Mathematics Extension 1

YEAR 12 TRIAL EXAMINATION
2008

Time allowed : Reading time – 5 minutes
Working time - 2 hours

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- Write using blue or black pen.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work
- Board approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.
- **Each question is to be started on a new page** and you are to write your name and teacher's name on each page.
- The marks allocated for each question are indicated

Name : ________________________ Class Teacher : ____________________

Q	1	2	3	4	5	6	7	Total
Mark	/12	/12	/12	/12	/12	/12	/12	/84

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Total marks – 84
Attempt Questions 1 – 7.
All questions are of equal value.

Start each question on a new page. Extra writing paper is available.

Question 1 (12 marks) Start a new page **Marks**

(a) Find $\int \frac{1}{x^2+3}\,dx$ **2**

(b) Ten people sit around a round table. How many arrangements are possible if three particular people want to sit together? **2**

(c) Find to the nearest degree the size of the acute angle between the lines **2**

$$x + 2y - 1 = 0$$
$$3x - 2y + 4 = 0$$

(d) If $y = \frac{3x+5}{x-4}$ **2**

Express the inverse function as a function of x

(e) For the points A (-5, 2) and B(2, 0)

(i) Write down the coordinates of P, the point that divides AB internally in the ratio k:1 **2**

(ii) If P lies on $xy=1$, show that $k^2-2k+11=0$ **2**

Question 2 (12 marks) Start a new page **Marks**

(a) Given that a root of the equation $e^{x}+e^{-x}-3=0$ is close to 1, use one application of Newton's Method of Approximation to find a second approximation to this root (correct to 2 decimal places) **2**

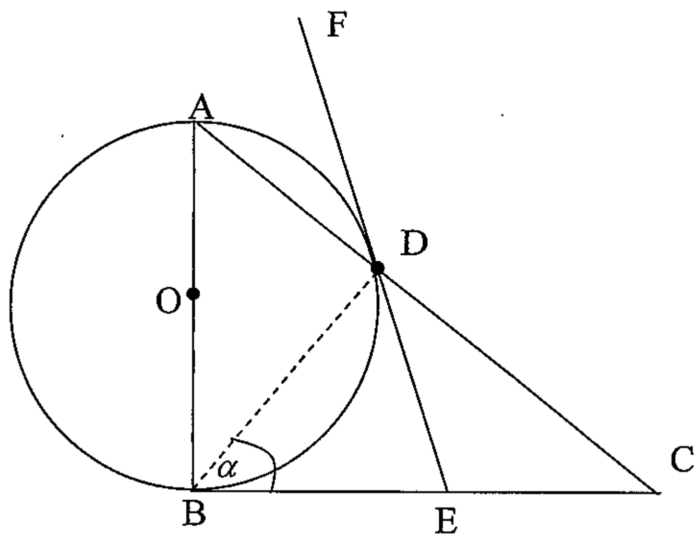
(b) Prove by Mathematical induction that $2^{2n+1}+1$ is divisible by 3 for all non-negative integral values of n. **3**

(c) (i) Express $\sin x-\cos x$ in the form $A\sin(x-\alpha)$, with A > O and $0<\alpha<\frac{\pi}{2}$ **2**

(ii) Hence or otherwise determine $\lim_{x\to\frac{\pi}{4}}\left(\frac{\sin x-\cos x}{x-\frac{\pi}{4}}\right)$ **1**

(d)

Not to scale



In the diagram, AB is a diameter of the circle, centre O, and BC is tangential to the circle at B. The line ADC intersects the circle at D. The tangent to the circle at D intersects BC at E. Let $\angle EBD=\alpha$

(i) Copy the diagram onto your page

(ii) Prove that $\angle EDC=\frac{\pi}{2}-\alpha$ **4**

Question 3 (12 marks) Start a new page **Marks**

(a) Consider the function $f(x) = \cos^{-1} x$

(i) Sketch the graph of $y = f(x)$, stating clearly its range and domain. **3**

(ii) Find the volume of the solid formed by rotating the arc of the curve $y = \cos^{-1} x$ that is in the positive quadrant about the y axis. **2**

(b) Evaluate $\int_0^{\frac{\pi}{2}} [(3\sin x) - 1]^2 \cos x \, dx$ using the substitution $u = (3\sin x) - 1$ **3**

