Cheltenham Girls High School

HN

TRIAL H.S.C. EXAMINATION, AUGUST, 1998

Mathematics – 3/4 Unit Common Paper

Name :- ____________________ Class 12M

Student Number:- ____________________

Time allowed:- 2 hours (+5minutes reading time)

Instructions

* All questions may be attempted.
* All questions are of equal value, except Question 2 (15 marks).
* All necessary working should be shown. Marks may not be awarded for careless or badly arranged work.
* Approved calculators may be used.
* Write your name, class and student number on this page, and your student number on each answer page.
* Hand in this Question Paper at the end of the examination together with your answer pages in 3 bundles:-

Bundle A – Questions 1,2,3;
Bundle B – Questions 4,5;
Bundle C – Questions 6,7.

* This paper constitutes 40% of the school assessment, but does not necessarily reflect the format and content of the H.S.C.

Table of Standard Integrals

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\quad n \neq -1;\ x \neq 0,\ \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left\{\frac{x}{a}\right\}\quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left\{\frac{x}{a}\right\}\quad a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\quad |x| > |a|$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

Question 1 (14 marks)

(a) The polynomial $P(x) = x^3 + 4x^2 + x + k$. When $P(x)$ is divided by $(x + 2)$, the remainder is 5. Find the value of k. [2]

(b) Differentiate $\ln[3\sin x]$. [3]

(c) If $\log_A B = 2$, find $\log_B A^3$. [3]

(d) Find the perpendicular distance between the lines $x - y + 3 = 0$ and $x - y + 1 = 0$. [3]

(e) Using the substitution $u = \ln x$, find $\displaystyle\int \frac{1}{x\sqrt{4-(\ln x)^2}}\,dx$. [3]

Question 2 (15 marks)

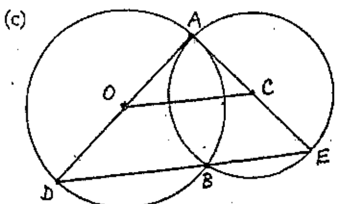
(a) The function $f(\theta) = \cos\theta - \theta$ has a root close to 1.

(i) Show that the root lies between 0.7 and 0.8. [2]

(ii) Taking $\theta = 0.7$ as an approximation for the solution to the equation $\cos\theta = \theta$, use one application of Newton's method to give a better approximation. [3]

(b) Show that $\displaystyle\int_0^3 \frac{dx}{9+x^2} = \frac{\pi}{12}$. [3]

(c)



Two circles, centres O and C, intersect at A and B. Diameter ECA is a tangent to the circle with centre O. AOD is a diameter.

(i) Show that D, B and E are collinear. [2]

(ii) Prove that triangle ADB is similar to triangle EAB. [3]

(iii) Show that $OC = \frac{1}{2}DE$. [2]

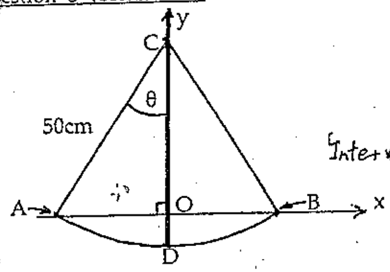
Question 3 (14 marks)

(a) The day before a test, the probability that Student A is absent is 0.7. The probability that Student B is absent is 0.2 and the probability that Student C is absent is 0.4. Find the probability that, on a day before a test, Student A is present but Students B and C are both absent. [2]

(b) Given that $\sin^{-1}[\cos x] = \theta$, where x, θ are acute, show that $\sin^{-1}[\cos x] = \cos^{-1}[\sin x]$. [4]

Question 3 (continued)

(c)



CA is a pendulum of length 50 cm oscillating uniformly about the vertical position, CD, so that the end of the pendulum describes the arc ADB and return in 4 seconds.

Interval AB = 60 cm and angle ACD = θ.

