Cheltenham Girls' High School

2020 TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen
- Approved calculators may be used
- A reference sheet is provided at the back of this paper
- In Questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks: 70

Section I – 10 marks (pages 2 – 7)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 8 – 13)

- Attempt Questions 11 – 14
- Allow about 1 hour and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 10.

1. What is the value of $\sin 2x$ given that $\sin x = \frac{2\sqrt{3}}{4}$ and x is obtuse?

A. $\frac{\sqrt{3}}{4}$

B. $\frac{\sqrt{3}}{2}$

C. $-\frac{\sqrt{3}}{4}$

D. $-\frac{\sqrt{3}}{2}$

2. Find the vector projection of $\underset{\sim}{p} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$ onto $\underset{\sim}{q} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

A. $\begin{pmatrix} -2 \\ 4 \end{pmatrix}$

B. $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$

C. $\begin{pmatrix} -2\sqrt{5} \\ 4\sqrt{5} \end{pmatrix}$

D. $\begin{pmatrix} 2\sqrt{5} \\ -4\sqrt{5} \end{pmatrix}$

3. Write the expression $y = \sin x + \sqrt{3} \cos x$ in the form $R \sin(x + \alpha)$.

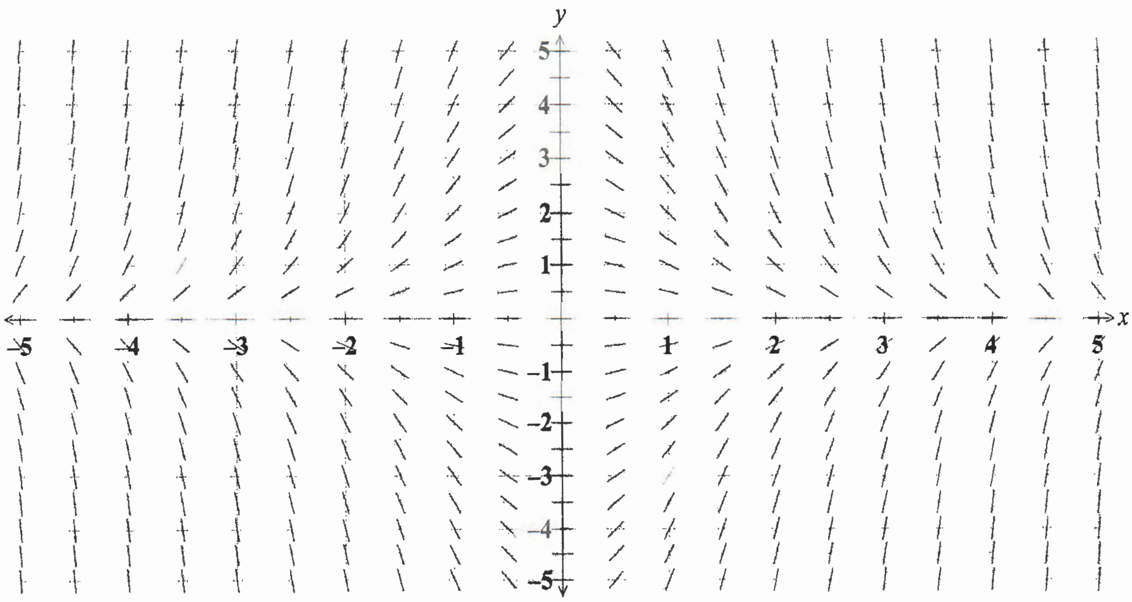
A. $2 \sin\left(x + \frac{\pi}{4}\right)$

B. $\sin\left(x + \frac{\pi}{3}\right)$

C. $2 \sin\left(x + \frac{\pi}{3}\right)$

D. $2 \sin\left(x - \frac{\pi}{3}\right)$

4. Which of the following differential equations could be represented by the slope field diagram below?



A. $y' = -xy$

B. $y' = xy$

C. $y' = -x^2y$

D. $y' = x^2y$

5. Given that $f(x) = e^x - 1$ and $y = f^{-1}(x)$, find an expression for $\dfrac{dy}{dx}$.

A. $\dfrac{1}{e^x - 1}$

B. $\dfrac{1}{x+1}$

C. $\ln x$

D. $\ln(x+1)$

6. If $y = \sin^{-1}\dfrac{a}{x}$ then $\dfrac{dy}{dx}$ equals:

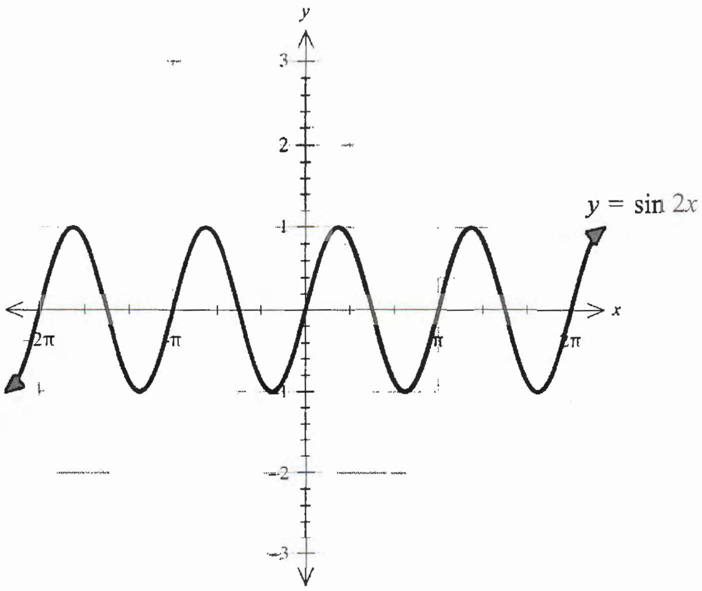
A. $\dfrac{-a}{x^2\sqrt{x^2 - a^2}}$

B. $\dfrac{x}{\sqrt{x^2 - a^2}}$

C. $\dfrac{-x}{\sqrt{x^2 - a^2}}$

D. $\dfrac{-a}{x\sqrt{x^2 - a^2}}$

7. The function $y = \sin 2x$ is shown in the diagram.



If this function is transformed using steps I, II and III as below:

I. Reflected about the $x-$ axis
II. Vertically translated 1 unit down
III. Dilated horizontally by a scale factor of 2.

Which equation would represent the transformed function?