(c) (i) If $t = \tan\frac{\theta}{2}$, write down expressions for $\sin\theta$ and $\cos\theta$ in terms of t **1**

(ii) Hence or otherwise solve the equation **3**

$\sqrt{3}\sin\theta = 1 + \cos\theta$ in the domain $0 \le \theta \le 2\pi$

Question 4 (12 marks) Start a new page **Marks**

(a)

(i) Show that the derivative of $x\tan x - \ln(\cos x)^{-1}$ is $x\sec^2 x$ **2**

(ii) Hence or otherwise evaluate $\int_0^{\frac{\pi}{4}} x\sec^2 x\,dx$ **2**

(b) A team of 6 girls is to be chosen from 10 girls.

(i) Find the number of ways that two particular girls, A and B, are **both included.** **1**

(ii) Find the number of ways that A and B are **both excluded.** **1**

(iii) Find the probability that either A is included or B is included, but A and B are not included together. **2**

(c) A particle is released from rest at the origin on a straight line, when x metres from the origin, its acceleration is given by

$$\frac{18}{(x-4)^2}\ m/s^2,\ \text{for } x < 4.$$

(i) In which direction will the particle first move? **1**

(ii) Find the particle's velocity when it reaches $x = 2$ **3**

Question 5 (12 marks) Start a new page **Marks**

(a) A particle moves along in a straight line such that its displacement x metres from an origin O at time t seconds is given by:

$$x = 4\sin\frac{\pi}{2}t$$

(i) Show that this motion is simple harmonic motion. **3**

(ii) State the amplitude and the period of this motion. **2**

(iii) Calculate the maximum speed attained by the particle. **3**

(b)

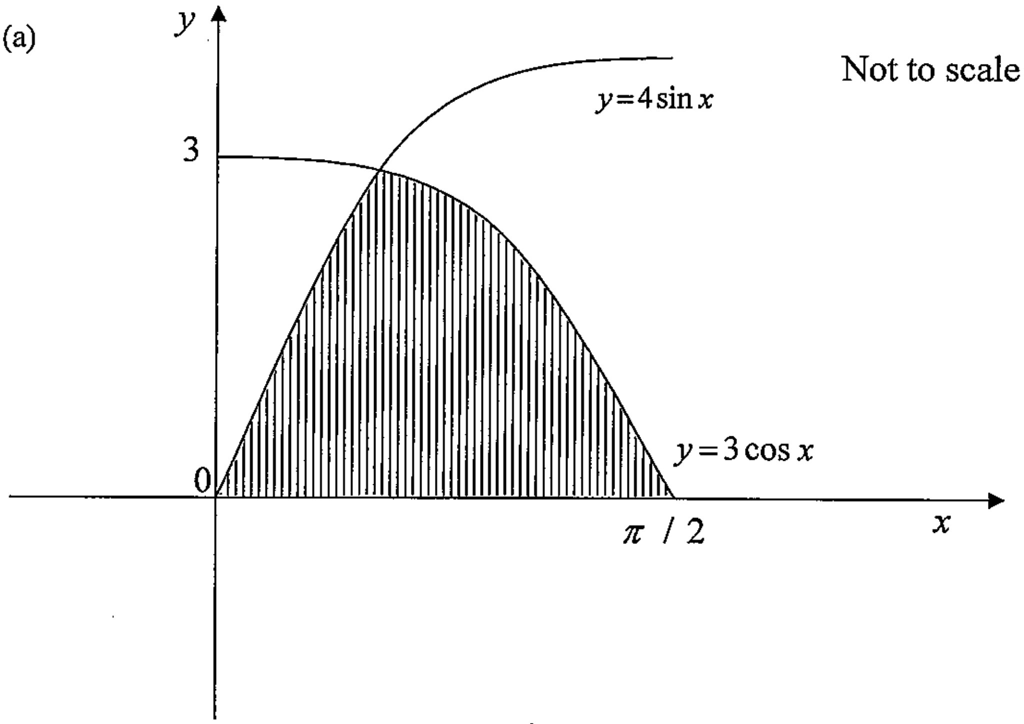
Use long division to divide the polynomial $f(x) = x^4 - x^3 + x^2 - x + 1$ by the polynomial $g(x) = x^2 - 3$. **2**

