

Caringbah High School

2013

Trial HSC Examination

Mathematics Extension I

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen (Black pen is preferred)
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 11–14, show relevant mathematical reasoning and/or calculations

Total marks – 70

Section I Pages 2 – 4

10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II Pages 5 – 8

60 marks

- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

Section I

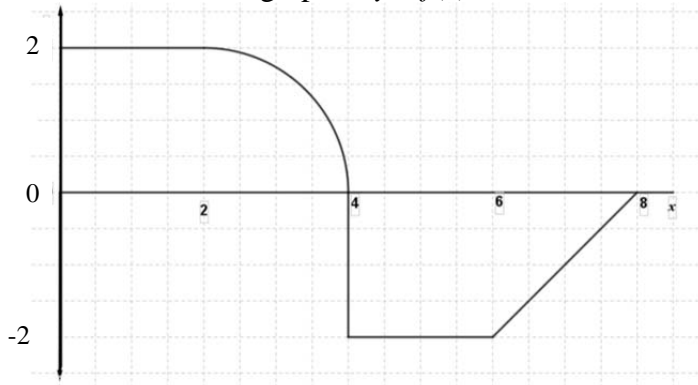
10 marks

Attempt Questions 1–10

Allow about 15 minutes for this section

For Questions 1–10, use the multiple-choice answer sheet on page 9. Please detach this from the exam paper and submit with your answer booklets.

1 This is the graph of $y = f(x)$, $0 \leq x \leq 8$.



The value of $\int_0^8 f(x) dx$ is

- A) $\frac{\pi}{4} - 2$ B) $\pi - 2$
C) $\frac{\pi}{4} + 10$ D) $10 + \pi$
- 2 Let $f(x) = 3\cos^{-1}\left(\frac{x}{2}\right)$.
The domain of the function $f(x)$ is given by
- A) $-\frac{1}{3} \leq x \leq \frac{1}{3}$ B) $-\frac{1}{2} \leq x \leq \frac{1}{2}$
C) $-2 \leq x \leq 2$ D) $-3 \leq x \leq 3$
- 3 The point $(3, -4)$ divides the interval AB externally in the ratio 3:2. If the coordinates of A are $(6, 5)$, then the coordinates of B are
- A) $(5, 2)$ B) $(8, 11)$
C) $(4, -1)$ D) $(1, -10)$

4

$$\frac{d(\tan^{-1}3x)}{dx} =$$

A) $\frac{3}{9+x^2}$

B) $\frac{1}{9+x^2}$

C) $\frac{1}{1+9x^2}$

D) $\frac{3}{1+9x^2}$

5 The variable point P ($5t, t^2$) lies on a parabola. The Cartesian equation for this parabola is

A) $y = \frac{x^2}{4}$

B) $x^2 = 10y$

C) $y = 25x^2$

D) $x^2 = 25y$

6 α, β and γ are roots of the equation $x^3 - 3x^2 + 1 = 0$.

The value of $\alpha\beta + \alpha\gamma + \beta\gamma$ is

A) -1

B) 0

C) 1

D) 3

7 A particle undergoes SHM about the origin. Its displacement in cm is given by

$$x = 3 \cos\left(2t + \frac{\pi}{3}\right).$$

The particle is at rest when

A) $x = -3$

B) $x = 0$

C) $t = \frac{\pi}{6}$

D) $t = 0$

8

$$\int \frac{-dx}{\sqrt{9-x^2}}$$

Which of the following may be a solution?

