1}{9\times 13}+\text{..........}+\frac{1}{(4n-3)(4n+1)}=\frac{n}{4n+1}$

When $n = 1, LHS=\frac{1}{1\times 5}=\frac{1}{5}; RHS=\frac{1}{4\times 1+1}=\frac{1}{5}$

$\therefore$ true for $n = 1.$ ☑

Assume true for $n = k$

i.e. $\frac{1}{1\times 5}+\frac{1}{5\times 9}+\frac{1}{9\times 13}+\text{..........}+\frac{1}{(4k-3)(4k+1)}=\frac{k}{4k+1}$

Prove true for $n = k + 1$

i.e. $S_k + T_{k+1} = S_{k+1}$

$S_k=\frac{k}{4k+1}, T_{k+1}=\frac{1}{(4k+1)(4k+5)}, S_{k+1}=\frac{k+1}{4k+5}$ ☑

$LHS = S_k + T_{k+1}$

$LHS=\frac{k}{4k+1}+\frac{1}{(4k+1)(4k+5)}$

$=\frac{k(4k+5)}{(4k+1)(4k+5)}+\frac{1}{(4k+1)(4k+5)}$ ☑

$=\frac{4k^2+5k+1}{(4k+1)(4k+5)}$

$=\frac{(4k+1)(k+1)}{(4k+1)(4k+5)}$

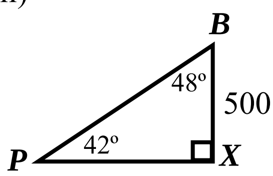
$=\frac{k+1}{4k+5}=S_{k+1}=RHS$ ☑

$\therefore$ If true for $n = k$, then true for $n = k + 1$.

Hence by the principle of mathematical induction, the result is true for all $n \geq 1$.

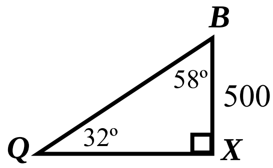
b) i) P is due south of X, i.e. at a bearing of 180º.
Q is at a bearing of 280º.
$\therefore$ the angle between P and Q on the ground is 280º - 180º = 100º. ☑

ii)



$\tan 48° = \frac{PX}{500}$

$\therefore PX = 500\tan 48°$ ☑



$\tan 58° = \frac{QX}{500}$

$\therefore QX = 500\tan 58°$

$\therefore$ using $\Delta\, QXP$:

$PQ^2 = PX^2 + QX^2 - 2\cdot PX\cdot QX\cdot\cos 100^\circ$ ☑

$= 500^2\tan^2 48° + 500^2\tan^2 58° - 2\times 500\tan 42°\times 500\tan 58°\times\cos 100^\circ$

$\therefore\ PQ^2 = 1\,102\,949.62 \rightarrow PQ \approx 1050\,m$ ☑

c) $v^2 = n^2(a^2 - x^2)$; $v=\sqrt{20}, x=1;\ v=4,\ x=2.$

$20 = n^2(a^2-1)$ ----$\boxed{1}$

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$16 = n^2(a^2-4)$ ----$\boxed{2}$

$\boxed{1} \div \boxed{2} \rightarrow \frac{5}{4}=\frac{a^2-1}{a^2-4}$ ☑

$\therefore 5a^2 - 20 = 4a^2 - 4$

$\therefore a^2 = 16 \rightarrow a = 4$

$\therefore \frac{16}{12}=\frac{4}{3}=n^2$

$\therefore n=\frac{2}{\sqrt{3}}$

hence the period $=\frac{2\pi}{n}=\frac{2\pi}{2/\sqrt{3}}=\pi\sqrt{3}$ seconds ☑

amplitude = 4 metres. ☑