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HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION
1996

MATHEMATICS

3 UNIT (ADDITIONAL)
AND
3/4 UNIT (COMMON)

Time allowed - Two hours
(Plus 5 minutes' reading time)

STUDENT NAME: ______________________

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- ALL questions are of equal value.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are printed
- Board-approved calculators may be used.
- Each question is to be returned in a separate Writing Booklet clearly labelled, showing your Student Name

Students are advised that this is a trial examination only and cannot in any way guarantee the content or the format of the 1996 Mathematics 3 Unit (Additional) and 3/4 Unit (Common) Higher School Certificate Examination.

UESTION 1. (Start a new page) **Marks**

By using a substitution of $U = e^{2x}$, or otherwise, solve the equation **3**

$$e^{2x} - 2e^{-2x} = 1.$$

By using the expansion of $\tan(\alpha + \beta)$, find the value of K such that **3**

$$\tan^{-1}(K) + \tan^{-1}(\tfrac{3}{5}) = \frac{\pi}{4}.$$

Evaluate $\int_0^3 \frac{2x^2}{\sqrt{4-x}}\,dx$ using the substitution $x = 4 - u^2$. **4**

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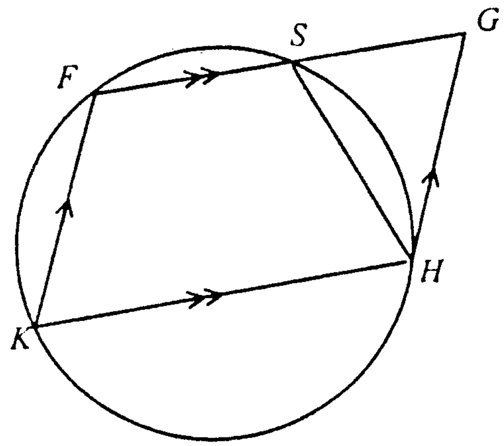


The figure $FGHK$ is a parallelogram. The point S lies on FG, and F, S, H, K lie on a circle.

Copy or trace the diagram into your Writing Booklet.

Prove that triangle HGS is isosceles.

QUESTION 2. (Start a new page) **Marks**

(a) Find $\frac{d}{dx}\left[x\tan^{-1}x-\frac{1}{2}\ln(1+x^2)\right]$ in simplified form. **3**

(b) Consider the equation $x^2-4+\log_e x=0$. **5**

(i) By drawing the graph of $y=\log_e x$, and another appropriate graph on the same set of axes, show that the equation has only one root.

(ii) Show, by means of calculations, that the root of the equation lies between 1 and 2.

(iii) Use one application of the 'halving the interval' method to find a smaller interval containing the root.

(iv) Taking the larger value of the interval from part (iii) as the first approximation, use Newton's method to find a second approximation of the root to 2 decimal places.

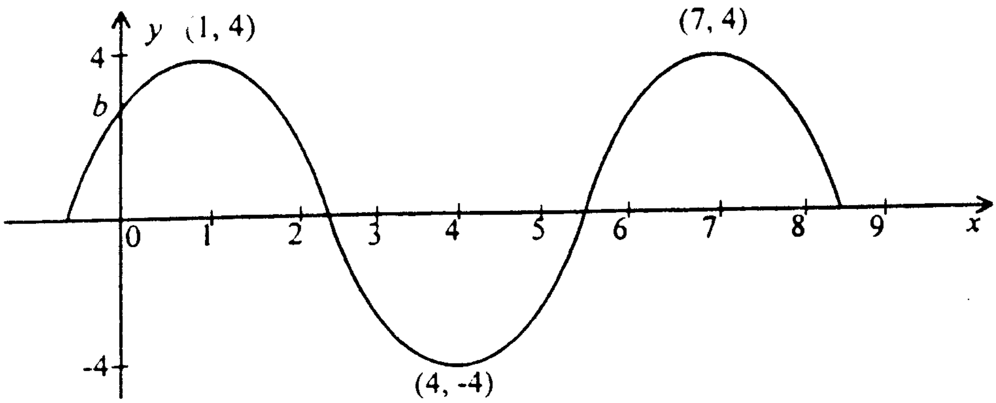
(c) A man standing 80 metres from the base of a high-rise building observes an external lift moving up the outside wall of the building at a constant rate of 7 metres per second. **4**

(i) If θ radians is the angle of elevation of the lift from the observer, find an expression for $\frac{d\theta}{dt}$ in terms of θ.

(ii) Evaluate $\frac{d\theta}{dt}$ at the instant when the lift is 30 metres above the observer's horizontal line of vision. Give your answer correct to 2 significant figures.