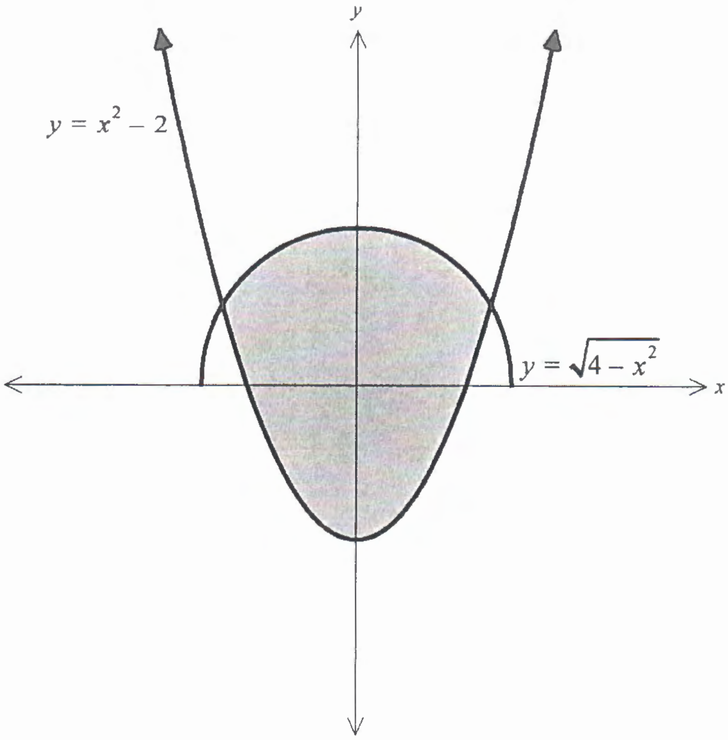
A. $y = -(\sin x + 1)$

B. $x = 2(\sin(2y) - 1)$

C. $y = -(\sin(4x)) + 2$

D. $y = -2\sin(2x) - 1$

8. The area enclosed by the curves $y = \sqrt{4 - x^2}$ and $y = x^2 - 2$ is shaded in the diagram below.



Which expression could be used to calculate this area?

A. $2\int_0^1 \left((4 - x^2)^{\frac{1}{2}} - x^2 + 2\right) dx$

B. $\int_2^{-2} \left((4 - x^2)^{\frac{1}{2}} - x^2 - 2\right) dx$

C. $2\int_0^{\sqrt{3}} \left((4 - x^2)^{\frac{1}{2}} - x^2 + 2\right) dx$

D. $\int_{-\sqrt{3}}^{\sqrt{3}} \left((4 - x^2)^{\frac{1}{2}} - x^2 - 2\right) dx$

9. A body of still water has suffered an oil spill and a circular oil slick is floating on the surface of the water.

The area of the oil slick is increasing by $0.1\ \text{m}^2/\text{minute}$.

At what rate is the radius increasing when the area is $0.3\ \text{m}^2$?

A. 0.0098 m/min

B. 0.03 m/min

C. 0.0515 m/min

D. 0.0531 m/min

10. A four-digit security code is to be created for a building alarm, using any selection of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

The code must be entered in the correct order to disarm the alarm when entering the building.

Digits ***may*** be repeated.

It has been decided that the code will contain exactly two different digits, for example 4224 or 7177.

If an intruder, who knew about this restriction, tried to guess the alarm code, what is the probability they would get it correct?

A. $\dfrac{1}{10\,000}$

B. $\dfrac{1}{5\,040}$

C. $\dfrac{1}{2\,100}$

D. $\dfrac{1}{630}$

Section II

60 marks

Attempt Questions 11 – 14.

Allow about 1 hour and 45 minutes for this section.

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11 – 14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the writing booklet.

(a) Find $\frac{d}{dx}(e^x \tan^{-1} x)$ **2**

(b) i) Write an expression for $\sin 5x \sin x$ in terms of $\cos 4x$ and $\cos 6x$. **2**

ii) Hence, find $\int_0^{\frac{\pi}{4}} \sin 5x \ \sin x \ dx$. **2**

(c) Evaluate $\int_{-8}^{0} \frac{x}{\sqrt{1-x}} dx$ using the substitution $u = 1 - x$. **3**

(d) The polynomial $x^4 + ax^3 - 3x^2 + bx - 2$ has roots -1 and 2, one of which is a triple root. **2**

Find the values of a and b.

Question 11 continues on page 9

Question 11 continued

(e) A particle is moving such that, at time, t seconds, its displacement, x metres, satisfies the equation $t = 2 - \dfrac{1}{e^{3x}}$.

i) Show that the velocity of the particle can be represented by **2**

$$\frac{dx}{dt} = \frac{e^{3x}}{3}\,\mathrm{ms}^{-1}$$

ii) Using the formula $a = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$, or otherwise, find the acceleration of the particle after 1 second. **2**

End of Question 11

Question 12 (15 marks) Use a NEW writing booklet.

(a) Consider vectors $\underset{\sim}{a} = 4\underset{\sim}{i} - 5\underset{\sim}{j}$ and $\underset{\sim}{b} = -2\underset{\sim}{i} + 4\underset{\sim}{j}$.

i) Find the magnitude and direction of $\underset{\sim}{a} + \underset{\sim}{b}$. **2**

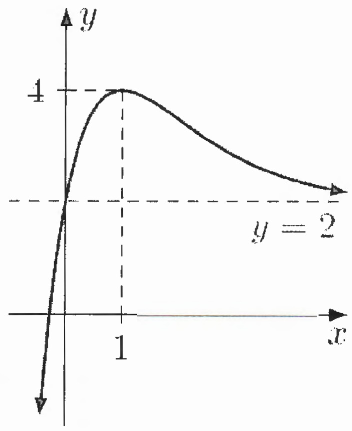
ii) Find the resultant vector of $2\underset{\sim}{b} - \underset{\sim}{a}$. **1**

iii) Calculate the dot product $\underset{\sim}{a} \cdot \underset{\sim}{b}$ **1**

iv) Find the angle between the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$. **2**

(b) The diagram below shows the graph of $y = f(x)$, sketch the graph of $y = \frac{1}{f(x)}$. **2**

Show all important features.



(c) Use mathematical induction to prove that $4 + 7 + 10 + \ldots + (3n + 1) = \frac{n(3n + 5)}{2}$ for all integers $n \geq 1$. **3**

(d) Show that $\sin(A + B) + \sin(A - B) = 2\sin A \cos B$. **4**

Hence, find $\int_0^{\pi} \sin(mx)\cos(nx)\, dx$ where m and n are both positive, even integers and $m \neq n$.

End of Question 12

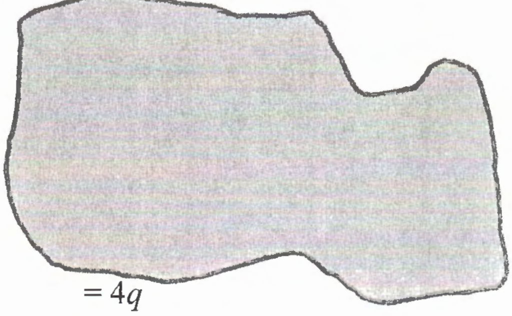
Question 13 (15 marks) Use the Question 13 writing booklet.

(a) Melinda aims to prove that $7^n - 3^n$ is divisible by 4 for all positive integral values of n.
Part of her proof is shown below.