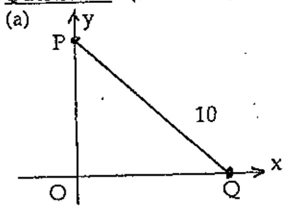
(i) Calculate the angle θ in radians. [1]

(ii) Derive an equation to describe the horizontal motion of the end of the pendulum between A and B, starting at position A. [4]

(iii) Calculate the area of the segment AOBD correct to two decimal places. [3]

Question 4 (14 marks)

(a)



PQ is a rod of fixed length 10 metres. Point P moves freely along the y-axis and point Q moves freely along the x-axis.

(i) Show that $\frac{dy}{dx} = -\frac{x}{\sqrt{100 - x^2}}$ [2]

(ii) Given that Q moves with constant velocity $\frac{dx}{dt} = 2$ m/s, determine the velocity of P when $x = 5$ metres. [4]

(b)(i) The function $f(x) = \cos x \, e^{\sin x}$. Complete this table of values for $y = f(x)$.

x	0	0.5	1	1.5	0.5π	2	2.5	3	π
f(x)									

Hence draw a sketch of $y = f(x)$ for $0 \le x \le \pi$. [4]

(ii) Calculate the area enclosed between the curve $y = \cos x \, e^{\sin x}$ and the x-axis from $x = 0$ to $x = \pi$. [4]

Question 5 (14 marks)

(a) In the time just before an election, the level of confidence in Candidate X is C, where C is expressed as a percentage. The rate of daily change in that confidence is given by $\frac{dC}{dt} = k(C + 0.1)$, where k is a constant.

(i) If C is initially 90%, show that $C = -0.1 + e^{kt}$. [3]

(ii) If C = 60% 10 days later, find the value of k. [1]

(iii) Determine the level of confidence in X at 20 days. [2]

(iv) How many days will it take for the level of confidence in X to reduce to 10% ? [2]

(b) (i) Sketch the curve $y = \cos^{-1}\left[\frac{x-2}{2}\right]$. [2]

(ii) Show that when this curve is rotated about the y-axis, the volume generated is $6\pi^2$ units^3. [4]

Question 6 (14 marks)

(a) \$A is borrowed at 12% per annum reducible interest, calculated monthly. The loan is repaid in equal monthly instalments of \$2400 over n months.

(i) Show that $A \times 1.01^n = 240000\,(1.01^n - 1)$ [3]

(ii) If $A = 200000$, calculate the period of the loan correct to the nearest month. [3]

(iii) Calculate the equivalent flat rate of interest per annum on the loan for the time calculated in part (ii). [2]

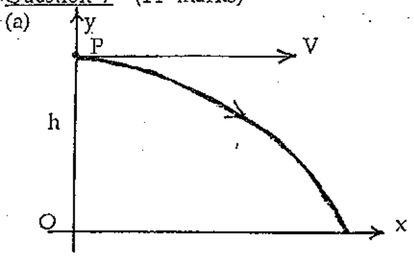
(b)(i) Show that $\frac{d}{d\theta}[\sec\theta + \tan\theta] = \sec\theta\tan\theta + \sec^2\theta$. [2]

(ii) Using this result, show that $\int \sec\theta \, d\theta = \ln[\sec\theta + \tan\theta] + c$. [2]

(iii) Using the substitution $x = \sec\theta$, evaluate $\int_2^3 \frac{dx}{\sqrt{x^2 - 1}}$ correct to two decimal places. [2]

Question 7 (14 marks)

(a)



A particle is projected horizontally from a point P , h metres above O , with a velocity of V metres/second . The equations of motion of the particle are

$$\ddot{x} = 0 \quad \text{and} \quad \ddot{y} = -10 .$$

(i) Show that the position of the particle at time t seconds is given by
$x = Vt$ and $y = -5t^2 + h$. [2]

A plane flying at a constant speed of 252 km/hr at a height of 145 metres above a horizontal stretch of land drops a package of supplies to a farm stranded by floods.

