AT
KW
LB

Name: ______________

Class: 12MTX______

Teacher: ______________

## CHERRYBROOK TECHNOLOGY HIGH SCHOOL

**2005**

**YEAR 12**

**AP4 EXAMINATION**

# MATHEMATICS EXTENSION 1

*Time allowed* - 2 HOURS
*(Plus 5 minutes' reading time)*

**DIRECTIONS TO CANDIDATES:**

- Attempt all questions.
- All questions are of equal value.
- Each question is to be commenced on a new page clearly marked Question 1, Question 2, etc on the top of the page. **
- Each question is to be returned in a separate bundle.
- All necessary working should be shown in every question. Full marks may not be awarded for careless or badly arranged work.
- Approved calculators may be used.

***Each page must show your name and your class. ***

**Question 1** (12 marks) **Marks**

(a) Let $A$ be the point $(-3,8)$ and let $B$ be the point $(5,-6)$. Find the coordinates of the point $P$ that divides the interval $AB$ internally in the ratio $1:3$. **2**

(b) What is the remainder when the polynomial $P(x)=x^3+3x^2-1$ is divided by $x-2$? **2**

(c) Use the table of standard integrals to find the exact value of **2**

$$\int_0^1 \frac{1}{\sqrt{x^2+9}}\,dx.$$

(d) Solve $\frac{2}{x+5}\leq 1$. **3**

(e) Use the substitution $u=x-1$ to evaluate $\int_2^4 \frac{x}{(x-1)^2}\,dx$. **3**

**Question 2** (12 marks)

(a) Sketch the graph of $y=2\sin^{-1}3x$ showing clearly the domain and range of the function as well as any intercepts. **2**

(b) Let $f(x)=4x^2-1$. **2**

Use the definition $f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ to find the derivative of $f(x)$ at $x=a$.

**Question 2** (continued) **Marks**

(c) Find $\frac{d}{dx}\left(3x^2 \cos^{-1} x\right)$ **2**

(d) Find $\int 4\cos^2 3x\,dx$. **2**

(e) Solve the equation $\sin 2\theta = \sqrt{2}\cos\theta$ for $0 \le \theta \le 2\pi$. **4**

**Question 3** (12 marks)

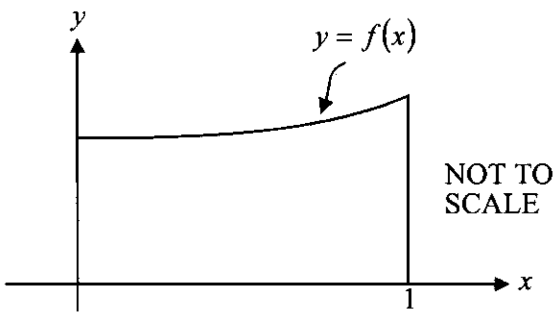
(a) The variable point $(2\cos\theta, 3\sin\theta)$ lies on a curve. Find the Cartesian equation of this curve. **2**

(b) The function $f(x) = \log_e x + 5x$ has a zero near $x = 0 \cdot 2$. **3**
Using $x = 0 \cdot 2$ as a first approximation, use one application of Newton's method to find a second approximation to the zero.
Write your answer correct to 3 decimal places.

(c) For the function $f(x) = \dfrac{1}{\sqrt{4 - x^2}}$

(i) Find the natural domain of the function. **1**

(ii) The sketch below shows part of the graph of $y = f(x)$. The area under the curve for $0 \le x \le 1$ is shaded.
Find the area of the shaded region. **2**




(d) A particle moves in simple harmonic motion about a fixed point $O$. The amplitude of the motion is 2 m and the period is $\dfrac{2\pi}{3}$ seconds. Initially the particle moves from $O$ with a positive velocity.

(i) Explain why the displacement $x$, in metres, of the particle at time $t$ seconds, can be given by **1**

$$x = 2\sin 3t$$

(ii) Find the speed of the particle when it is $\sqrt{3}$ m from $O$. **2**

(iii) What is the maximum speed reached by the particle? **1**

**Question 4** (12 marks) **Marks**

(a) Use mathematical induction to prove that **3**

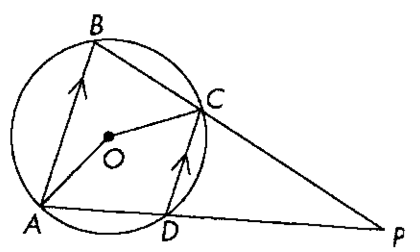
$$1+6+15+\ldots+n(2n-1)=\frac{1}{6}n(4n-1)(n+1)$$

for all positive integers $n$.