A) $\cos^{-1}\frac{x}{3}$

B) $\sin^{-1}\frac{x}{3}$

C) $\cos^{-1}3x$

D) $\sin^{-1}3x$

- 9** The vertical asymptote on the graph of $y = \frac{3x}{x-2}$ is
- A) $x = 2$ B) $y = 0$
C) $x = 3$ D) $y = 3$
- 10** Given that $(2x - 3)$ is a factor of $P(x) = 2x^3 - 3x + c$, the value of c is
- A) -45 B) $-2\frac{1}{4}$
C) $2\frac{1}{4}$ D) 45

Section II

60 marks

Attempt Questions 11–14

Allow about 1 hour and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

- a) Evaluate $\lim_{x \rightarrow \infty} \frac{3x + 2}{5 - x}$ 1
- b) Differentiate $\sin^{-1}(x^2)$, with respect to x . 2
- c) Sketch the graph of $y = 3 \sin\left(\frac{x}{2}\right)$ in the domain $0 \leq x \leq 2\pi$. 2
- d) Using the substitution $u = 4 - x^3$, 3
Evaluate $\int_{-1}^1 x^2 \sqrt{4 - x^3} dx$
- e) The line $y = 2x - 3$ intersects with the curve $y = 2x^3 - 15$ at the point (2,1). Find the size of the angle between the line and the curve at the point of intersection. (Answer to nearest degree) 3
- f) Find all values for x that satisfy $\frac{5}{x - 4} \leq x$ 2
- g) The function $f(x) = x^2 - e^x$ has a root near $x = 3$. Use one application of Newton's method to find a better approximation. 2

Question 12 (15 marks) Use a SEPARATE writing booklet.

a) Evaluate $\lim_{x \rightarrow 0} \frac{3x}{\sin 2x}$ **1**

b) Find the general solution to **2**

$$\sqrt{3} \tan x - 1 = 0$$

Answer as an exact value in radians.

c) A pendulum swings freely due to gravity and is friction free. When viewed from above, the end of the pendulum executes simple harmonic motion, with a period of π seconds and an amplitude of 1.2 m.

i) Explain why the acceleration, \ddot{x} , of the pendulum is given by $\ddot{x} = -4x$, where x is the position at any time, t . **1**

ii) Using part i), show that the maximum velocity of the end of the pendulum is $2 \cdot 4 \text{ m s}^{-1}$. **2**

d) i) $x^2 + 8x + 20$ can be expressed in the form $(x + a)^2 + b^2$. **2**
Find values for a and b .

ii) Hence or otherwise find **2**

$$\int \frac{1}{x^2 + 8x + 20} dx$$

e) A spherical beach ball is being inflated at a rate of 12 mm^3 per second. **2**
Calculate the rate that the radius is increasing when the surface area is 5000 mm^2 . (NB. $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$)

f) The half-life of a substance is the time taken for half of the substance to decay. The carbon isotope ^{14}C decays at a rate proportional to its mass. It has been shown that ^{14}C has a half-life of 5580 years. **3**

A fossil that was tested contained 40% of the ^{14}C it would have originally contained.

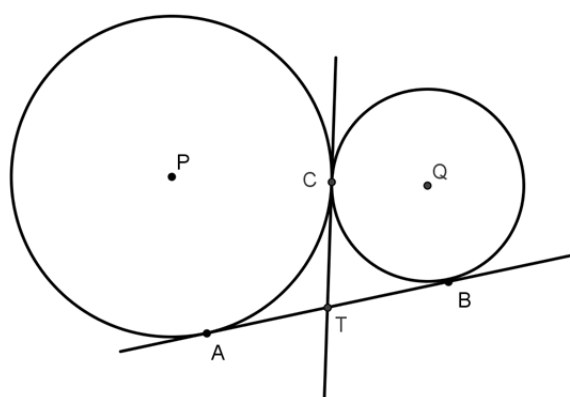
Estimate the age of the fossil.

Question 13 (15 marks) Use a SEPARATE writing booklet.

- a) Solve, $\cos 2x = \sqrt{3} \cos x - 1$, for $0 \leq \theta \leq \frac{\pi}{2}$. 2
- b) i) Differentiate $x \sin^{-1}(x) + \sqrt{1-x^2}$ 2
- ii) Hence evaluate $\int_0^1 \sin^{-1} x \, dx$ 2
- c) The 2 points P $(2ap, ap^2)$ and Q $(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$, with $a > 0$.