(ii) How far will the package travel horizontally before hitting the ground ? [3]

(iii) If the package must clear a line of trees 20 metres high at the perimeter of the farm, what is the maximum horizontal distance the plane must be from the trees when it drops the package ? [3]

(b)(i) If $f(x) = 1 - x + x^2 - x^3 + \ldots\ldots\ldots\ldots + (-1)^n x^n + \ldots\ldots\ldots$, where $0 < |x| < 1$, find a simpler expression for $f(x)$. [1]

(ii) Find $\int_0^u f(x)\,dx$ and derive a series expansion for $\ln(1+u)$, where $0 < u < 1$. [3]

(iii) Use the first four terms of this expansion to find an approximation for ln 1.6 . Do not use the calculator value of ln 1.6, as your error will be detected . [2]

τηε ενδ οφ τηισ εξαμινατιον εϖεν ιφ ιτ ισ νοτ τηε ενδ οφ ψου αδιευ

1)(a) $P(x) = x^3 + 4x^2 + x + k$
$P(-2) = -8 + 16 - 2 + k = 5$
$6 + k = 5$
$\underline{k = -1}$ [2]

(b) $\frac{d}{dx}\{\ln[3\sin x]\} = \frac{3\cos x}{3\sin x}$
$= \underline{\cot x}$ [3]

(c) $\log_A B = 2$, so $B = A^2$
$A = B^{1/2}$
$\therefore \log_B A^3 = \log_B B^{3/2}$ (14)
$= \underline{\frac{3}{2} \text{ or } 1.5}$ [3]

(d) On $x - y + 1 = 0$, $x = 0 \to y = 1$.
Distance from $(0, 1)$ to $x - y + 3 = 0$
is $d = \left|\frac{0 - 1 + 3}{\sqrt{1^2 + (-1)^2}}\right| = \frac{2}{\sqrt{2}}$
or
$\underline{\sqrt{2}}$ [3]

(e) $I = \int \frac{dx}{x\sqrt{4 - (\ln x)^2}}$, $u = \ln x$, $du = \frac{dx}{x}$
$\therefore I = \int \frac{du}{\sqrt{4 - u^2}}$
$= \sin^{-1}\left(\frac{u}{2}\right) + c$, c const.
$= \underline{\sin^{-1}\left(\frac{\ln x}{2}\right) + c.}$ [3]

2)(a) $f(\theta) = \cos\theta - \theta$
(i) $f(0.7) = 0.0648\ldots$
$f(0.8) = -0.1033\ldots$ [2]
$\therefore f(\theta)$ changes sign between 0.7 and 0.8.
$\therefore f(\theta) = 0$ for $0.7 < \theta < 0.8$.

(ii) $\theta_2 = \theta_1 - \frac{f(\theta_1)}{f'(\theta_1)}$ [3]
$= 0.7 - \frac{\cos 0.7 - 0.7}{-\sin 0.7 - 1}$
$= 0.7 - \frac{0.064842}{-1.64422}$
$= 0.7394$

(b) $\int_0^3 \frac{dx}{9 + x^2} = \int_0^3 \frac{dx}{3^2 + x^2}$
$= \left[\frac{1}{3}\tan^{-1}\frac{x}{3}\right]_0^3$
$= \left[\tan^{-1} 1 - \tan^{-1} 0\right] \times \frac{1}{3}$
$= \frac{\pi}{12}$

(c)(i) $\angle ABD = 90°$ ($\angle$ in semicircle)
$\angle ABE = 90°$ ($\angle$ in semicircle)
$\therefore \angle DBE = 180°$
$\therefore$ DBE is a sraight line. [2]
ie D, B, E are collinear.

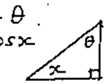
(ii) In Δ's ADB, EAB
$\angle ABD = \angle ABE = 90°$ (proven)
[illegible]

$\therefore \Delta$'s ADB, AEB are equiangular
ie $\Delta ADB \,|||\, AEB$ (AAA sim). [3]

(iii) In Δ's AOC, ADE,
$\angle DAE$ is common
$\frac{AO}{AD} = \frac{AC}{AE} = \frac{1}{2}$ [3]
(Radii = $\frac{1}{2}$ Diameters)
$\therefore \Delta AOC \,|||\, \Delta ADE$ (SAS sim)
$\therefore \frac{OC}{DE} = \frac{1}{2}$ (sides in propn)
$\therefore OC = \frac{1}{2}DE$