QUESTION 3. (Start a new page) **Marks**

a) Part of the graph of a periodic function is shown below. The graph cuts the y axis at $(0, b)$, and has turning points shown at $(1, 4)$, $(4, -4)$, and $(7, 4)$. **5**



The function can be written in the form $y = A\cos(nx + \alpha)$, where A, n and α are constants.

(i) State the value of A.

(ii) State the period of the function.

(iii) Show that $n = \frac{\pi}{3}$.

(iv) Find α, given $0 \le \alpha \le 2\pi$.

(v) Evaluate b, the y intercept.

b) Consider the function $f(x) = \sin^{-1}(x - 1)$. **7**

(i) Evaluate $f(0)$.

(ii) Draw the graph of $y = f(x)$.

(iii) State the domain and range of $y = f(x)$.

(iv) If the area bounded by the curve $y = f(x)$, the y axis and the line $y = \frac{\pi}{2}$ is rotated about the y axis, find the volume of the solid formed.

Marks

QUESTION 4. (Start a new page)

(a) When the cubic polynomial $y = P(x)$ is graphed it cuts the x axis at $-\frac{1}{3}$ and touches it at 1. Also, the graph passes through the point $(2, -14)$.

(i) Find $P(x)$ in expanded form.

(ii) Determine the values of x for which $P(x) < 0$.

(b) A particle is moving in a straight line. At time t seconds its velocity, v metres per second, and displacement, x metres, are such that

$$v^2 = 48 - 3x^2.$$

(i) By using differentiation, show that the motion is simple harmonic.

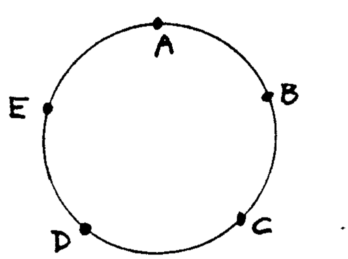
(ii) Find the amplitude of the motion.

(iii) Determine the particle's maximum speed.

(iv) Determine the particle's maximum acceleration.

(c) Find the general solution to the equation $\sin 2x = 2\cos x$. 2

(d) (i) Five points are evenly spaced around a circle. 2



How many triangles can be formed using these points as vertices ?

(ii) How many of these do not include the point A as one of its vertices ?

UESTION 5. (Start a new page) **Marks**

) The acceleration of a raindrop which at time t seconds is falling with speed v metres per second, is given by the equation : 6

$$\frac{dv}{dt} = -\frac{1}{3}(v - 3g), \text{ where } g \text{ is a constant.}$$

(i) Show that $v = 3g + Ae^{-\frac{1}{3}t}$, where A is a constant, satisfies the above equation.

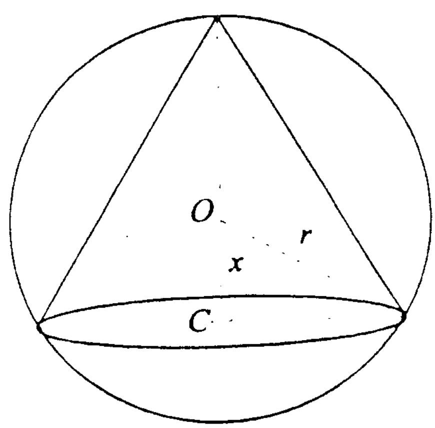
(ii) Given that the initial velocity has a value of g, evaluate A.

(iii) After how many seconds is the raindrop falling with a speed of $2g$ metres per second? (Give the answer correct to one decimal place.)

(iv) What limit does v approach as $t \to \infty$?

(v) Does the limiting value for the speed of the falling raindrop depend on the raindrop's initial speed? Briefly justify your answer.

b) The diagram shows a cone inscribed in a sphere. The sphere has centre O and radius r units. The axis of the cone meets the base of the cone at point C, as well as passing through the centre of the sphere. The interval OC is x units in length. 6



(i) Find both the radius of the base of the cone and the height of the cone in terms of x and r.

(ii) Show that for any given value of r, the cone has greatest volume where $x = \frac{r}{3}$.

(iii) What fraction of the sphere's volume is occupied by the cone with greatest volume?

Mark

QUESTION 6. (Start a new page)

(a) (i) Show that $\sin\theta + \cos\theta = \sqrt{2}\,\sin(\theta + \frac{\pi}{4})$.

(ii) Hence show that $\frac{d}{dx}\left[e^x \sin\left(x + \frac{n\pi}{4}\right)\right] = \sqrt{2}\,e^x \sin\left(x + \frac{(n+1)\pi}{4}\right)$,

where n is a positive integer.

(iii) Use mathematical induction to prove that if $y = e^x \sin x$ then

$$\frac{d^n y}{dx^n} = (\sqrt{2}\,)^n\, e^x \sin\left(x + \frac{n\pi}{4}\right) \quad \text{for } n = 1, 2, 3, \ldots\ldots,$$

where $\frac{d^n y}{dx^n}$ represents the n^{th} derivative of y with respect to x.