Assume true for $n = k$ i.e $7^k - 3^k = 4p$

Show that $7^{k+1} - 3^{k+1} = 4q$

LHS $= 7^{k+1} - 3^{k+1}$



$= 4q$

$\therefore$ if true for $n = k$ also true for $n = k + 1$
therefore by induction it is also true for all positive integral values of n.

i) Melinda spilled some ink on her work. **2**
Write what you think Melinda might have written in those unreadable lines.

ii) Apart from the obscured lines, another step has been left out of the proof. **1**

Write the missing step

(b) Consider the function $y = \cos^{-1}(\sin x)$.

i) Find $\dfrac{dy}{dx}$. **1**

ii) What does your answer to part (a) tell you about stationary points for this function? **1**

iii) State the domain and range for this function. **2**

iv) Sketch the function over the domain $0 \leq x \leq 2\pi$. **2**

Question 13 continues on page 12

Question 13 continued

(c) Sienna intends to row her boat from the south bank of a river to meet with her friends on the north bank. The river is 100 metres wide. Sienna's rowing speed is 5 metres per second when the water is still. The river is flowing east at a rate of 4 metres per second. Sienna's boat is also being impacted by a wind blowing from the south-west, which pushed the boat at 8 metres per second. She starts rowing across the river by steering the boat such that it is perpendicular to the south bank.

i) Show that the velocity of Sienna's boat can be expressed as the component vector: **2**

$$(4 + 4\sqrt{2})\underset{\sim}{i} + (5 + 4\sqrt{2})\underset{\sim}{j}$$

ii) Calculate the speed of the boat, correct to 2 decimal places. **1**

iii) Determine the distance rowed from Sienna's starting point to her landing point and how long it will take her to reach the north bank. **3**

End of Question 13

Question 14 (15 marks) Use the Question 14 writing booklet.

(a) At time t seconds the length of the side of a cube is x cm, the surface area of the cube is S cm^2, and the volume of the cube is V cm^3. The surface area of the cube is increasing at a constant rate of 8 cm^{2}s^{-1}.

i) Show that $\frac{dx}{dt} = \frac{k}{x}$ where k is a constant. **1**

ii) Show that $\frac{dV}{dt} = 2V^{\frac{1}{3}}$ **2**

iii) Given that $V = 8$ when $t = 0$, solve the differential equation in part (b), and find the value of t when $V = 16\sqrt{2}$. **3**

(b) A golf ball is hit at a velocity of 110 ms^{-1} at an angle θ to the horizontal.

The position vector *s(t)*, from the starting point, of the ball after t seconds is given by

$$s = 110t\cos\theta\, \underset{\sim}{\mathbf{i}} + (110t\sin\theta - 4{\cdot}9\,t^2)\underset{\sim}{\mathbf{j}}$$

i) Using gravity of $9{\cdot}8$ ms^{-2} show that the maximum horizontal range of the ball is $\frac{12100\sin 2\theta}{9{\cdot}8}$ metres. **2**

ii) To ensure that the ball lands on the green, it must travel between 400 and 450 metres. **1**

What values of θ, correct to the nearest minute, would allow this to happen?

iii) The golfer aims accurately and hits the ball directly towards the green. **3**

After 3.4 seconds of flight, at a point 8 metres above the ground, the ball hits a low flying TV drone.

If it had not hit the drone or any other obstacles, would the ball have landed on the green?

(c) For the differential equation $y' = \frac{6}{5x^2 + 4x - 1}$:

i) Show that $y' = \frac{5}{5x - 1} - \frac{1}{x + 1}$. **1**

ii) Find the solution to the differential equation; given that when $x = \frac{1}{2}$, $y = 3$. **2**

End of Paper

Cheltenham Girls' High School

2020 Trial Higher School Certificate Examination
Mathematics Extension 1

Name ____________________________ Teacher ____________________

Section I – Multiple Choice Answer Sheet

Allow about 25 minutes for this section

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: $2 + 4 =$ (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ⊗ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word **correct** and drawing an arrow as follows.

A ⊗ B ⊗ correct C ○ D ○

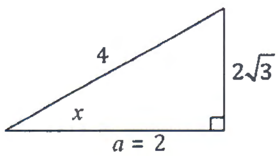
1. A ○ B ○ C ○ D ○
2. A ○ B ○ C ○ D ○
3. A ○ B ○ C ○ D ○
4. A ○ B ○ C ○ D ○
5. A ○ B ○ C ○ D ○
6. A ○ B ○ C ○ D ○
7. A ○ B ○ C ○ D ○
8. A ○ B ○ C ○ D ○
9. A ○ B ○ C ○ D ○
10. A ○ B ○ C ○ D ○

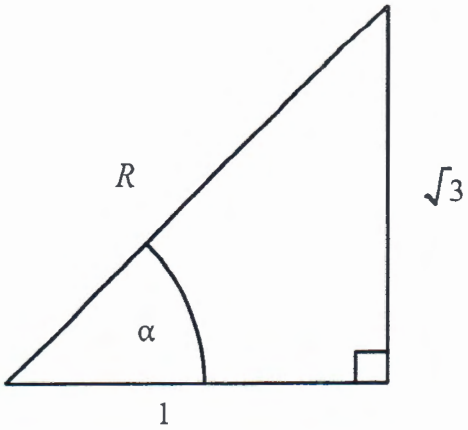
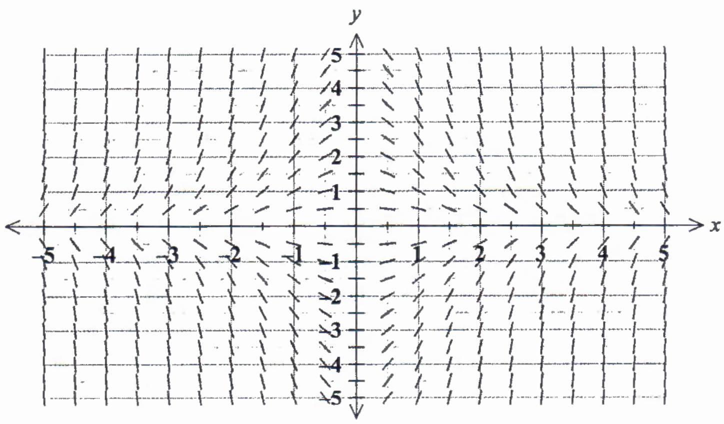
Cheltenham Girls HS