Express your answer in the form $f(x) = g(x).q(x) + r(x)$

(c) Differentiate $\cos^{-1}(e^{-x})$ with respect to x, putting your answer in simplest form. **2**

Question 6 (12 marks) Start a new page **Marks**

(a)



The diagram shows the graphs of $y = 4\sin x$ and $y = 3\cos x$. Show that the area of the shaded region in the diagram is 2 *units*^{2}. **3**

(b) A particle projected from ground level at an angle θ to the horizontal has its position at time t given by the coordinates

$$x = Vt\cos\theta \ , \ y = Vt\sin\theta - \frac{1}{2}gt^2$$ **(DO NOT PROVE THESE)**

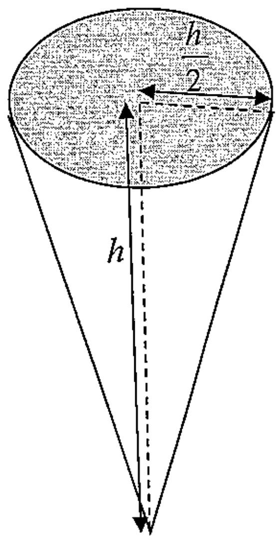
(i) Using these equations, find the maximum height reached in terms of V, g and θ. **2**

(ii) What is the speed of the object at its maximum height? **1**

Question 6 continues on page 9

Marks

(c)



A cone has a depth of h and radius of $\frac{h}{2}$, as shown.

(i) Show that the volume of the cone is given by $V = \frac{1}{12}\pi h^3$ **1**

(ii) Water is poured in at a rate of $10\,mm^3/s$. Find the rate at which the depth, h *mm*, is increasing when the depth of water in the cone is 50 *mm*. **2**

The cone is filled to a depth of 100*mm* and pouring then stops. A hole is then opened at the vertex of the cone and water flows out at a rate of $\pi h^2\ mm^3/s$

(iii) Find $\frac{dh}{dt}$ in *mm/s* **2**

(iv) Hence find how long it takes to empty the cone. **1**

End of Question 6

Marks

Question 7 (12 marks) Start a new page

(a) The mass, M, of a radioactive element decreases at a rate proportional to the mass,

ie. $\frac{dM}{dt} = -kM$ where k is a constant.

(i) Show that the function $M = M_0 e^{-kt}$, where M_0 is the initial mass, provides such a rate. **1**

(ii) The "half-life" period of an element is the time taken for any given mass to be reduced by half. If the half-life period is T, prove that $k = \frac{\ln 2}{T}$ **3**

Question 7 continues on page 11

(b) **Marks**

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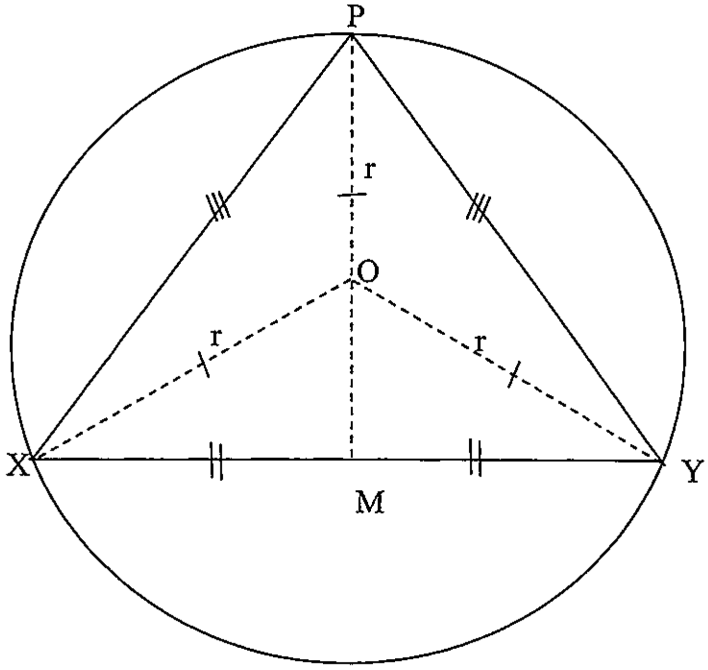


In the diagram above, O is the centre of a circle of constant radius r. A variable chord XY subtends an angle 2θ at the centre O. Let P be the point on the major arc XY so that ΔXPY is isosceles with $XP = YP$. Let PO produced meet XY at M so that PM is the perpendicular bisector of the chord XY.