The chord PQ passes through the focus of the parabola.

- i) Show that $pq = -1$. 1
- ii) The tangent at P has the equation $y = px - ap^2$ 2
 The tangents from P and Q intersect at T. Show that T lies on the directrix, $y = -a$.
- d) Find all values of θ , $0 \leq \theta \leq \pi$ such that 3
 $\sqrt{2} \sin \theta + \cos \theta = 1$
- e) Two circles with centres P and Q touch externally at C and have a common tangent that touches at A and B, as shown. The common tangent at C meets AB in T. 2



- i) Show that T is the midpoint of AB. 1
- ii) Show that C, T, A and P are concyclic. 2

Question 14 (15 marks) Use a SEPARATE writing booklet.

a) Find $\int \sin^2 x \, dx$ 2

b) Let $f(x) = \frac{1}{1+x^3}$ for all x . 1

Find an expression for the inverse function $f^{-1}(x)$, in terms of x .

c) i) Sketch the curve $y = \cos^{-1} x$ 1

ii) The area between the curve $y = \cos^{-1} x$, the line $x = -1$ and the x -axis is rotated about the x -axis. Use Simpson's rule with 5 function values to approximate the volume of the solid formed. 3

d) Prove by mathematical induction that 3

$$7^{2n} - 3^{3n} \text{ is divisible by } 11, \text{ for all integers } n \geq 1.$$

e) If an archer fires an arrow with a velocity of 50 ms^{-1} at an angle of θ to the horizontal, it can be shown that the equations of motion are given by

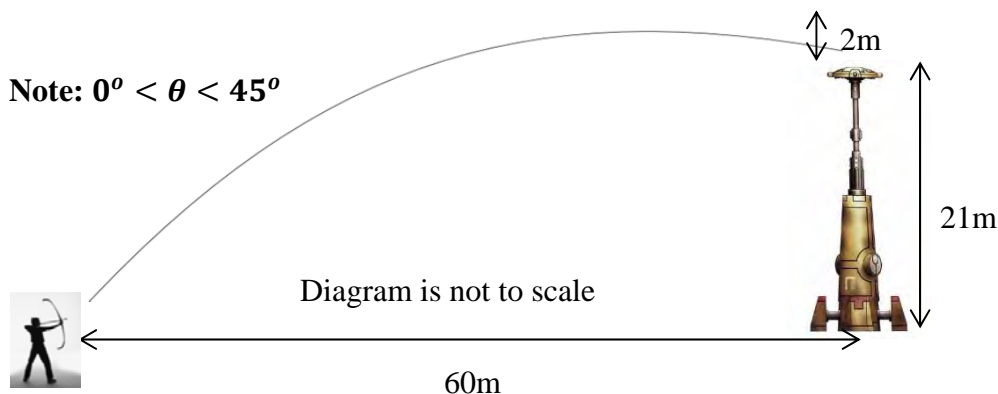
$$x = 50t \cos \theta \quad \text{and} \quad y = \frac{-gt^2}{2} + 50t \sin \theta \quad (\text{do not prove this}).$$

i) Show the Cartesian equation for the flight of the arrow is given by 2

$$y = x \tan \theta - \frac{gx^2}{5000} \sec^2 \theta$$

ii) In the 1992 Olympic Games in Barcelona, paralympian Antonio Rebello lit the Olympic cauldron in a most unique manner. From a horizontal distance of 60 metres from the base of the cauldron he fired a lit arrow across the top of the cauldron. The top of the cauldron was 21 metres higher than him. He had to shoot the arrow to within 2 metres above the cauldron to ignite the rising gas.