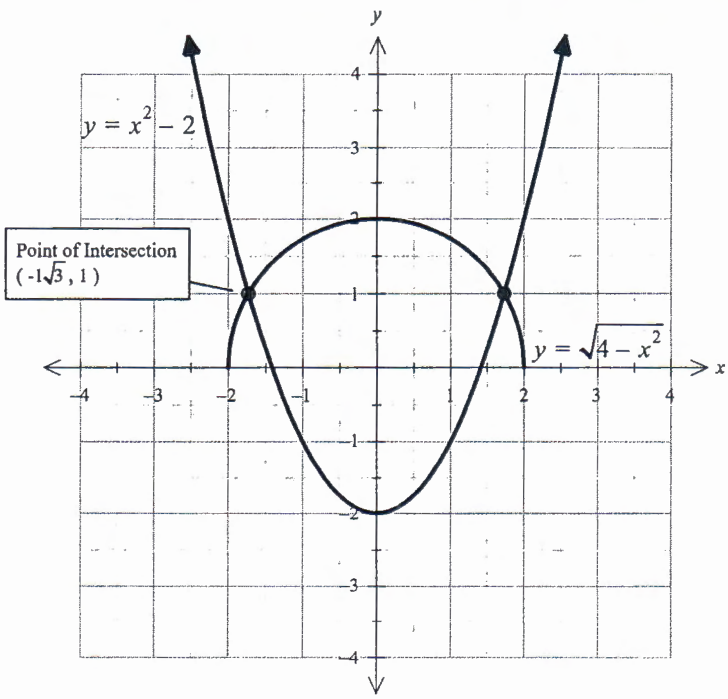
2020 | TRIAL HSC EXAMINATION

Mathematics Extension 1 Solutions

Section I

No	Working	Answer
1.	$4^2 = \left(2\sqrt{3}\right)^2 + a^2$ $a^2 = 16 - 12$ $a = 2$ $\sin 2x = 2\sin x\cos x$ $= 2 \times \frac{2\sqrt{3}}{4} \times -\frac{2}{4} = -\frac{\sqrt{3}}{2}$ 	**D**
2.	$\text{Proj}_{\underset{\sim}{q}}(\underset{\sim}{p}) = \frac{\underset{\sim}{p}.\underset{\sim}{q}}{\lvert \underset{\sim}{q}\rvert^2} \cdot \underset{\sim}{q}$ $\underset{\sim}{p} = \begin{pmatrix}4\\-3\end{pmatrix}$ onto $\underset{\sim}{q} = \begin{pmatrix}-1\\2\end{pmatrix}$. $\underset{\sim}{p}.\underset{\sim}{q} = 4 \times -1 + \ -3 \times 2 = -10$ $\lvert \underset{\sim}{q}\rvert^2 = (-1)^2 + 2^2 = 5$ $\text{Proj}_{\underset{\sim}{q}}(\underset{\sim}{p}) = \frac{-10}{5} \cdot \begin{pmatrix}-1\\2\end{pmatrix}$ $= -2\begin{pmatrix}-1\\2\end{pmatrix}$ $= \begin{pmatrix}2\\-4\end{pmatrix}$	**B**

No	Working	Answer
3.	 $\alpha = \dfrac{\pi}{3}$ $\therefore \sin x + \sqrt{3}\cos x = 2\sin\left(x + \dfrac{\pi}{3}\right)$	C
4.	 The slope in the 1st and 3rd quadrants is always negative. The slope in the 2nd and 4th quadrants is always positive. This rules out options B, C and D.	A
5.	$f(x) = e^x - 1$ Let $y = e^x - 1$ For $f^{-1}(x)$ we have $x = e^y - 1$ which also gives $e^y = x + 1$ $\dfrac{dx}{dy} = e^y$ $\dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}}$ $= \dfrac{1}{e^y}$ $= \dfrac{1}{x+1}$	B

No	Working	Answer
6.	$y = \sin^{-1}\dfrac{a}{x}$ $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-\left(\frac{a}{x}\right)^2}} \times -\dfrac{a}{x^2}$ $= \dfrac{-a}{x\sqrt{x^2-a^2}}$	**D**
7.	Step I has the function reflecting about the x-axis, so it is the negative of the original function, ie. $y = -\sin(2x)$. Step II has us moving the function moving down by 1 unit vertically, which means we must subtract 1 from the function. ie. $y = -\sin(2x) - 1$ or $y = -(\sin(2x) + 1)$ Step III has us stretching the graph horizontally by a factor of 2, so it's frequency will be halved. ie. $y = -\left(\sin\left(\dfrac{2x}{2}\right) + 1\right)$ $y = -(\sin x + 1)$	**A**
8.	 The area can be found by doubling the integral between zero and $\sqrt{3}$ of the difference between the two curves, since the points of intersection are at $(\pm\sqrt{3}, 1)$. $2\displaystyle\int_0^{\sqrt{3}} \left((4-x^2)^{\frac{1}{2}} - x^2 + 2\right) dx$	**C**

No	Working	Answer
9.	$\frac{dA}{dt} = 0.1\ \text{m}^2/\text{min}$ $A = \pi r^2$ $\frac{dA}{dr} = 2\pi r$ $\therefore \quad \frac{dr}{dA} = \frac{1}{2\pi r}$ $\frac{dr}{dt} = \frac{dA}{dt} \times \frac{dr}{dA}$ $= 0.1 \times \frac{1}{2\pi r}$ $= \frac{0.05}{\pi r}$ $When\ A = 0.3$ $\pi r^2 = 0.3$ $r = \sqrt{\frac{0.3}{\pi}}$ $\frac{dr}{dt} = \frac{0.05}{\pi\sqrt{\frac{0.3}{\pi}}}$ $= 0.05150322694$ ≈ 0.0515 m/min	**C**
10.	We have $^{10}\mathrm{C}_2$ ways of choosing the pair of numbers to use in the code. With the restriction, we have 14 ways of arranging any pair of digits. Eg. If we used 0 and 1 We could have 2 of each digit: 0011 0110 0101 1100 1001 1010 (6 arrangements) We could have 3 x 0 and 1 x 1 0001 0010 0100 1000 (4 arrangements) Or we could have 3 x 1 and 1 x 0 1110 1101 1011 0111 (4 arrangements) So we have $^{10}\mathrm{C}_2 \times 14 = 630$ arrangements with exactly 2 digits. Probability of guessing the correct code is $\frac{1}{630}$.	**D**

Trial HSC Examination 2020
Mathematics Extension 1

Name ____________________ Teacher ____________________

Section I – Multiple Choice Answer Sheet

Allow about 15 minutes for this section

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample:	2 + 4 =	(A) 2	(B) 6	(C) 8	(D) 9
		A ○	B ●	C ○	D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ⊗ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word **correct** and drawing an arrow as follows.

correct

A ⊗ B ⊗ C ○ D ○

1.	A ○	B ○	C ○	D ●
2.	A ○	B ●	C ○	D ○
3.	A ○	B ○	C ●	D ○
4.	A ●	B ○	C ○	D ○
5.	A ○	B ●	C ○	D ○
6.	A ○	B ○	C ○	D ●
7.	A ●	B ○	C ○	D ○
8.	A ○	B ○	C ●	D ○
9.	A ○	B ○	C ●	D ○
10.	A ○	B ○	C ○	D ●