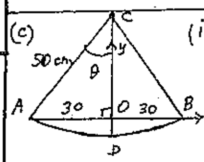
3)(a) $P(\bar{A}, B, C) = 0.3 \times 0.2 \times 0.4$
$= \underline{0.024}$ [2]

(b) $\sin^{-1}[\cos x] = \theta$
$\therefore \sin\theta = \cos x$
$\therefore \theta = \frac{\pi}{2} - x$



$\therefore \cos\theta = \sin x$
ie. $\cos^{-1}[\sin x] = \theta$. [4]
Hence $\sin^{-1}[\cos x] = \cos^{-1}[\sin x]$.

(c)

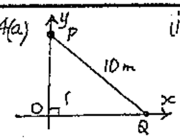


(i) $AO = \frac{1}{2}AB$ [1]
$= \underline{30 \text{ cm}}$
$\therefore \sin\theta = \frac{30}{50}$
$= 0.6$
$\therefore \underline{\theta = 0.6435}$

(ii) Since motion is simple harmonic, equation is of form $x = a\cos(nt + \varepsilon)$.
Amplitude, $a = 30$.
Period $\frac{2\pi}{n} = 4$ (14)
$\therefore n = \frac{\pi}{2}$.
$\therefore x = 30\cos\left(\frac{\pi}{2}t + \varepsilon\right)$
When $t = 0$, $x = -30$.
$\therefore -30 = 30\cos\varepsilon$
$\cos\varepsilon = -1$ [4]
$\varepsilon = \pi$
$\therefore x = 30\cos\left(\frac{\pi}{2}t + \pi\right)$
OR $\underline{x = -30\cos\frac{\pi}{2}t}$
Alternatively, using $x = a\sin(nt + \varepsilon)$
$x = 30\sin\left(\frac{\pi}{2}t + \varepsilon\right)$
$-30 = 30\sin\varepsilon$
$\sin\varepsilon = -1$
$\varepsilon = -\pi/2$
$\therefore \underline{x = 30\sin\frac{\pi}{2}(t - 1)}$.

3(c)(iii) Segment Area $= \frac{1}{2}r^2(2\theta) - \frac{1}{2}r^2\sin$ [illegible]
$= \frac{1}{2} \times 50^2(1.287 - \sin 1.28$ [illegible]
$= \underline{408.75 \text{ cm}^2}$. [illegible]

4(a)

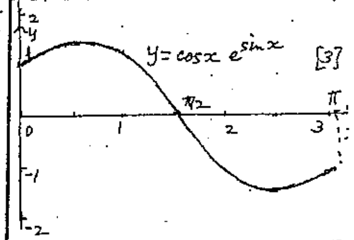


(i) $x^2 + y^2 = 100$
$\therefore y = \sqrt{100 - x^2}$
$\therefore \frac{dy}{dx} = \frac{1}{2}(100 - x^2)^{-\frac{1}{2}} \times -2x$
$\therefore \underline{\frac{dy}{dx} = -\frac{x}{\sqrt{100 - x^2}}}$ [illegible]

(ii) $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$
$= \frac{-x}{\sqrt{100 - x^2}} \times 2$ (14)
At $x = 5$, $\frac{dy}{dt} = \frac{-5}{\sqrt{100 - 25}} \times 2$ [4]
$= \frac{-10}{5\sqrt{3}}$
$= \underline{\frac{-2}{\sqrt{3}} \text{ or } \frac{-2\sqrt{3}}{3} \text{ m/s}}$.