(b) (i) Show that $\dfrac{1}{1-\tan x}-\dfrac{1}{1+\tan x}=\tan 2x$ **2**

(ii) Evaluate $\dfrac{1}{1-\tan\frac{\pi}{6}}-\dfrac{1}{1+\tan\frac{\pi}{6}}$ in simplest exact form **1**

(c) In the diagram below, $O$ is the centre of the circle and $AB \parallel DC$. $AD$ and $BC$ meet at $P$.




Prove: (i) $CP = DP$. **2**

(ii) $\Delta ABP$ is isosceles. **2**

(iii) $OAPC$ is a cyclic quadrilateral. **2**

**Question 5** (12 marks) **Marks**

(a) Solve for $x$

$x^{\log_2 x} = 8x^2 (x > 0)$ **3**

(b) Consider the function $f(x) = x(x-2)^2$, $x \leq a$ where $a$ is a constant.

(i) Find the values of $a$ given that the inverse function $f^{-1}(x)$ of $f(x)$ exists. **2**

(ii) State the domain of $f^{-1}(x)$. **1**

(c) If $\alpha$, $\beta$ and $\gamma$ are the roots of the cubic equation $x^3 - 4x^2 + 3x + 2 = 0$, find $\alpha^2 + \beta^2 + \gamma^2$. **2**

(d) Factorise $m^3 - 3m + 2$ and solve the equation $(3x-4)^3 - 9x + 14 = 0$. **4**

**Question 6** (12 marks) **Marks**

(a) A particle moves in a straight line with an acceleration given by

$$\frac{d^2x}{dt^2} = 9(x-2)$$

where $x$ is the displacement in metres from an origin $O$ after $t$ seconds. Initially, the particle is 4 metres to the right of $O$ so that $x = 4$ and has velocity $v = -6$.

(i) Show that $v^2 = 9(x-2)^2$. **2**

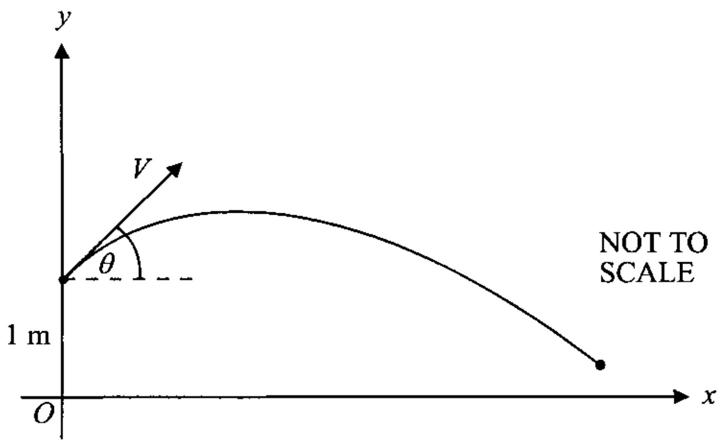
(ii) Find an expression for $v$ and hence find $x$ as a function of $t$. **2**

(iii) Explain whether the velocity of the particle is ever zero. **2**

*Question 6 continues on the next page.*

**Question 6** (continued) **Marks**

(b) A boy throws a ball and projects it with a speed of $V$ m s$^{-1}$ from a point 1 metre above the ground. The ball lands on top of a flowerpot in a neighbour's yard.




The angle of projection is $\theta$ as indicated in the diagram.
The equations of motion of the ball are

$$\ddot{x} = 0 \text{ and } \ddot{y} = -10$$

where $x$ and $y$ are shown on the axes on the diagram. The position of the ball $t$ seconds after it is thrown by the boy is described by the coordinates $(x, y)$.

It has been found that $y = Vt\sin\theta - 5t^2 + 1$.