Question 11	2020

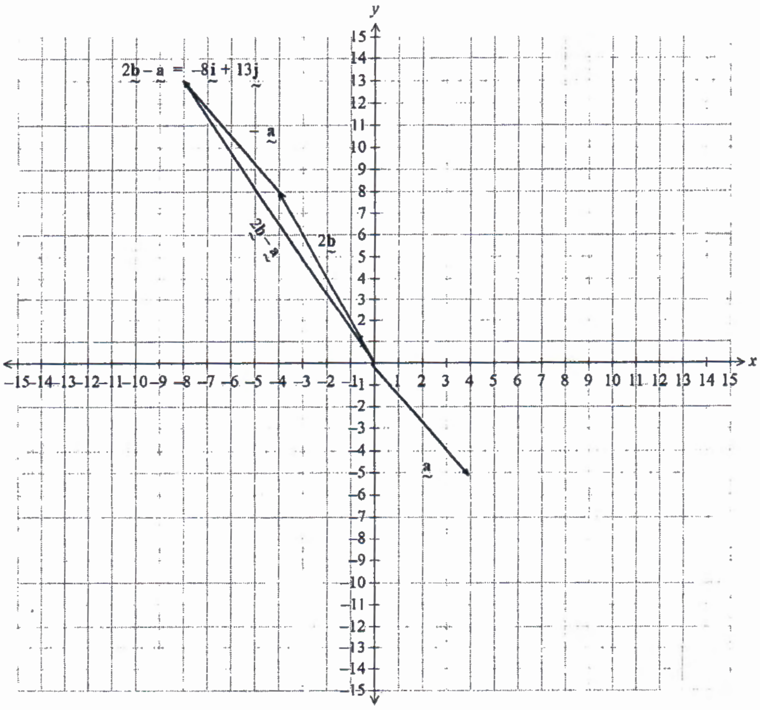
	Solution	Marks	Allocation of marks
(a)	$\dfrac{d}{dx}(e^x \tan^{-1}x) = e^x \tan^{-1}x + \dfrac{e^x}{1+x^2}$	2	2 Marks for correct answer. 1 Mark for using the product rule
(b)	i)Using $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$ $\sin 5x \sin x = \frac{1}{2}(\cos(5x-x) - \cos(5x+x))$ $= \frac{1}{2}(\cos 4x - \cos 6x)$	2	2 marks for correct derivation using sums and differences 1 mark for using sums and differences to work towards incorrect or incomplete proof
	ii) $\displaystyle\int_0^{\frac{\pi}{4}} \sin 5x \ \sin x \, dx = \frac{1}{2}\int_0^{\frac{\pi}{4}} \cos 4x - \cos 6x \ dx$ $= \dfrac{1}{2}\left[\dfrac{\sin 4x}{4} - \dfrac{\sin 6x}{6}\right]_0^{\frac{\pi}{4}}$ $= \dfrac{1}{2}\left[\dfrac{\sin(\pi)}{4} - \dfrac{\sin\left(\frac{3\pi}{2}\right)}{6} - 0\right]$ $= \dfrac{1}{2}\left(0 - \left(-\dfrac{1}{6}\right)\right)$ $= \dfrac{1}{12}$	2	2 marks for the correct answer with sufficient working 1 mark for use of proven identity from part a) to work towards solution with error or omission

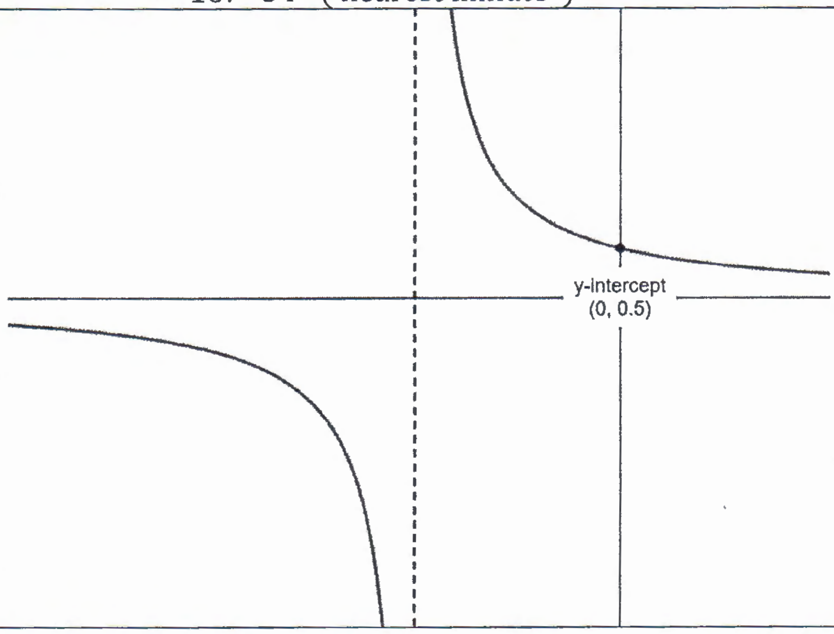
	Solution	Marks	Allocation of marks
(c)	Let $u = 1 - x$ $du = -\,dx$ $x = 1 - u$ when $x = 0$, $u = 1$ when $x = -8$, $u = 9$ $\int_{-8}^{0} \frac{x}{1-x}\,dx = \int_{9}^{1} \frac{1-u}{\sqrt{u}} \; - \; du$ $= \int_{1}^{9} \frac{1}{\sqrt{u}} - \frac{u}{\sqrt{u}}\,du$ $= \int_{1}^{9} u^{-\frac{1}{2}} - u^{\frac{1}{2}}\,du$ $= \left[2u^{\frac{1}{2}} - \frac{2}{3}u^{\frac{3}{2}} \right]_{1}^{9}$ $= \left[2\sqrt{9} - \frac{2}{3} \times 9\sqrt{9} - \left(2 - \frac{2}{3}\right) \right]$ $= 6 - 18 - 1\frac{1}{3}$ $= -\frac{40}{3}$ or $-13\frac{1}{3}$	3	3 marks for correct answer obtained from correct use of given substitution 2 marks for obtaining correct integral in terms of u, including upper and lower bounds or equivalent merit 1 mark for some relevant work towards using the given substitution

	Solution	Marks	Allocation of marks
(d)	For $x^4 + ax^3 - 3x^2 + bx - 2$ the triple root cannot be 2 since the constant term is -2. A triple root of 2 would lead to a constant of -8. So the triple root is $x = -1$. Substituting -1 into the polynomial gives: $1 - a - 3 - b - 2 = 0$ $- a - b - 4 = 0$ (1) First derivative of the polynomial is $4x^3 + 3ax^2 - 6x + b$ The second derivative is $12x^2 + 6ax - 6$. Substituting $x = -1$ into second derivative gives: $12 - 6a - 6 = 0$ $-6a = -6$ $a = 1$ Substituting $a = 1$ into (1) gives: $-1 - b - 4 = 0$ $- b = 5$ $b = -5$ OR **substituting x = -1 and 2 into f(x) and solving simultaneously works.**	2	2 marks for correct answer with valid working 1 mark for some progress towards the solution