(i) Prove that the area, A, of ΔXPY is given by: **3**

$$A = r^2 \sin\theta\,(1 + \cos\theta)$$

(ii) Show that $\dfrac{dA}{d\theta} = r^2(2\cos^2\theta + \cos\theta - 1)$ **2**

(iii) Show that ΔXPY has maximum area when it is an equilateral triangle. You may assume it is a maximum – **you are NOT required to test** $\dfrac{d^2A}{d\theta^2}$ **3**

End of paper

Standard Integrals

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Question 1

(a) $\int \frac{1}{x^2+3}\,dx = \int \frac{dx}{x^2+(\sqrt{3})^2}$

$= \frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} + c$

(b) (x x x) $7! \times 3! = 30240$

(c) $x + 2y - 1 = 0 : m_1 = -\frac{1}{2}$

$3x - 2y + 4 = 0 : m_2 = \frac{3}{2}$

$\tan\theta = \left| \frac{-\frac{1}{2} - \frac{3}{2}}{1 + -\frac{1}{2} \times \frac{3}{2}} \right|$

$= \frac{2}{1/4} = 8$

$\theta = 82.87\ldots$

$\approx 83°$

(d) $f : y = \frac{3x+5}{x-4}$

$f^{-1} : x = \frac{3y+5}{y-4}$

$xy - 4x = 3y + 5$

$-4x - 5 = 3y - xy$

$-4x - 5 = y(3 - x)$

$\therefore y = \frac{4x+5}{x-3}$

ie $f^{-1}(x) = \frac{4x+5}{x-3}$

(e) $A(-5, 2)$ $B(2, 0)$

$k : 1$

(i) $P = \left(\frac{2k-5}{k+1}, \frac{2}{k+1} \right)$

(ii) $xy = 1 \Rightarrow \frac{2k-5}{k+1} \times \frac{2}{k+1} = 1$

$4k - 10 = (k+1)^2$

$= k^2 + 2k + 1$

$\therefore 0 = k^2 - 2k + 11$

Question 2

(a) $y = e^x + e^{-x} - 3$

$y' = e^x - e^{-x}$

$a_1 = 1 - \frac{e^1 + e^{-1} - 3}{e^1 - e^{-1}}$

$= 0.9633\ldots$

≈ 0.96

(b) $2^{2n+1} + 1$

check $n=1$: $2^{2+1} + 1 = 9$, which is divisible by 3

assume $2^{2k+1} + 1 = 3P$, where P is an integer

show $2^{2(k+1)+1} + 1$ is divisible by 3.

proof $2^{2(k+1)+1} + 1 = 2^{2k+3} + 1$

$= 2^{2k+1+2} + 1$

$= 4 \cdot 2^{2k+1} + 1$

$= 4(3P - 1) + 1$, by assumption.

$= 12P - 4 + 1$

$= 12P - 3$

$= 3(4P - 1)$

$= 3Q$

ie $2^{2(k+1)+1} + 1$ is divisible by 3,

$\therefore 2^{2n+1} + 1$ is divisible by 3 by mathematical induction

(c) (i) Let $\sin x - \cos x = A\sin(x - \alpha)$

$= A\sin x\cos\alpha - A\cos x\sin\alpha$

$\therefore 1 = A\cos\alpha$ and $1 = A\sin\alpha$

$\therefore 1 = \tan\alpha$

$\alpha = \pi/4$

$\therefore A = 1 \div \sin\frac{\pi}{4}$

$= \sqrt{2}$

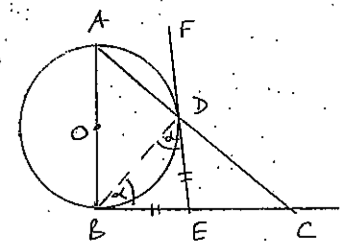
ie $\sin x - \cos x = \sqrt{2}\sin\left(x - \frac{\pi}{4}\right)$

(ii) $\lim_{x \to \frac{\pi}{4}} \frac{\sin x - \cos x}{x - \frac{\pi}{4}}$

$= \lim_{x \to \frac{\pi}{4}} \frac{\sqrt{2}\sin\left(x - \frac{\pi}{4}\right)}{\left(x - \frac{\pi}{4}\right)}$

$= \sqrt{2}.$

(d) (i)



(ii) $BE = ED$ (equal tangents to ⊙ from E)

$\therefore B\hat{D}E = \alpha$.

