HB.
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KW

Name: ______________

Class: 12MT3__________

Teacher: ______________

CHERRYBROOK TECHNOLOGY HIGH SCHOOL

2000 AP3

YEAR 12 HALF YEARLY HSC

MATHEMATICS
3/4 UNIT (COMMON)

Time allowed – 1.5 HOURS
(plus 5 minutes' reading time)

DIRECTIONS TO CANDIDATES:

* Attempt ALL questions.
* The value for each question is indicated
* All necessary working should be shown in every question.
  Full marks may not be awarded for careless or badly arranged work.
* Standard Integrals are provided. Approved calculators may be used.
* Each question attempted is to be returned on a new page clearly marked Question 1, Question 2, etc on the top of the page.

***Each page must show your class and your name.**

Marks

Question 1: (12 Marks)

(a) Differentiate the following: **4**
  (i) $\log_e(e^{3x}+2)$
  (ii) $x^3\cos 3x$.

(b) Find the following indefinite integrals: **4**
  (i) $\int \frac{dx}{(7x+4)^5}$.
  (ii) $\int \sin 6x\,dx$
  (iii) $\int 4xe^{x^2}\,dx$.

(c) Solve for x: **2**
  $\log_c 8+\log_a 16 = x\log_a 2$.

(d) Find the exact value of $\cos 105^0$. **2**

Question 2: (Start a New Page) (12 Marks)

(a) Simplify $\frac{\sin x}{\cos x-\sin x}+\frac{\sin x}{\cos x+\sin x}$ **3**

(b) Simplify $\sec x+\tan x$, in terms of t, where $t=\tan\frac{x}{2}$. **3**

(c) Use the substitution $u=x^2-1$ to find **3**
  $\int x^3(x^2-1)dx$

(d) Consider the curve $y=\sin x$, for $0\le x\le 2\pi$. **3**
  For what values of x is the gradient equal to $\frac{1}{2}$?

Question 3: (Start a New Page) (12 Marks) **Marks**

(a) The quartic expression x^4+ax^2+b has factors $(x+1)$ and $(x-2)$. Find the values of a and b. **3**

(b) If $x=c$ is a double root of $P(x)$, show that $x=c$ is a root of $P'(x)$. **3**

(c) p, q and r are the roots of the cubic equation $x^3+2x^2+3x+5=0$. Evaluate: **4**

(i) $p+q+r$.

(ii) $p^{-1}+q^{-1}+r^{-1}$.

(d) The equation $e^x-4x-8=0$ has a root close to $x=3$. Using 3 as a first approximation and one application of Newton's Method to find a better approximation for this root. Give your answer correct to three decimal places. **2**

Question 4: (Start a New Page) (12 Marks)

(a)

(i) Find R and α such that $2\cos\theta-\sin\theta=R\cos(\theta+\alpha)$. (Note: $R>0$ and $0^0<\alpha<90^0$.) **4**

(ii) Hence, solve $2\cos\theta=\sin\theta+1$, for $0^0\le\theta\le 360^0$

(b) The curve $y=\cos x$, from $-\frac{\pi}{2}\le x\le\frac{\pi}{2}$, is rotated about the x-axis. **4**

Find the volume of the solid formed. Leave your answer in exact form.

(c) **4**

(i) Find $\frac{d}{dx}(x\log_e x)$.

(ii) Prove that $\int_e^{e^2}\frac{1+\log_e x}{x\log_e x}dx = 1+\log_e 2$.

Question 5: (Start a New Page) (9 Marks) **Marks**

(a)

(i) Sketch $y=\sin 2x$, for $0\le x\le 2\pi$. **4**

(ii) By drawing a suitable straight line, state the number of values of x, in this domain, such that $\sin 2x=\frac{x}{2\pi}$.

(iii) Can there be further solutions beyond $x=2\pi$? Briefly justify your answer.

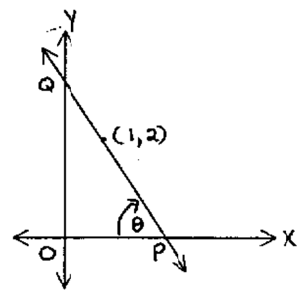
(b) $A(t,e^t)$ and $B(-t,e^{-t})$ are points on the curve $y=e^x$ and $t>0$. **5**

The tangents at A and B form an angle of 45^0.