Using $g = 10$, find the range of angles from the horizontal that Antonio Rebello could aim through to successfully light the Olympic flame. 3



Name: _____

Multiple Choice Answer Sheet

Sample: $2 + 4 =$ (A) 2 (B) 6 (C) 8 (D) 9
A B C D

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A B C D

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word **correct** and drawing an arrow as follows.

A B C D
correct

- | | | | |
|-----------------------------|-------------------------|-------------------------|-------------------------|
| 1) A <input type="radio"/> | B <input type="radio"/> | C <input type="radio"/> | D <input type="radio"/> |
| 2) A <input type="radio"/> | B <input type="radio"/> | C <input type="radio"/> | D <input type="radio"/> |
| 3) A <input type="radio"/> | B <input type="radio"/> | C <input type="radio"/> | D <input type="radio"/> |
| 4) A <input type="radio"/> | B <input type="radio"/> | C <input type="radio"/> | D <input type="radio"/> |
| 5) A <input type="radio"/> | B <input type="radio"/> | C <input type="radio"/> | D <input type="radio"/> |
| 6) A <input type="radio"/> | B <input type="radio"/> | C <input type="radio"/> | D <input type="radio"/> |
| 7) A <input type="radio"/> | B <input type="radio"/> | C <input type="radio"/> | D <input type="radio"/> |
| 8) A <input type="radio"/> | B <input type="radio"/> | C <input type="radio"/> | D <input type="radio"/> |
| 9) A <input type="radio"/> | B <input type="radio"/> | C <input type="radio"/> | D <input type="radio"/> |
| 10) A <input type="radio"/> | B <input type="radio"/> | C <input type="radio"/> | D <input type="radio"/> |

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln \left(x + \sqrt{x^2 - a^2} \right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln \left(x + \sqrt{x^2 + a^2} \right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

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Section 1

- 1) B 2) C 3) A 4) D 5) D
- 6) B 7) A 8) A 9) A 10) C

Section 2

Question 11

a) -3

b) $2x$

$\sqrt{1-x^4}$

c) $\frac{y}{3}$



d) $u = 4 - x^3$ $u = 1$ $u = 3$

$du = -3x^2 dx$ $x = -1$ $u = 5$

$\int_{-1}^1 x^2 \sqrt{4-x^3} dx = \int_5^3 \sqrt{u} \cdot \frac{-du}{3}$

$= -\frac{1}{3} \int_5^3 u^{1/2} du$

$= -\frac{1}{3} \left[\frac{2}{3} u^{3/2} \right]_5^3$

$= -\frac{2}{9} \left[\sqrt{3^3} - \sqrt{5^3} \right]$

$= \frac{2}{9} \left[5\sqrt{5} - 3\sqrt{3} \right]$

e) $y = 2x^3 - 15$

$y_1 = 6x^2$

$m_1 = 24$ $m_2 = 2$

$\tan \theta = \frac{24-2}{1+24 \times 2}$

$= \frac{22}{49}$

$\therefore \theta = 24^\circ$

- f) Critical values
- $x = 4$ $x^2 - 4x = 5$ T
 - $x = 5$ $x = -1$ F
 - $x = 6$ T



$\therefore -1 < x < 4$ or $x > 6$

a) $f(x) = x^2 - e^{2x}$ $f(3) = 9 - e^3$

$f'(x) = 2x - e^{2x}$ $f'(3) = 6 - e^3$

$x_2 = 3 - \frac{(9-e^3)}{(6-e^3)}$

≈ 2.21 (3 sig fig)