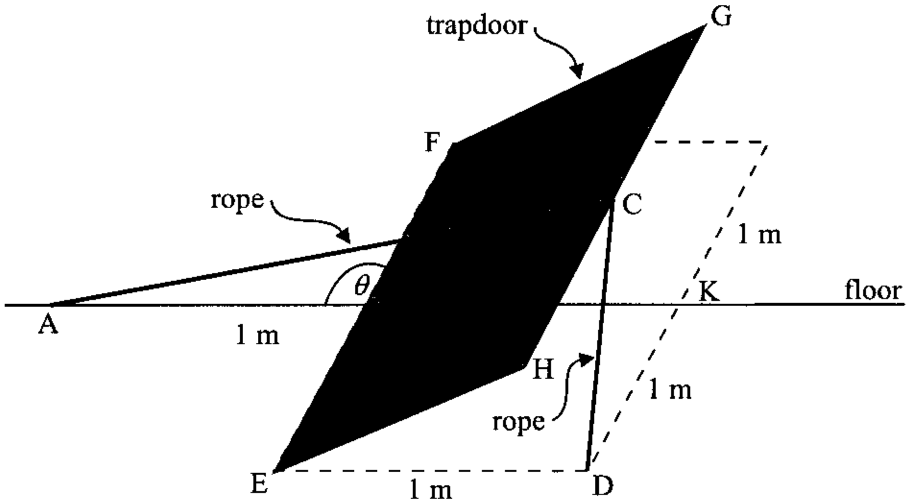
(i) Show that $x = Vt\cos\theta$. **2**

(ii) When the ball is at its maximum height above the ground, it passes directly above a 1.5 metre high fence and clears the fence by 0.5 metres. **2**
Find an expression for $V$ in terms of $\theta$.

(iii) Find the value of $V$ given that $\theta = \tan^{-1}\dfrac{9}{40}$. **2**
Give your answer in m s$^{-1}$, correct to 2 decimal places.

**Question** 7 (12 marks) **Marks**

(a)




A rectangular trapdoor is shown in the diagram as $EFGH$ where $EH = 1$ m, $EF = 2$ m and $BC$ divides the trapdoor in half.

A rope is anchored at point $A$ on the floor 1 metre from point $B$ and points $A$, $B$ and $K$ lie on a straight line.

The rope passes through a small loop on the edge of the trapdoor at point $C$ and is anchored to the floor at point $D$.

As the trapdoor is being opened or closed, the rope running from $A$ through $C$ to $D$ is kept taut by pulling it tight or letting it out through anchor point $A$.

Let $\angle ABC = \theta$, $0^\circ \le \theta \le 180^\circ$.

(i) Show that $AC = \sqrt{2 - 2\cos\theta}$. **1**

(ii) Show that $CD = \sqrt{3 + 2\cos\theta}$. **2**

(iii) Let $l$ equal the length of the rope from $A$ through $C$ to $D$. find the maximum value of $l$. Justify your answer. **4**

**Question** 7 (continued) **Marks**

(b) A cup of soup with a temperature $95^0$C is placed in a room which has a temperature of $20^0$C. In 10 minutes the cup of soup cools to $70^0$C. Assuming the rate of heat loss is proportional to the excess of its temperature above room temperature, that is

$$\frac{dT}{dt} = -k(T - 20),$$

(i) show that $T = 20 + Ae^{-kt}$ is a solution of **1**

$$\frac{dT}{dt} = -k(T - 20).$$

(ii) find the temperature of the soup after a further 5 min. to the nearest degree. **2**

(iii) how long will it take the soup to cool to $35^0$C? Give your answer correct to the nearest minute. **1**

(iv) find the rate of cooling when the soup is $35^0$C. Give your answer correct to 1 decimal place. **1**

# TRIAL SOLUTIONS
# AP4 EXT 1 2005

## Question 1

a) $P = \left(\dfrac{1\times 5 + 3\times -3}{4}, \dfrac{1\times -6 + 3\times 8}{4}\right)$ 1

$= \left(-1, 4\frac{1}{2}\right)$ 1

b) $P(x) = x^3 + 3x^2 - 1$

using remainder Theorem

$P(2) = 8 + 12 - 1$ 1

$= 19$ 1

∴ the remainder is 19.

c) $\displaystyle\int_0^1 \frac{1}{\sqrt{x^2+9}}\,dx = \left[\ln\left(x + \sqrt{x^2+9}\right)\right]_0^1$