(i) Prove that $e^t-\frac{1}{e^t}=2$.

(ii) Solve this equation to show that $e^t=1+\sqrt{2}$.

Question 6: (Start a New Page) (10 Marks) **10**



A straight line passes through the point (1,2) and meets the x and y axes at P and Q respectively, as shown. The angle OPQ is θ.

(a) Show that the equation of the line PQ is given by $y=\tan\theta+2-x\tan\theta$.

(b) Show that the area (A) of ΔOPQ is given by $A=\frac{\tan\theta}{2}+2+\frac{2}{\tan\theta}$.

(c) Prove that the area is a minimum when $\tan\theta=2$.

(d) Hence, find the minimum area.

End of Exam

CTHS 3 Unit AP3 (April 2000)

Question 1

a). $y = \log_e(e^{3x}+2)$

$$\frac{dy}{dx} = \frac{3e^{3x}}{e^{3x}+2} \quad (1) \ (1)$$

(ii) $y = x^3\cos 3x$

$y' = \cos 3x \times (3x^2) + x^3 \times (-3\sin 3x)$ (1)

$= 3x^2(\cos 3x - x\sin 3x)$ (1)

b) (i) $\int \frac{dx}{(7x+4)^5}$

$= \int (7x+4)^{-5}\,dx$

$= \frac{(7x+4)^{-4}}{-4\times 7} + c$

$= \frac{-1}{28(7x+4)^4} + c$

(ii) $\int \sin 6x\,dx$

$= -\frac{\cos 6x}{6} + c$ (1)

(iii) $\int 4x e^{x^2}\,dx$

$= 2 \times \int 2x e^{x^2}\,dx$ (1)

$= 2e^{x^2} + c$ (1)

c) $\log_a 8 + \log_a 16 = x\log_a 2$

$3\log_a 2 + 4\log_a 2 = x\log_a 2$ (1)

$7\log_a 2 = x\log_a 2$

$\therefore x = 7$ (1)

Question 1 (cont)

d) $\cos 105°$

$= \cos(60° + 45°)$

$= \cos 60°\cos 45° - \sin 60°\sin 45°$ (1)

$= \frac{1}{2}\times\frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}}$

$= \frac{1-\sqrt{3}}{2\sqrt{2}}$ (1)

$\left[= \frac{\sqrt{2}(1-\sqrt{3})}{4}\right]$

Question 2

a) $\frac{\sin x}{\cos x - \sin x} + \frac{\sin x}{\cos x + \sin x}$

$= \frac{\sin(\cos x + \sin x + \cos x - \sin x)}{(\cos x - \sin x)(\cos x + \sin x)}$ (1)

$= \frac{2\sin x\cos x}{\cos^2 x - \sin^2 x}$

$= \frac{\sin 2x}{\cos 2x}$ (1)

$= \tan 2x$ (1)

b) $\sec x + \tan x$

$= \frac{1+t^2}{1-t^2} + \frac{2t}{1-t^2}$ (1)

$= \frac{1+2t+t^2}{1-t^2}$

$= \frac{(1+t)^2}{1-t^2}$ (1)

$= \frac{(1+t)^2}{(1+t)(1-t)} = \frac{1+t}{1-t}$ (1)

CTHS 3 Unit AP3 (April 2000) Page 2

Question 2 (cont)

c) $u = x^2 - 1$

$\frac{du}{dx} = 2x$

$\frac{du}{2} = x\,dx$ (1)

And $x^2 = u+1$

Now $\int x^3(x^2-1)\,dx$

$= \int x^2(x^2-1)x\,dx$

$= \int (u+1)\cdot u\cdot\frac{du}{2}$

$= \frac{1}{2}\int (u^2+u)\,du$

$= \frac{1}{2}\left(\frac{u^3}{3} + \frac{u^2}{2}\right) + c$ (1)

$= \frac{u^2}{2}\left(\frac{u}{3} + \frac{1}{2}\right) + c$

$= \frac{u^2}{2}\left(\frac{2u+1}{6}\right) + c$

$= \frac{(x^2-1)^2(2(x^2-1)+3)}{12} + c$

$= \frac{(x^2-1)^2(2x^2+1)}{12} + c$ (1)