	Solution		Marks	Allocation of marks
(e)	i) $t = 2 - \frac{1}{e^{3x}}$ $= 2 - e^{-3x}$ $\frac{dt}{dx} = 3e^{-3x}$ $= \frac{3}{e^{3x}}$ $\therefore \frac{dx}{dt} = \frac{e^{3x}}{3}$ ie $v = \frac{e^{3x}}{3}$	OR $t = 2 - \frac{1}{e^{3x}}$ $t - 2 = -\frac{1}{e^{3x}}$ $e^{3x} = -\frac{1}{t-2}$ $= \frac{1}{2-t}$ $3x = \ln\left(\frac{1}{2-t}\right)$ $x = \frac{1}{3}\ln\left(\frac{1}{2-t}\right)$ $\frac{dx}{dt} = \frac{1}{3}\left(\frac{\frac{1}{(2-t)^2}}{\frac{1}{2-t}}\right)$ $= \frac{1}{3}\left(\frac{1}{2-t}\right)$ $= \frac{e^{3x}}{3}$	2	2 marks- correctly finds $\frac{dt}{dx}$ or rearranges to make t the subject and finds $\frac{dx}{dt}$ 1 mark - attempts to differentiate t in terms of x or correctly expresses x in terms of t
	ii) $a = \frac{d}{dx}\left(\frac{v^2}{2}\right)$ $= \frac{d}{dx}\left(\frac{e^{6x}}{18}\right)$ $= \frac{e^{6x}}{3}$ when $t = 1$ $1 = 2 - \frac{1}{e^{3x}}$ $-1 = -\frac{1}{e^{3x}}$ $e^{3x} = 1$ $3x = \ln 1$ $x = 0$ $\therefore a = \frac{e^0}{3}$ $= \frac{1}{3}\text{ms}^{-2}$	OR $\frac{dx}{dt} = \frac{1}{3(2-t)} = \frac{1}{3}(2-t)^{-1}$ $a = \frac{d^2x}{dt^2} = \frac{-1}{3}(2-t)^{-2}$ $\times -1$ $= \frac{1}{3}\left(\frac{1}{(2-t)^2}\right)$ When t = 1 $a = \frac{1}{3} \times \frac{1}{(2-1)^2} = \frac{1}{3}\text{ms}^{-2}$	2	2 marks - correct answer after obtaining correct expression for acceleration in terms of t or x, or equivalent 1 mark - attempt to use acceleration formula $\frac{d}{dx}\frac{v^2}{2}$ or to find acceleration in terms of t.

Question 12	**2020**

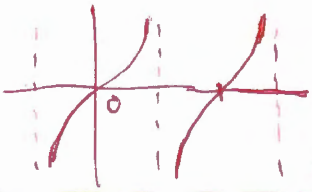
	Solution	Marks	Allocation of marks
(a)	i) $4\underset{\sim}{i} - 5\underset{\sim}{j} + \left(-2\underset{\sim}{i} + 4\underset{\sim}{j}\right) = 2\underset{\sim}{i} - \underset{\sim}{j}$ magnitude of $a + b = \sqrt{2^2 + (-1)^2}$ $= \sqrt{5}$ direction $=$ α where $\tan\alpha = \left(-\frac{1}{2}\right)$ $\alpha = \tan^{-1}\left(-\frac{1}{2}\right)$ $= -26.56505118$ $\approx -26°34'$	2	2 marks for both direction and magnitude correct 1 mark for one answer correct
	ii) Graphically  Algebraically $2\underset{\sim}{b} - \underset{\sim}{a} = 2 \times \left(-2\underset{\sim}{i} + 4\underset{\sim}{j}\right) - \left(4\underset{\sim}{i} - 5\underset{\sim}{j}\right)$ $= -4\underset{\sim}{i} + 8\underset{\sim}{j} - 4\underset{\sim}{i} + 5\underset{\sim}{j}$ $= -8\underset{\sim}{i} + 13\underset{\sim}{j}$	1	1 mark for correct answer graphically or algebraically
	iii) $\underset{\sim}{a} \cdot \underset{\sim}{b} = x_1 x_2 + y_1 y_2$ $= 4 \times -2 + \; -5 \times 4$ $= -8 - 20$ $= -28$	1	1 mark for correct answer

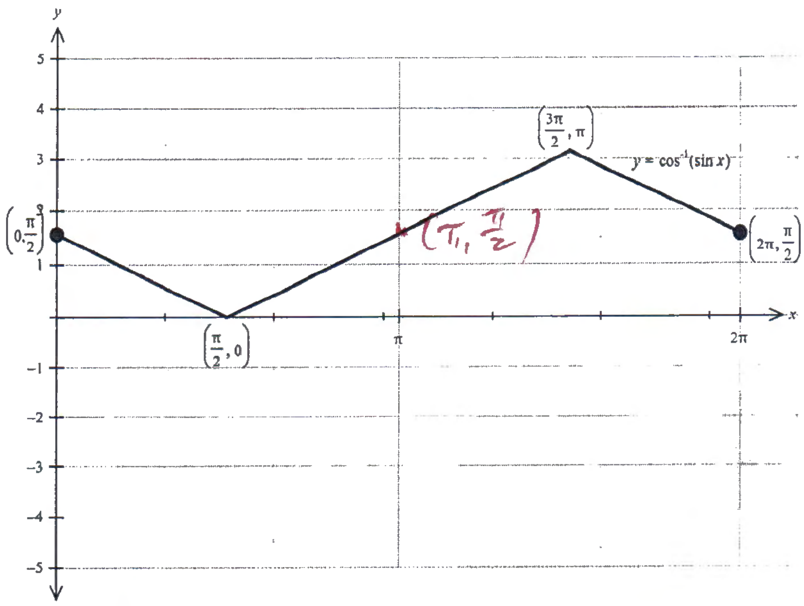
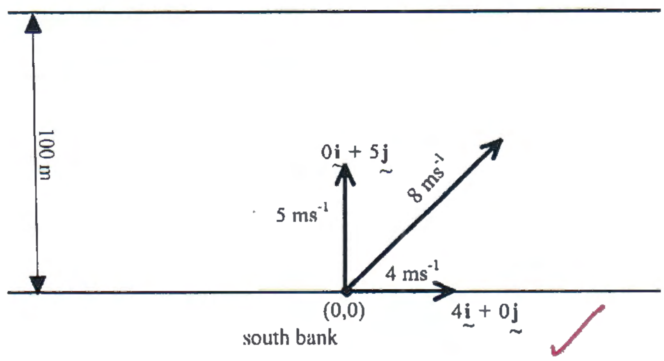
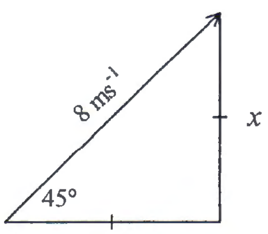
	Solution	Marks	Allocation of marks
	iv) $\underset{\sim}{a} \cdot \underset{\sim}{b} = \|\underset{\sim}{a}\|\|\underset{\sim}{b}\| \cos\theta$ where θ is the angle between the vectors. $\|\underset{\sim}{a}\| = \sqrt{4^2 + (-5)^2} = \sqrt{41}$ $\|\underset{\sim}{b}\| = \sqrt{(-2)^2 + 4^2} = \sqrt{20}$ $\underset{\sim}{a} \cdot \underset{\sim}{b} = \sqrt{41} \times \sqrt{20} \cos\theta$ $\underset{\sim}{a} \cdot \underset{\sim}{b} = -28$ (from iii)) $\therefore \sqrt{820}\cos\theta = -28$ $\cos\theta = -\dfrac{28}{\sqrt{820}}$ $\theta = \cos^{-1}\left(-\dfrac{28}{\sqrt{820}}\right)$ $\theta = \cos^{-1}(-0.9778)$ $= 167.905$ $= 167° 54'$ (nearest minute)	2	2 marks for correct solution 1 mark for equating the expressions for scalar product of vectors but with an error in algebra or calculation
(b)		2	2 marks for the correct shape and intercept/s 1 mark for the correct shape or intercept/s