$= \ln(1+\sqrt{10}) - \ln(0+3)$ 1

$= \ln\left(\dfrac{1+\sqrt{10}}{3}\right)$ 1

d) $\dfrac{2}{(x+5)} \le 1 \quad x \ne -5$

$2(x+5) \le (x+5)^2$ 1

$2x + 10 \le x^2 + 10x + 25$

$0 \le x^2 + 8x + 15$

$0 \le (x+5)(x+3)$ 1

∴ $x \le -5$ or $x \ge -3$

but $x \ne -5$

∴ $x < -5$ or $x \ge -3$ 1

e) $\displaystyle\int_2^4 \frac{x}{(x-1)^2} = \int_1^3 (u+1)\,u^{-2}\,du$

$\displaystyle = \int_1^3 (u^{-1} + u^{-2})\,du$ 1

$= \left[\log_e u - u^{-1}\right]_1^3$ 1

$= (\log 3 - \frac{1}{3}) - (\log_e 1 - 1)$

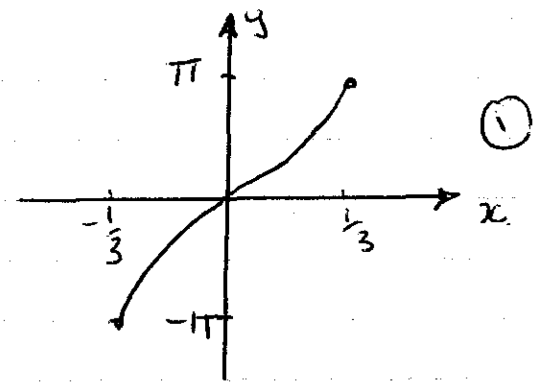
$= \log_e 3 + \frac{2}{3}$ 1

## Question 2

a) $-\pi \le y \le \pi$

$-\frac{1}{3} \le x \le \frac{1}{3}$

1 for either




1

b) $f'(a) = \lim\limits_{h \to 0} \dfrac{4(a+h)^2 - 1 - [4a^2 - 1]}{h}$ 1

$= \lim\limits_{h \to 0} \dfrac{4a^2 + 8ah + 4h^2 - 1 - 4a^2 + 1}{h}$

$= \lim\limits_{h \to 0} \dfrac{8ah + 4h^2}{h}$

$= \lim\limits_{h \to 0} 8a + 4h$

$= 8a$ 1

## Question 2 (continued)

c) $\frac{d}{dx}\left(3x^2 \cos^{-1} x\right)$

$= 6x\cos^{-1}x - \frac{3x^2}{\sqrt{1-x^2}}$ (1)

$u = 3x^2 \qquad v = \cos^{-1} x$

$u' = 6x \qquad v' = \frac{-1}{\sqrt{1-x^2}}$ (1)

d) $\int 4\cos^2 3x \, dx$

$= 4\int\left(\frac{1}{2} + \frac{1}{2}\cos 6x\right)dx$ (1)

since $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$

$= 4\left[\frac{x}{2} + \frac{1}{12}\sin 6x\right] + c$

$= 2x + \frac{1}{3}\sin 6x + c$ (1)

e) $\sin 2\theta = \sqrt{2}\cos\theta$

$2\sin\theta\cos\theta = \sqrt{2}\cos\theta$ (1)

$\sqrt{2}\sin\theta\cos\theta - \cos\theta = 0$

$\cos\theta\left(\sqrt{2}\sin\theta - 1\right) = 0$ (1)

$\therefore \cos\theta = 0$ or $\sin\theta = \frac{1}{\sqrt{2}}$

(1)

$\therefore \theta = \frac{\pi}{2}, \frac{3\pi}{2} \qquad \theta = \frac{\pi}{4}, \frac{3\pi}{4}$ (1)

$\therefore \theta = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{3\pi}{2}$

## Question 3

a) $x = 2\cos\theta \Rightarrow \cos\theta = \frac{x}{2}$ (1)

$y = 3\sin\theta \Rightarrow \sin\theta = \frac{y}{3}$

since $\sin^2\theta + \cos^2\theta = 1$

$\therefore \frac{x^2}{4} + \frac{y^2}{9} = 1$ (1)

b) $f(x) = \log_e x + 5x$

$f'(x) = \frac{1}{x} + 5$

$x_0 = 0.2$

Now $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ (1)

$= 0.2 - \frac{\log_e 0.2 + 5 \times 0.2}{\frac{1}{0.2} + 5}$ (1)

$= 0.261$ (correct to 3 dec pl.) (1)

c) i) $4 - x^2 \geq 0$ so $-2 \leq x \leq 2$

but $\sqrt{4-x^2} \neq 0 \therefore x \neq -2, 2$

$\therefore$ Natural domain is

$-2 < x < 2$

Question 3 (continued)

c) ii)

$$A = \int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx$$

$$= \left[\sin^{-1}\left(\frac{x}{2}\right)\right]_0^1 \quad 1$$

$$= \sin^{-1}\frac{1}{2} - \sin^{-1}0$$

$$= \frac{\pi}{6}. \quad 1$$

$\therefore$ Area $= \frac{\pi}{6}$ units$^2$.

d) Particle starts from centre of motion with positive velocity (ie: $x=0$.)