$B\hat{D}A = \frac{\pi}{2}$ (∠ in semi ⊙)

$\therefore B\hat{D}C = \frac{\pi}{2}$ (adj supp ∠s)

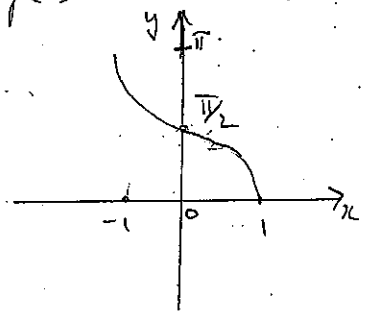
$\therefore E\hat{D}C = \frac{\pi}{2} - \alpha$ (adj compl. ∠s)

$\therefore$

Question 3.

a) $f(x) = \cos^{-1} x$

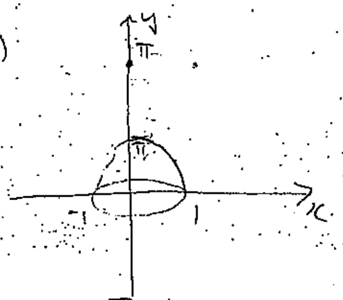
(i)



D : $-1 \le x \le 1$

R : $0 \le y \le \pi$.

(ii)



$V = \pi \int_0^{\pi/2} x^2 \, dy$

$= \pi \int_0^{\pi/2} \cos^2 y \, dy$

$= \pi \int_0^{\pi/2} \frac{1}{2}(\cos 2y + 1) \, dy$

$= \frac{\pi}{2} \left[\frac{\sin 2y}{2} + y \right]_0^{\pi/2}$

$= \frac{\pi}{2} \left[\frac{\pi}{2} \right]$

$= \frac{\pi^2}{4}$ units^3

(b) Let $u = 3\sin x - 1$

$\frac{du}{dx} = 3\cos x$

$du = 3\cos x \cdot dx$

when $\begin{cases} x = 0, & u = -1 \\ x = \pi/2, & u = 2 \end{cases}$

$\int_0^{\pi/2} (3\sin x - 1)^2 \cos x \, dx = \frac{1}{3} \int_{-1}^{2} u^2 \, du$

$= \frac{1}{9} \left[u^3 \right]_{-1}^{2}$

$= \frac{1}{9} [8 + 1]$

$= 1$

(c) (i) $\sin\theta = \frac{2t}{1+t^2}$

$\cos\theta = \frac{1-t^2}{1+t^2}$

(ii)

$\sqrt{3} \sin\theta = 1 + \cos\theta$ — ①

$\sqrt{3} \times \frac{2t}{1+t^2} = 1 + \frac{1-t^2}{1+t^2}$

$\frac{2\sqrt{3}t}{1+t^2} = \frac{1+t^2+1-t^2}{1+t^2}$

$\therefore 2\sqrt{3}t = 2$

$t = \frac{1}{\sqrt{3}}$

$\therefore \tan\frac{\theta}{2} = \frac{1}{\sqrt{3}}$ $\quad 0 \le \frac{\theta}{2} \le \pi$

$\therefore \frac{\theta}{2} = \frac{\pi}{6}$

$\therefore \theta = \frac{\pi}{3}$

test $\theta = \pi$:

LHS of ① $= \sqrt{3}\sin\pi = 0$

RHS of ① $= 1 + \cos\pi = 0$

$\therefore$ Solution are

$\theta = \frac{\pi}{3}$ or π.