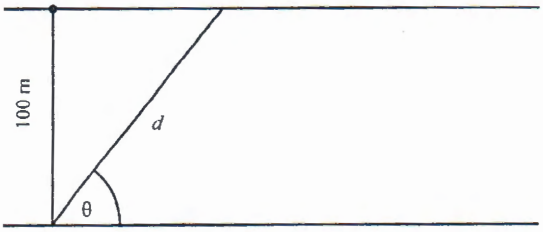
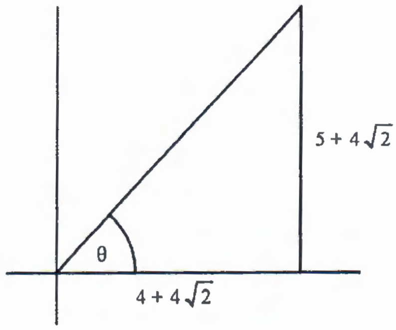
(c)	Step 1: Prove true for $n=1$ $\text{LHS} = 3n + 1$ $= 3(1) + 1$ $= 4$ $\text{RHS} = \dfrac{n(3n+5)}{2}$ $= \dfrac{1(3(1)+5)}{2}$ $= 4$ $= \text{LHS}$ $\therefore$ true for $n = 1$ Step 2: Assume true for $n = k$ where k is an integer ≥ 1 $4 + 7 + 10 + \ldots + (3k+1) = \dfrac{k(3k+5)}{2}$ Step 3: Prove true for $n = k + 1$ using the assumption that it is true for $n = k$. RTP: $4 + 7 + 10 + \ldots + (3(k+1)+1) = \dfrac{(k+1)(3(k+1)+5)}{2}$ $\text{LHS} = \dfrac{k(3k+5)}{2} + 3k + 4$ $= \dfrac{3k^2 + 5k + 6k + 8}{2}$ $= \dfrac{3k^2 + 11k + 8}{2}$ $= \dfrac{3k^2 + 3k + 8k + 8}{2}$ $= \dfrac{3k(k+1) + 8(k+1)}{2}$ $= \dfrac{(k+1)(3k+8)}{2}$ $= \dfrac{(k+1)(3(k+1)+5)}{2}$ $= \text{RHS}$ $\therefore$ true for $n = k + 1$ when true for $n = k$ Since true for $n = 1$ and true for $n = k + 1$ when true for $n = k$, proven true for all $n \geq 1$.	3	3 marks for a correct proof that includes all necessary steps 2 marks for an incorrect proof that has minor error or omissions 1 mark for working which shows knowledge of the steps required and completes at least one correctly

(d)	$\sin(A+B)+\sin(A-B)=\sin A\cos B+\cos A\sin B+\sin A\cos B-\cos A\sin B$ $=2\sin A\cos B$ OR $\text{Sin } A\cos B = \frac{1}{2}(\sin(A+B)+\sin(A-B))$ $\sin(A+B)+\sin(A-B) = 2\sin A\cos B$ $\int_0^{\pi} \sin mx\cos nx\,dx$ $=\frac{1}{2}\int_0^{\pi} 2\sin mx\ \cos nx\,dx$ $=\frac{1}{2}\int_0^{\pi} \sin(mx+nx)+\sin(mx-nx)\ dx$ $=\frac{1}{2}\int_0^{\pi} \sin(m+n)x\ +\sin(m-n)x\,dx$ $=\frac{1}{2}\left[-\frac{\cos(m+n)x}{m+n}-\frac{\cos(m-n)x}{m-n}\right]_0^{\pi}$ $=-\frac{1}{2}\left[\left(\frac{\cos(m+n)\pi}{m+n}+\frac{\cos(m-n)\pi}{m-n}\right)-\left(\frac{\cos 0}{m+n}+\frac{\cos 0}{m-n}\right)\right]$ When m and n are both even, $(m+n)$ and $(m-n)$ are even, : $\cos 2\pi = \cos 4\pi = \cos 6\pi = \ldots.. \cos(m+n)\pi = 1$ $=-\frac{1}{2}\left[\left(\frac{1}{m+n}+\frac{1}{m-n}\right)-\left(\frac{1}{m+n}+\frac{1}{m-n}\right)\right]$ $=0$	4	4 marks for all correct solutions with justification 3 marks for one of the solutions or for significant progress to 2 solutions or equivalent merit 2 marks for some progress 1 mark for some correct and relevant expressions or for correct derivation

Question 13	2020

	Solution	Marks	Allocation of marks
(a)	i) $\text{LHS} = 7^{k+1} - 3^{k+1}$ $= 7^k \times 7 - 3^k \times 3$ $= 4 \times 7^k + 3 \times 7^k - 3 \times 3^k$ $= 4 \times 7^k + 3(7^k - 3^k)$ $= 4 \times 7^k + 3(4p)$ $= 4(7^k + 3p)$ $= 4q$	2	2 marks for any correct working to prove LHS = $4q$ 1 mark for working with a minor error, or incomplete
	ii) When $n = 1,\ 7^k - 3^k = 7^1 - 3^1$ $= 7 - 3$ $= 4$ Which is divisible by 4 $\therefore$ true for $n = 1$	1	1 mark for correct step
(b)	i) $y = \cos^{-1}(\sin x)$ $\dfrac{dy}{dx} = -\dfrac{\cos x}{\sqrt{1-\sin^2 x}}$ $= -\dfrac{\cos x}{\sqrt{\cos^2 x}}$ $= \mp 1$ ✱ e.g. $\cos^{-1}(\tan x)$  $-1 \le \tan x \le 1$ $-\frac{\pi}{4} \le x \le \frac{\pi}{4}$ R is unchanged.	1	Correct use of formula for derivative of $\cos^{-1}(f(x))$
	ii) Since $\dfrac{dy}{dx} \neq 0$ this function has no stationary points / no points where the graph is horizontal.	1	1 mark for correct answer
	iii) Range: Using $-1 \le \sin x \le 1$ when $\sin x = 1$ $y = \cos^{-1}(1)$ $= 0$ when $\sin x = -1$ $y = \cos^{-1}(-1)$ $= \pi$ $\therefore$ range is $[0,\pi]$ Domain: $\{(-\infty,\infty)\}$ ✱ R & D of composite functions!	2	2 marks for correct domain and range 1 mark for one of these correct

	Solution	Marks	Allocation of marks
	iv)  	2	2 marks for correct graph with all important points correct 1 mark for graph that matches given domain and range
(c)	i)  Component vectors of the wind  $8^2 = 2x^2$ $x^2 = 32$ $x = \sqrt{32}$ $= 4\sqrt{2}$ Wind vector is $4\sqrt{2}\,\underset{\sim}{i} + 4\sqrt{2}\,\underset{\sim}{j}$ Rowing vector is $0\,\underset{\sim}{i} + 5\,\underset{\sim}{j}$ Water flow vector is $4\,\underset{\sim}{i} + 0\,\underset{\sim}{j}$ Resultant velocity $= 4\underset{\sim}{i} + 0\underset{\sim}{j} + 0\underset{\sim}{i} + 5\underset{\sim}{j} + 4\sqrt{2}\,\underset{\sim}{i} + 4\sqrt{2}\,\underset{\sim}{j}$ $= (4 + 4\sqrt{2})\underset{\sim}{i} + (5 + 4\sqrt{2})\underset{\sim}{j}$	2	2 marks for correct component vector for wind and addition of the three vectors 1 mark for correct component vector for the wind or valid attempt at adding the three vectors