Question 12

a) $\frac{3}{2}$ $\tan x = \frac{1}{\sqrt{3}}$ acute $x = \frac{\pi}{6}$

b) $x = n\pi + \frac{\pi}{6}$ $n = 0, \pm 1, \pm 2, \dots$

c) $\ddot{x} = -n^2 x$ defines SHM. $T = \frac{2\pi}{n}$

$T = \pi$ then $n = 2$ so $\ddot{x} = -4x$

i) $V^2 = n^2 (a^2 - x^2)$ $n = 2$, $a = 1.2$

max v when $x = 0 \therefore V = na$

$= 2.4 \text{ ms}^{-1}$

d) $x^2 + 8x + 20 = x^2 + 8x + 16 + 4$

$= (x+4)^2 + 4$

$a = 4$ $b = 2$

ii) $\int \frac{1}{(x+4)^2 + 4} dx = \frac{1}{2} \tan^{-1} \left(\frac{x+4}{2} \right) + C$

e) $V = \frac{4}{3} \pi r^3$

$\frac{dV}{dr} = 4\pi r^2$ $\frac{dV}{dt} = 12 \text{ mm}^3/\text{s}$

$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$

$= \frac{1}{4\pi r^2} \times 12$ but $4\pi r^2 = 5000$

$= \frac{12}{5000}$

$= \frac{3}{1250} \text{ mm/s}$ $(2.4 \times 10^{-3} \text{ mm/s})$

f) $M = M_0 e^{-kt}$

$\frac{1}{2} M_0 = M_0 e^{-5580k}$ $(k = 1.2422)$

$k = -\frac{\ln(\frac{1}{2})}{5580}$

$0.4 M_0 = M_0 e^{-kt}$

$t = -\frac{\ln(0.4)}{k}$

$= 7376 \text{ years}$

Question 13

a) $\cos 2x = 2 \cos^2 x - 1$

so $2 \cos^2 x - 1 = \sqrt{3} \cos x - 1$

$2 \cos^2 x - \sqrt{3} \cos x = 0$

$\cos x (2 \cos x - \sqrt{3}) = 0$

$\cos x = 0$ or $\cos x = \frac{\sqrt{3}}{2}$

$x = \frac{\pi}{2}$ or $x = \frac{\pi}{6}$

b) let $y = x \sin x + (1-x^2)^{1/2}$

$y' = x \cdot \frac{1}{\sqrt{1-x^2}} + \sin x + \frac{1}{2} (1-x^2)^{-1/2} \cdot (-2x)$

$= \frac{x}{\sqrt{1-x^2}} + \sin x - \frac{x}{\sqrt{1-x^2}}$

$= \sin x$

ii) $\int_0^1 \sin x dx = [-\cos x]_0^1$

$= -\cos(1) + \cos(0) = 1 - \cos(1)$

$= \frac{\pi}{2} - 1$

c) focus $S(0, a)$

need mps = mas $\frac{a^2 - a}{2ap} = \frac{a^2 - a}{2ag}$

$\frac{a^2 - a}{a} = \frac{a^2 - a}{g}$

$a^2 - a = a^2 - a$

$a^2 - a = a^2 - a$

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ii) $y = px - ap^2$ $x = \frac{y + ap^2}{p}$

$y = q^2 - aq^2$ $x = \frac{y + aq^2}{q}$

$\therefore \frac{y + ap^2}{p} = \frac{y + aq^2}{q}$

$qy + ap^2q = py + apq^2$ as $pp = qq$

$qy - ap = py - qa$

$y(q-p) = -a(q-p)$

$y = -a$

d) $\sqrt{2} \sin \theta + \cos \theta = R \sin(\theta + \alpha)$

$R \cos \alpha = \sqrt{2}$ $R \sin \alpha = 1$

$R = \sqrt{3}$ $\tan \alpha = \frac{1}{\sqrt{2}}$

$\alpha = 0.615$

so $\sqrt{3} \sin(\theta + 0.615) = 1$

$\theta + 0.615 = \sin^{-1} \left(\frac{1}{\sqrt{3}} \right)$

$\theta = 0.615 - 0.615$ or $\theta = (\pi - 0.615) - 0$

as $0 \leq \theta \leq 2\pi$ $\theta = 0, \pi.911$

e) i) $AT = CT$ and $BT = CT$ (tangents from point)

$\therefore AT = BT \Rightarrow T$ is midpoint of AB

ii) join PA and CA

$\angle PAT = \angle PCT = 90^\circ$ (radius \perp tangent)

$\therefore \angle PAT + \angle PCT = 180^\circ$

$\therefore S, T, A$ and P are concyclic. (opp angles supp.)