Question 4

a) (i)

$\frac{d}{dx}\left[x\tan x - \ln(\cos x)^{-1}\right] = \frac{d}{dx}\left(x\tan x + \ln\cos x\right)$

$= \tan x + x\sec^2 x + \frac{1}{\cos x} \times (-\sin x)$

$= \tan x + x\sec^2 x - \tan x$

$= 1.x\sec^2 x$

(ii) $\int_0^{\pi/4} x\sec^2 x\,dx$

$= \left[x\tan x - \ln\sec x\right]_0^{\pi/4}$

$= \frac{\pi}{4}\tan\frac{\pi}{4} - \ln\left(\frac{1}{\cos\frac{\pi}{4}}\right)$

$- 0 + \ln\left(\frac{1}{\cos 0}\right)$

$= \frac{\pi}{4} - \ln(\sqrt{2})$

(b) (i) $^8C_4 = 70$

(ii) $^8C_6 = 28$

(iii) $\frac{^{10}C_6 - 70 - 28}{^{10}C_6}$ 160

$= \frac{112}{210}$

$= \frac{8}{15}$

(c) $\ddot{x} = \frac{18}{(x-4)^2}$ ms^{-2}

(i) to the right

(ii) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{18}{(x-4)^2}$

$\frac{1}{2}v^2 = \frac{18(x-4)^{-1}}{-1} + c$

when $x=0$, $v=0$

$\therefore 0 = \frac{-18}{-4} + c$

$c = -\frac{18}{4}$

$\therefore \frac{1}{2}v^2 = \frac{-18}{x-4} - \frac{18}{4}$

when $x = 2$

$\frac{1}{2}v^2 = \frac{-18}{-2} - \frac{18}{4}$

$= 4\frac{1}{2}$

$\therefore v = 3$ ms^{-1}

Question 5

$x = 4\sin\frac{\pi}{2}t$

(i) $\dot{x} = \frac{4\pi}{2}\cos\frac{\pi}{2}t$

$\ddot{x} = -\frac{4\pi^2}{4}\sin\frac{\pi}{2}t$

$= -\frac{\pi^2}{4}x$

$\Rightarrow$ SHM, $n = \frac{\pi}{2}$

(ii) amplitude $= 4$

period $= \frac{2\pi}{\pi/2}$

$= 4$

(iii) max speed is when $x = 0$

$x = 0$

$0 = 4\sin\frac{\pi}{2}t$

$\therefore \frac{\pi}{2}t = \pi$

$t = 2$

$\dot{x} = \frac{4\pi}{2}\cos\pi$

$= -2\pi\ \text{m s}^{-1}$

$\therefore$ max speed $= 2\pi\ \text{m s}^{-1}$

(b) (i)

$$\begin{array}{r} x^2 - x + 4 \\ x^2+0x-3 \overline{\smash{)}\, x^4 - x^3 + x^2 - x + 1} \\ x^4 + 0x^3 - 3x^2 \\ \hline -x^3 + 4x^2 - x \\ -x^3 + 0x^2 + 3x \\ \hline 4x^2 - 4x + 1 \\ 4x^2 + 0x - 12 \\ \hline -4x + 13 \end{array}$$

$x^4 - x^3 + x^2 - x + 1 = (x^2-3)(x^2-x+4) - 4x + 13$

(c) $\frac{d}{dx}\cos^{-1}(e^{-x})$

$= \frac{-1}{\sqrt{1-e^{-2x}}} \times (-1)(e^{x})^{-2} \times e^{x}$

$= \frac{e^{x}}{e^{2x}\sqrt{1-e^{-2x}}}$

$= \frac{1}{e^{x}\sqrt{1-e^{-2x}}}$

$= \frac{1}{e^{x}\sqrt{\frac{e^{2x}-1}{e^{2x}}}}$

$= \frac{1}{e^{x}\frac{\sqrt{e^{2x}-1}}{e^{2x}}}$

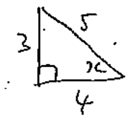
$= \frac{1}{\sqrt{e^{2x}-1}}$

Question 6

a) At Pnt of int:

$4\sin x = 3\cos x$

$\tan x = \frac{3}{4}$



Let $x = \theta$ at pnt of intersection

$A = \int_0^{\theta} 4\sin x\,dx + \int_{\theta}^{\pi/2} 3\cos x\,dx$

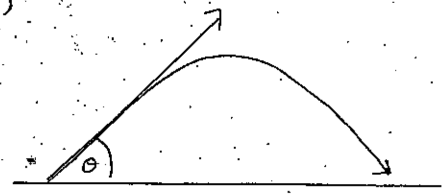
$= 4[\cos x]_0^{\theta} + 3[\sin x]_{\theta}^{\pi/2}$

$= -4[\cos\theta - \cos 0] + 3[\sin\frac{\pi}{2} - \sin\theta]$

$= -4[\frac{4}{5} - 1] + 3[1 - \frac{3}{5}]$

$= 2\ u^2$

(b)