(b) (i) $f(x) = \cos x \cdot e^{\sin x}$ [1]

x	0	0.5	1	1.5	0.5π	2	2.5	3	π
$f(x)$	1	1.42	1.25	·19	0	-1.03	-1.46	-1.44	[illegible]



[3]

(ii) Area $= \int_0^{\pi/2} \cos x\, e^{\sin x}\,dx + \left|\int_{\frac{\pi}{2}}^{\pi} \cos x\, e^{\sin x}\,dx\right|$
$= \left[e^{\sin x}\right]_0^{\pi/2} + \left|\left[e^{\sin x}\right]_{\pi/2}^{\pi}\right|$
$= (e - 1) + |1 - e|$
$= \underline{2e - 2 \text{ units}^2}$ [4]
$\doteqdot \underline{3.4366 \text{ units}^2}$

5. (a)

$c = -0.1 + e^{kt}$

$\frac{dc}{dt} = ke^{kt}$

$= k(-0.1 + e^{kt} + 0.1)$

$= k(c + 0.1)$

When $t = 0$

$c = -0.1 + 1$

$= 0.9$

$= 90\%$

(ii) When $t = 10$; $c = 0.6$.

$0.6 = -0.1 + e^{10k}$ [1]

$\therefore k = \frac{1}{10}\ln 0.7$

$\doteqdot -0.035667494$

(iii) When $t = 20$, $c = -0.1 + e^{2\ln 0.7}$ [2]

$\therefore \underline{c = 0.39}$

$\therefore$ Level of confidence is $\underline{39\%}$.

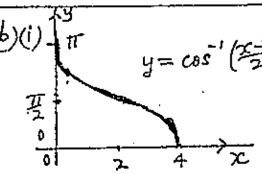
(iv) $c = 0.1$ $\therefore 0.1 = -0.1 + e^{\frac{1}{10}\ln 0.7\, t}$

$\therefore t = \frac{10 \ln 0.2}{\ln 0.7}$ [2]

$\underline{t = 45.123 \text{ days}}$.

$\therefore$ Level of confidence reaches 10% on the $\underline{46\text{th day}}$.

(b)(i)



$y = \cos^{-1}\left(\frac{x-2}{2}\right)$ [2]

(ii) Volume $= \int_0^{\pi} \pi x^2\, dy$.

Now $\frac{x-2}{2} = \cos y$

$x = 2\cos y + 2$

$\therefore V = \pi \int_0^{\pi} (4\cos^2 y + 4\cos y + 4)\, dy$

$V = \pi \int_0^{\pi} (2\cos 2y + 4\cos y + 6)\, dy$

$= \pi\left[\sin 2y + 4\sin y + 6y\right]_0^{\pi}$

$= \pi\{(0+0+6\pi) - (0+0+0)\}$

$= \underline{6\pi^2 \text{ units}^3}$.

6)(a) Let A_n be amount owing at end of n months.

(i) $A_1 = 1.01A - 2400$

$A_2 = 1.01A_1 - 2400$

$= 1.01^2 A - 2400(1.01 + 1)$ [3]

etc.

$A_n = 1.01^n A - 2400(1.01^{n-1} + \cdots + 1.01 + 1)$

$A_n = 0$.

$\therefore A \times 1.01^n = \frac{2400 \times 1 \times (1.01^n - 1)}{1.01 - 1}$

$\underline{A \times 1.01^n = 240000(1.01^n - 1)}$

(ii) $A = 200000$.

$\therefore 200000 \times 1.01^n = 240000(1.01^n - 1)$

$\therefore 40000 \times 1.01^n = 240000$

$1.01^n = 6$.

$n = \frac{\log 6}{\log 1.01}$ (14)

$= 180.07$

$\therefore$ Period of loan is 180 months ie $\underline{15 \text{ years}}$. [3]

(iii) Interest $= \$2400 \times 180 - \$200000$

$= \$232000$

Interest pa % $= \frac{232000}{15} \times \frac{100\%}{200000}$

$= \underline{7.73\%}$. [2]

(b)(i) $\frac{d}{d\theta}[\sec\theta + \tan\theta] = \frac{d}{d\theta}[(\cos\theta)^{-1} + \tan\theta]$

$= -(\cos\theta)^{-2}.(-\sin\theta) + \sec^2\theta$

$= \frac{\sin\theta}{\cos\theta} \cdot \frac{1}{\cos\theta} + \sec^2\theta$ [1]

$= \underline{\sec\theta\tan\theta + \sec^2\theta}$

(ii) $\int \sec\theta\, d\theta = \int \frac{\sec\theta(\sec\theta + \tan\theta)\, d\theta}{(\sec\theta + \tan\theta)}$

$= \int \frac{\sec\theta\tan\theta + \sec^2\theta}{\sec\theta + \tan\theta}\, d\theta$ [2]

$= \underline{\ln[\sec\theta + \tan\theta] + c}$, where c is a constant.