	Solution	Marks	Allocation of marks
	ii) The velocity is equal to the magnitude of the resultant velocity vector from (a). $v = \sqrt{(4 + 4\sqrt{2})^2 + (5 + 4\sqrt{2})^2}$ $= 14.38135517$ $\approx 14.38\ \text{ms}^{-1}$	1	1 mark for correct answer
	iii)  NOT TO SCALE  $\tan\theta = \dfrac{5 + 4\sqrt{2}}{4 + 4\sqrt{2}}$ $\theta = \tan^{-1}\left(\dfrac{5 + 4\sqrt{2}}{4 + 4\sqrt{2}}\right)$ $= 47.81827238°$ $\approx 48°$ The distance from start to finish is given by *d*. $\sin\theta = \dfrac{100}{d}$ $d = \dfrac{100}{\sin 48°}$ $= 134.9493465$ m ≈ 135 m Time taken $= 134.9493465\ \text{m} \div 14.38\ \text{ms}^{-1}$ $= 9.384516448$ ≈ 9.4 seconds	3	3 marks for correct distance and time 2 marks for correct calculation of angle and work towards finding distance and time 1 mark for correct calculation of angle or use of distance and velocity to attempt to find time taken or working towards using vectors to calculate distance travelled

Question 14	**2020**

	Solution	Marks	Allocation of marks
(a)	i) Surface area of a cube with a side length of x is $6x^2$ $S = 6x^2$ $\frac{dS}{dx} = 12x$ Given $\frac{dS}{dt} = 8$ $\frac{dx}{dt} = \frac{dS}{dt} \div \frac{dS}{dx} = \frac{8}{12x}$ $= \frac{\frac{2}{3}}{x}$ with $k = \frac{2}{3}$	1	1 mark for the correct answer
	ii) $V = x^3$ (volume of a cube with side length x) $\frac{dV}{dx} = 3x^2$ $\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$ $= 3x^2 \times \frac{2}{3x} = 2x$ $x = \frac{1}{2}\frac{dV}{dt}$ $x = V^{\frac{1}{3}}$ $\frac{1}{2}\frac{dV}{dt} = V^{\frac{1}{3}}$ $\frac{dV}{dt} = 2V^{\frac{1}{3}}$	2	2 mark for the correct answer 1 mark for some progress towards the answer
	iii) $\frac{dV}{dt} = 2V^{\frac{1}{3}}$ $V^{-\frac{1}{3}}dV = 2\mathrm{dt}$ $\int V^{-\frac{1}{3}}\,dV = \int 2\,\mathrm{dt}$ $\frac{3}{2}V^{\frac{2}{3}} = 2t + C$ Given that $V = 8$ when $t = 0$ $\frac{3}{2}8^{\frac{2}{3}} = 2 \times 0 + C$ or $C = 6$ $\frac{3}{2}V^{\frac{2}{3}} = 2t + 6$ Find t when $V = 16\sqrt{2}$ $\frac{3}{2}(16\sqrt{2})^{\frac{2}{3}} = 2t + 6$ $12 = 2t + 6$ $t = 3$	3	3 Marks for correct answer. 2 Marks for finding the expression for V and t. 1 Mark for separating the variables and attempts to integrate both sides

(b)	i) Max range when $y = 0$ $110t\sin\theta - 4{\cdot}9t^2 = 0$ $t(110\sin\theta - 4{\cdot}9t) = 0$ $t = 0$ or $110\sin\theta = 4{\cdot}9t$ $t = \dfrac{110\sin\theta}{4{\cdot}9}$ When $t = \dfrac{110\sin\theta}{4{\cdot}9}$ $x = 110\left(\dfrac{110\sin\theta}{4{\cdot}9}\right)\cos\theta$ $= \dfrac{110^2\sin\theta\cos\theta}{4{\cdot}9}$ $= \dfrac{12100 \times 2\sin\theta\cos\theta}{9{\cdot}8}$ $= \dfrac{12100\sin 2\theta}{9{\cdot}8}$	2	2 marks for correct proof showing solving quadratic equation and several steps of the proof, including $2\sin\theta\cos\theta$ 1 mark for working towards proof either using solution of quadratic equation or double angle result
	ii) We want max range to be between 400 and 450 metres. $400 < \dfrac{12100\sin 2\theta}{9{\cdot}8} < 450$ $\dfrac{400 \times 9{\cdot}8}{12100} < \sin 2\theta < \dfrac{450 \times 9{\cdot}8}{12100}$ $18°54'10{\cdot}8'' < 2\theta < 21°22'28{\cdot}3''$ $9°27' < \theta < 10°41'$	1	1 marks for correct range of angles
	iii) When t=3.4 sec, y=8 m To find the angle the ball was hit at: $y = 110t\sin\theta - 4{\cdot}9t^2$ $8 = 110(3{\cdot}4)\sin\theta - 4{\cdot}9(3{\cdot}4)^2$ $\sin\theta = \dfrac{8 + 4{\cdot}9(3{\cdot}4)^2}{110(3{\cdot}4)}$ $\theta = 9°57'11{\cdot}77''$ To find the distance travelled using this angle, using time of flight from (a): $t = \dfrac{110\sin\theta}{4{\cdot}9}$ $= \dfrac{110\sin 9°57''}{4{\cdot}9}$ $= 3{\cdot}880192077$ sec $x = 110 \times 3{\cdot}880192077\cos 9°57'11{\cdot}77''$ $= 420{\cdot}3970664$ m Therefore, if the ball hadn't hit the drone, it would have made the green.	3	3 marks for correct horizontal distance calculated, showing all working 2 marks for using given information to find angle and time 1 mark for using given information to find angle

(c)	i) $\dfrac{5}{5x-1} - \dfrac{1}{x+1} = \dfrac{5x+5-5x+1}{5x^2+4x-1}$ $= \dfrac{6}{5x^2+4x-1}$	1	Adequate steps to show equality
	ii) $y = \displaystyle\int \dfrac{6}{5x^2+4x-1}\,dx = \int \dfrac{5}{5x-1} - \dfrac{1}{x+1}\,dx$ $= \ln(5x-1) - \ln(x+1) + C$ $= \ln\left(\dfrac{5x-1}{x+1}\right) + C$ when $x = \dfrac{1}{2}, y = 3$ $3 = \ln\left(\dfrac{\frac{5}{2}-1}{\frac{1}{2}+1}\right) + C$ $3 = \ln\left(\dfrac{\frac{3}{2}}{\frac{3}{2}}\right) + C$ $3 = \ln(1) + C$ $3 = 0 + C$ $\therefore C = 3$ $y = \ln\left(\dfrac{5x-1}{x+1}\right) + 3$	2	2 marks - correct answer with valid working 1 mark - Attempt integration using the separate fractions and resulting in an expression involving logs