Question 14

a) $\int \sin^2 x dx = \frac{1}{2} \int (1 - \cos 2x) dx$

$= \frac{1}{2} \left[x - \frac{1}{2} \sin 2x \right] + c$

$= \frac{x}{2} - \frac{1}{4} \sin 2x + c$

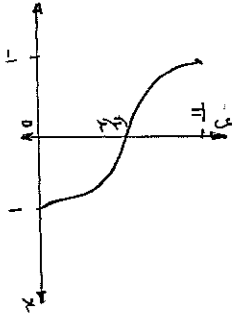
b) $y = \frac{1}{1+x^3}$ inverse $x = \frac{1}{1-y^3}$

$1-y^3 = \frac{1}{x}$

$y^3 = \frac{1}{x} - 1$

$y = \sqrt[3]{\frac{1-x}{x}}$

9)



$$ii) V = \int_{-1}^1 [\cos^{-1} x]^2 dx$$

x	-1	-1/2	0	1/2	1
y^2	$\frac{\pi^2}{4}$	$\frac{\pi^2}{9}$	$\frac{\pi^2}{4}$	$\frac{\pi^2}{9}$	0

$$h = \frac{1}{2}$$

$$h = \frac{1}{6}$$

$$V = \frac{\pi^2}{6} \left[\left(\frac{\pi^2}{4} + \frac{\pi^2}{9} + 4x \cdot \frac{\pi^2}{9} \right) + \left(\frac{\pi^2}{4} + 0 + 4x \cdot \frac{\pi^2}{4} \right) \right]$$

$$= \frac{67\pi^3}{108} \quad (19.24)$$

ii) $n=1$ $7^2 - 3^3 = 22$ \therefore true for $n=1$

Assume true for $n=k$

$$7^{2k} - 3^{3k} = 11M$$

Prove true for $n=k+1$

$$7^{2(k+1)} - 3^{3(k+1)} = 7^2 [7^{2k} - 3^{3k}] + 7^2 \cdot 3^k - 3^{3k+3}$$

$$= 7^2 [11M + 3^k (7^2 - 3^3)]$$

$$= 11 (49M + 2 \times 3^k)$$

\therefore If true for $n=k$, then also true for $n=k+1$.

As it is true for $n=1$, then by mathematical induction it is true for all $n=1, 2, 3, \dots$

e) $x = 50t \cos \theta$

$$t = \frac{x}{50 \cos \theta}$$

$$y = \frac{-g}{2} \left(\frac{x}{50 \cos \theta} \right)^2 + 50 \left(\frac{x}{50 \cos \theta} \right) \sin \theta$$

$$= \frac{-g x^2}{5000} \sec^2 \theta + x \tan \theta$$

ii) When $x=60$ $21 \leq y \leq 23$

$$23 = 60 \tan \theta - \frac{(60)^2 \sin^2 \theta}{5000} (1 + \tan^2 \theta)$$

$$\frac{36}{5} \tan^2 \theta - 60 \tan \theta + (23 + \frac{36}{5}) = 0$$

$$A = \frac{36}{5} \quad B = -60 \quad C = (23 + \frac{36}{5})$$

$$\tan \theta = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

as $\theta < 45^\circ$

$$\tan \theta = 0.5381$$

$$\theta = 28.3^\circ \quad (28^\circ 17')$$

using $C = (21 + \frac{36}{5})$

$$\tan \theta = \frac{1}{2}$$

$$\theta = 26.6^\circ \quad (26^\circ 34')$$

\therefore Angle must be between 26.6° and 28.3°