Question 2 (cont)

d) $y = \sin x$

$\frac{dy}{dx} = \cos x$ (1)

$\cos x = \frac{1}{2}$ (1)

when $x = \frac{\pi}{3},\ 2\pi - \frac{\pi}{3}$

$= \frac{\pi}{3},\ \frac{5\pi}{3}$ (1)

Question 3

(a) $P(x) = x^4 + ax^2 + b$

$P(-1) = 0$

$\therefore 1 + a + b = 0$ ① (1)

$b = -a - 1$

and $P(2) = 0$

$16 + 4a + b = 0$ ② (1)

Sub. in ①

$\therefore 16 + 4a - a - 1 = 0$

$15 + 3a = 0$

$\left.\begin{array}{l} a = -5 \\ \therefore b = 4 \end{array}\right\}$ (1)

b) $P(x) = (x-c)^2 . Q(x)$ (1)

$\therefore P'(x) = Q(x). 2(x-c) + (x-c)^2 . Q'(x)$ (1)

$= (x-c)[2.Q(x) + (x-c)Q'(x)]$

$\therefore x = c$ is a root of $P'(x)$ (1)

Question 3 (cont)

c)

$x^3 + 2x^2 + 3x + 5 = 0$

$a = 1,\ b = 2,\ c = 3,\ d = 5$

i) $p + q + r = -\frac{b}{a}$

$= -2$ (1)

ii) $p^{-1} + q^{-1} + r^{-1}$

$= \frac{1}{p} + \frac{1}{q} + \frac{1}{r}$

$= \frac{qr + pr + pq}{pqr}$ (1)

$= \frac{c/a}{-d/a}$

$= -\frac{c}{d}$ (1)

$= -\frac{3}{5}$ (1)

d) $f(x) = e^x - 4x - 8$

$f'(x) = e^x - 4$

$a_2 = a_1 - \frac{f(a_1)}{f'(a_1)}$ (1)

$= 3 - \frac{f(3)}{f'(3)}$

$= 3 - \frac{e^3 - 20}{e^3 - 4}$ (1)

$= 2.99468\ldots$

$= 2.995$ (3 dp)

Question 4

a) i) $R = \sqrt{2^2 + 1^2}$

$= \sqrt{5}$ (1)

$\therefore \frac{2}{\sqrt{5}}\cos\theta - \frac{1}{\sqrt{5}}\sin\theta$

$= \cos\theta\cos\alpha - \sin\theta\sin\alpha$

$\therefore \cos\alpha = \frac{2}{\sqrt{5}}$

$\alpha = 26°34'$ (1)

$\therefore R = \sqrt{5},\ \alpha = 26°34'$

ii) $2\cos\theta - \sin\theta = 1$

$\sqrt{5}\cos(\theta + \alpha) = 1$

$\cos(\theta + \alpha) = \frac{1}{\sqrt{5}}$ (1)

$(\theta + \alpha) = 63°26',\ 296°34'$

$\theta = 63°26' - 26°34'$

$296°34' - 26°34'$

$\theta = 36°52',\ 270°$ (1)

b) $V = \pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 x\, dx$ (1)

$= 2\pi\int_0^{\frac{\pi}{2}} \cos^2 x\, dx$

$= 2\pi\int_0^{\frac{\pi}{2}} \frac{\cos 2x + 1}{2}\, dx$ (1)

$= \pi\int_0^{\frac{\pi}{2}} (\cos 2x + 1)\, dx$

$= \pi\left[\frac{\sin 2x}{2} + x\right]_0^{\frac{\pi}{2}}$ (1)

$= \pi\left(\frac{\sin\pi}{2} + \frac{\pi}{2}\right) = \frac{\pi^2}{2}$ (1)

Question 4 (cont)

c) (i)

$\frac{d}{dx}(x\log_e x)$

$= (\log_e x)\times 1 + x \times \frac{1}{x}$

$= 1 + \log_e x$ (1)

(ii) $\int_e^{e^2} \frac{1 + \log_e x}{x\log_e x}\, dx$

$= \left[\log_e(x\log_e x)\right]_e^{e^2}$ (1)