(iii) $I = \int_2^3 \frac{dx}{\sqrt{x^2 - 1}}$

Let $x = \sec\theta$, so $\frac{dx}{d\theta} = \sec\theta\tan\theta$

Then $I = \int_{\sec^{-1}2}^{\sec^{-1}3} \frac{\sec\theta\tan\theta\, d\theta}{\sqrt{\sec^2\theta - 1}}$

$= \int_{\sec^{-1}2}^{\sec^{-1}3} \frac{\sec\theta\tan\theta}{\tan\theta}\, d\theta$ [3]

$= \int_{\sec^{-1}2}^{\sec^{-1}3} \sec\theta\, d\theta$

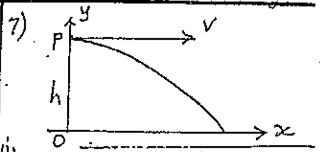
$= \left[\ln(\sec\theta + \tan\theta)\right]_{\sec^{-1}2}^{\sec^{-1}3}$

$= \ln(3 + 2\sqrt{2}) - \ln(2 + \sqrt{3})$

$\doteqdot 1.7627 - 1.3170$

$\doteqdot \underline{0.45}$ (to 2 dec pl).

7)



(i) $\ddot{x} = 0$ $\quad$ $\ddot{y} = -10$

$\dot{x} = V$ $\quad$ $\dot{y} = -10t$

$x = Vt + c_1$ $\quad$ $y = -5t^2 + c_2$

$t = 0 \rightarrow x = 0$, $\quad$ $y = h$

$\therefore c_1 = 0$, $\quad$ $c_2 = h$. [

$\therefore \underline{x = Vt}$ $\quad$ $\underline{y = -5t^2 + h}$

(ii) $V = 252\text{ km/h} = 252 \div 3.6 = 70\text{ m/}$

$h = 145\text{ m}$.

At ground, $y = 0$.

$\therefore 0 = -5t^2 + 145$ (14

$5t^2 = 145$

$t^2 = 29$

$t = 5.385$ sec.

$\therefore x = 70 \times 5.385$ [3

$= 376.96$ m.

$\therefore$ Package takes 376.96 m or approximately $\underline{377\text{ m}}$.

(iii) When $y = 20$, $20 = -5t^2 + 145$

$5t^2 = 125$

$t = 5$ sec.

$\therefore x = 70 \times 5$ [3

$= \underline{350\text{ m}}$.

$\therefore$ Plane must be no more than 350 m horizontally fr the trees.

(b)(i) $f(x) = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots$

Since $0 < |x| < 1$, this is a GP with a limiting sum.

$\therefore \underline{f(x) = \frac{1}{1+x}}$ [1

(ii) $\int_0^u f(x)\, dx = \int_0^u \frac{1}{1+x}\, dx$

$= [\ln(1+x)]_0^u$ [3

$= \underline{\ln(1+u)}$

But $\int_0^u f(x)\, dx = \int_0^u (1 - x + x^2 - x^3 + \cdots)\, d$

$= \left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^n x^{n+1}}{n+1} + \cdots\right.$

$= u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \cdots + \frac{(-1)^n u^{n+1}}{n+1} + \cdots$

$\therefore \ln(1+u) = u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \cdots + \frac{(-1)^n u^n}{n+}$ $+ \cdots$

(iii) $\ln 1.6 \doteqdot 0.6 - \frac{0.6^2}{2} + \frac{0.6^3}{3} - \frac{0.6^4}{4}$

$= \underline{0.4596}$. [2]

Total = 100 marks