(iv) Hence write an expression for $\frac{d^2 y}{dx^2}$ in simplified form.

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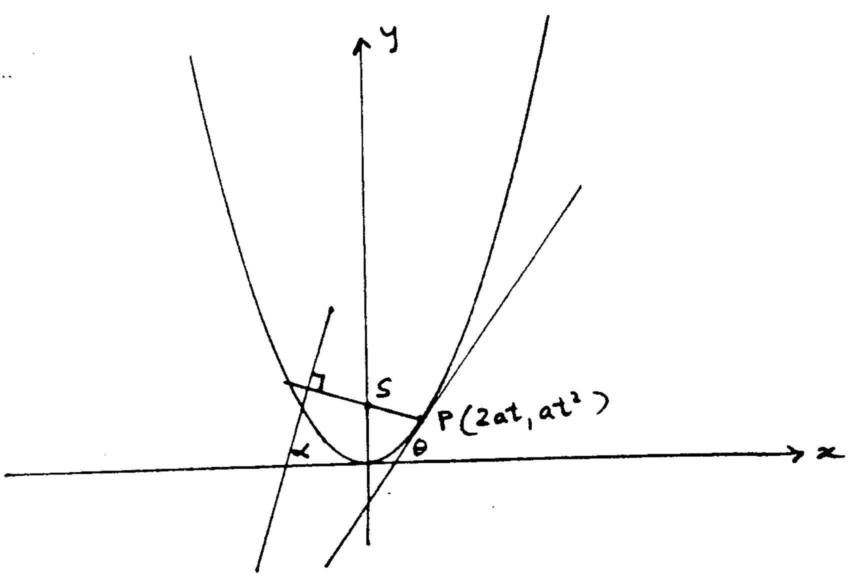
(b)



The diagram shows a point P $(2at, at^2)$ on the parabola $x^2 = 4ay$ where t is positive and $t \neq 1$. The tangent at P makes an angle θ with the positive x-axis.

(i) Show that the tangent at P has gradient t.

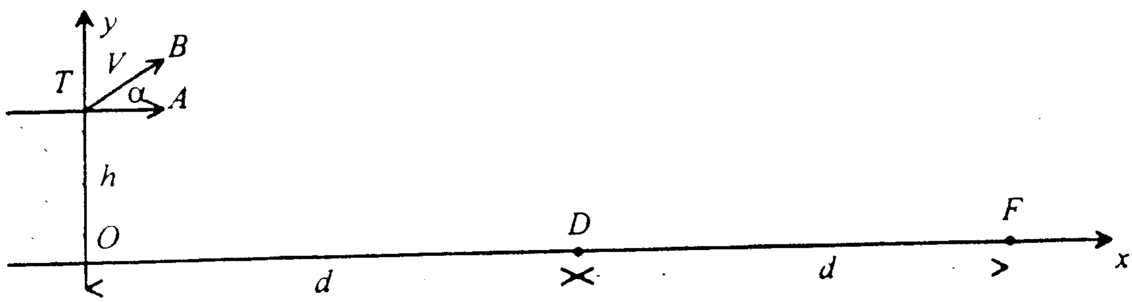
(ii) Calculate the gradient of SP where S (0,a) is the focus.

(iii) A perpendicular to SP makes an angle of α with the

Marks

QUESTION 7. (Start a new page)

T is the top of a building, h metres high. The points O, D, F are in the same line on flat, level ground. O is at the base of the building. D is d metres from O, and F is a further d metres from D.
At time $t = 0$, two particles A and B, are projected with the same initial velocity V metres per second from T. Particle A is projected horizontally, and particle B is projected in the same direction, but at an angle α, $\alpha > 0$, to the horizontal.



(a) The equations of motion of both particles are $\ddot{x} = 0$, $\ddot{y} = -g$. **3**

Using calculus, show that at time t seconds, the position of particle B is given by

$$x = (V\cos\alpha)\, t, \quad y = -\frac{1}{2} g t^2 + (V\sin\alpha)\, t + h.$$

(b) (i) Assuming that the position of particle A at time t is given by **2**

$$x = Vt, \qquad y = -\frac{1}{2}gt^2 + h,$$

show that the trajectory of particle A is given by $y = h - \dfrac{g}{2V^2}x^2$

(ii) Show that the trajectory of particle B is given by

$$y = -\frac{g}{2V^2\cos^2\alpha}x^2 + (\tan\alpha)x + h.$$

(c) If both particles land on the ground at D, show that $\tan\alpha = \dfrac{d}{h}$. **4**

(d) If particle A lands on the ground at D and particle B lands on the ground at F, **3**

show that $d \geq 2\sqrt{3}\, h$.