$= \log_e(e^2 . 2) - \log_e(e)$

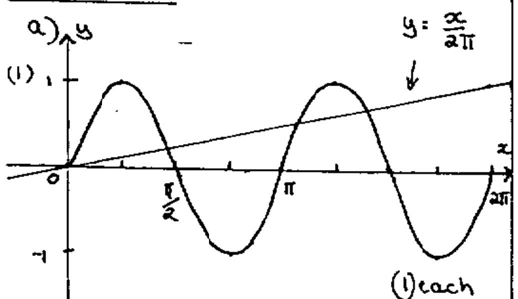
$= \log_e e^2 + \log_e 2 - 1$ (1)

$= 2 + \log_e 2 - 1$

$= 1 + \log_e 2$ (1)

Question 5

a)



ii) There are 4 values (1)

iii) No. $\frac{x}{2\pi} > 1$ when $x > 2\pi$ (1)

$\therefore$ no further solutions because max. value of $\sin 2x$ is 1.

Question 5 (cont)

b)

(i) Let θ = angle between the tangents.

At A, $m_1 = e^t$

B, $m_2 = e^{-t}$ (1)

$\therefore \tan\theta = \left|\frac{e^t - (e^{-t})}{1 + e^t . e^{-t}}\right|$ (1)

$1 = \frac{e^t - e^{-t}}{2}$

i.e. $2 = e^t - e^{-t}$

or $e^t - \frac{1}{e^t} = 2$ (1)

(ii) $e^t - \frac{1}{e^t} - 2 = 0$

$(e^t)^2 - 2e^t - 1 = 0.$ (1)

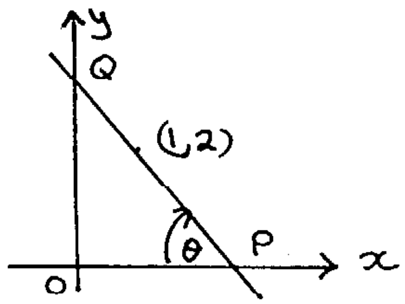
$\therefore e^t = \frac{2 \pm \sqrt{4 + 4}}{2}$ ($\frac{1}{2}$)

$= 1 \pm \sqrt{2}$ ($\frac{1}{2}$)

(Quest 6 on next page)

Question 6

a)



For PQ $m = -\tan\theta$ (1)

Now, $y - y_1 = m(x - x_1)$

$y - 2 = -\tan\theta\,(x - 1)$

$y - 2 = -x\tan\theta + \tan\theta$

OR $y = 2 + \tan\theta - x\tan\theta$ (1)

b) $A = \frac{1}{2} \times OP \times OQ$

At P $y = 0$

$\therefore \quad 0 = \tan\theta + 2 - x\tan\theta$

$x\tan\theta = \tan\theta + 2$

$x = 1 + \frac{2}{\tan\theta}$ (1)

$\therefore OP = 1 + \frac{2}{\tan\theta}$

At Q, $x = 0$

$\therefore y = 2 + \tan\theta$

$\therefore OQ = 2 + \tan\theta$ (1)

$\Rightarrow A = \frac{1}{2}\left(1 + \frac{2}{\tan\theta}\right)(2 + \tan\theta)$

$= \frac{1}{2}\left(2 + \tan\theta + \frac{4}{\tan\theta} + 2\right)$

$= \frac{\tan\theta}{2} + 2 + \frac{2}{\tan\theta}$ (1)

Question 6 (cont)

c) Let $t = \tan\theta$

$\therefore A = \frac{t}{2} + 2 + \frac{2}{t}$

Now $\frac{dA}{dt} = \frac{1}{2} - \frac{2}{t^2}$ (1)

$\frac{d^2A}{dt^2} = \frac{4}{t^3}$ (1)

$\frac{dA}{dt} = 0$ when $\frac{1}{2} = \frac{2}{t^2}$

$t^2 = 4$

$t = \pm 2$

But $t > 0$, since $A > 0$ (1)

$\therefore$ when $t = 2$, $\frac{d^2A}{dt^2} = \frac{4}{2^3} > 0$ (1)

$\therefore$ min. value when $t = 2$

d) When $t = 2$

$A = \frac{2}{2} + 2 + \frac{2}{2}$

$= 4$ (1)

$\therefore$ Min area is 4 sq. units.