$x = Vt\cos\theta$

$y = Vt\sin\theta - \frac{1}{2}gt^2$

(i) Max height when $\dot{y} = 0$

$\dot{y} = V\sin\theta - gt = 0$

when $t = \frac{V\sin\theta}{g}$

then $y = V\sin\theta \times \frac{V\sin\theta}{g} - \frac{g}{2} \times \frac{V^2\sin^2\theta}{g^2}$

$= \frac{V^2\sin^2\theta}{2g}$

(ii) $\dot{x} = V\cos\theta$ at max. height

(c)

i) $V = \frac{1}{3}\pi R^2 H$

$= \frac{1}{3}\pi\left(\frac{H}{2}\right)^2 \times H$

$= \frac{1}{12}\pi H^3$

ii) $\frac{dV}{dt} = 10$

$\frac{dh}{dt} = ?$ when $h = 50$.

$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$

$\frac{dV}{dh} = \frac{3}{12}\pi h^2$

$= \frac{1}{4}\pi h^2$

$\frac{dh}{dt} = \frac{4}{\pi h^2} \times 10$

$= \frac{40}{\pi \times 2500}$ when $h = 50$

$= \frac{2}{125\pi}$

iii) $\frac{dV}{dt} = -\pi h^2$

$\frac{dh}{dt} = \frac{4}{\pi h^2} \times (-\pi h^2)$

$= -4$

iv) When $h = 0$, $t = ?$

$h = 100$ when $t = 0$

$\frac{dt}{dh} = -\frac{1}{4}$

$t = -\frac{1}{4}h + c$

Sub $t = 0$, $h = 100$

$0 = -25 + c$

$c = 25$

$\therefore t = -\frac{1}{4}h + 25$

$= 25$ when [illegible]

→ alternative:

If $\frac{dh}{dt} = -4$ mm/s

To lower by 100 mm at 4 mm/s takes 25 s

Question 7

a) $\frac{dM}{dt} = -kM$

i) $M = M_0 e^{-kt}$

$\frac{dM}{dt} = -kM_0 e^{-kt}$

$= -kM$

(ii) when $t = T$

$M = \frac{1}{2}M_0$

$\therefore \frac{1}{2}M_0 = M_0 e^{-kT}$

$\frac{1}{2} = e^{-kT}$

$\ln\frac{1}{2} = -kT$

$\therefore k = \frac{-\ln\left(\frac{1}{2}\right)}{T}$

$= \frac{\ln 2}{T}$

b)

(i) $\hat{P} = \theta$ ($\angle$ at circumf = $\frac{1}{2} \times \angle$ at centre)

and $X\hat{P}O = \frac{\theta}{2}$.

$\therefore P\hat{O}X = 180 - \theta$ (isos $\triangle XPO$, $\angle$ sum of $\triangle = 180$)

$\therefore A$ = area $\triangle XOY$ + 2 × area $\triangle XPO$

$= \frac{1}{2} r^2 \sin 2\theta + 2 \times \frac{1}{2} r^2 \sin(180 - \theta)$

$= \frac{1}{2} r^2 \times 2 \sin\theta \cos\theta + r^2 \sin\theta$

$= r^2 \sin\theta(\cos\theta + 1)$

(ii) $A = r^2 \sin\theta(1 + \cos\theta)$

$\frac{dA}{d\theta} = r^2 [\cos\theta(1 + \cos\theta) + \sin\theta(-\sin\theta)]$

$= r^2 [\cos\theta + \cos^2\theta - \sin^2\theta]$

$= r^2 [\cos\theta + \cos^2\theta - (1 - \cos^2\theta)]$

$= r^2 [2\cos^2\theta + \cos\theta - 1]$

(iii) For max area $\frac{dA}{d\theta} = 0$

$\therefore 2\cos^2\theta + \cos\theta - 1 = 0$

$(2\cos\theta - 1)(\cos\theta + 1) = 0$

$\therefore \cos\theta = \frac{1}{2}$ ✓ or $\cos\theta = -1$ ✗

$\therefore \theta = 60^\circ$

ie $X\hat{P}Y = 60^\circ$

Since $\triangle XPY$ is isosceles, it must be equilateral.