$\therefore$ general form of displacement time function is $x = a\sin nt$.

Now $a=2$, period $= \frac{2\pi}{n} = \frac{2\pi}{3}$

$\therefore n = 3$.

$\therefore$ required equation is

$$x = 2\sin 3t.$$

1 for correctly derived eqn.

ii)

$$v^2 = n^2(a^2 - x^2)$$

$$= 9(4 - x^2)$$

when $x = \sqrt{3}$

$$v^2 = 9(4-3) \quad 1$$

$$= 9$$

$\therefore v = \pm 3$

initially positive velocity

$\therefore v = 3$ m/s $\quad 1$.

iii)

For SHM max speed occurs at centre of motion i.e. $x = 0$

$$\therefore v^2 = n^2(a^2 - x^2)$$

$$= 9(4-0)$$

$$= 36$$

$\therefore v = \pm 6$ m/s

$\therefore$ Maximum speed is 6 m/s. $\quad 1$

Question 4

a) For $n=1$

$LHS = 1$

$RHS = \frac{1}{6}[(4-1)(1+1)$

$= \frac{1}{6}(3 \times 2)$

$= 1.$

$\therefore$ true for $n=1$ 1

Assume true for $n=k$.

$\therefore 1+6+15+\dots+k(2k-1)$

$= \frac{1}{6}k(4k-1)(k+1)$

For $n=k+1$

$1+6+15+\dots+k(2k-1)+(k+1)(2k+1)$

$= \frac{1}{6}k(4k-1)(k+1) + (k+1)(2k+1)$ 1

$= \frac{1}{6}(k+1)[k(4k-1)+6(2k+1)]$

$= \frac{1}{6}(k+1)[4k^2-k+12k+6]$

$= \frac{1}{6}(k+1)[4k^2+11k+6]$

$= \frac{1}{6}(k+1)(4k+3)(k+2)$

$\therefore$ If true for $n=k$ it is true for $n=k+1$ but true for $n=1$, $\therefore$ true for $n=2$ and hence 3, 4, 5 ...

$\therefore$ true for all integers $n \geq 1$. 1

b) i) $\frac{1}{1-\tan x} - \frac{1}{1+\tan x}$

$= \frac{1+\tan x-(1-\tan x)}{1-\tan^2 x}$ 1

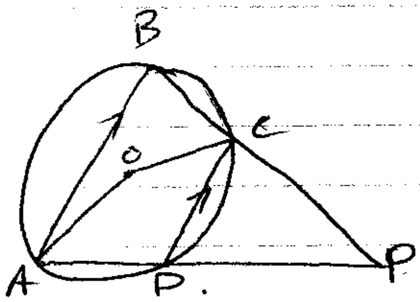
$= \frac{2\tan x}{1-\tan^2 x}$ 1

$= \tan 2x.$

ii) $\frac{1}{1-\tan\frac{\pi}{6}} - \frac{1}{1+\tan\frac{\pi}{6}} = \tan\frac{\pi}{3}$

$= \sqrt{3}$ 1

c)




i) $\angle PCD = \angle ABC$. 1 (corresp. $\angle$'s; $AB \parallel DC$)

$\angle PDC = \angle ABC$ 1 (exterior $\angle$ of cyclic Quad)

$\therefore \angle PCD = \angle PDC$

$\therefore \triangle PCD$ is isosceles.

$\therefore PC = PD$.

1

ii) $\angle PCD = \angle BAD$ (corres $\angle$'s; $AB \parallel C$

$\angle PCD = \angle ABC$ (exterior angle of cyclic Quad) 1

$\therefore \angle BAD = \angle ABC$

$\therefore \triangle ABP$ is isosceles

## Question 4 (continued)

iii) $\angle AOC = 2\angle ABC$ (1)

($\angle$ at centre twice angle at circumference)

$= \angle PDC + \angle PCD$

but $\angle PDC + \angle PCD + \angle CPD = 180^\circ$

($\angle$'s of a $\Delta = 180^\circ$)

$\therefore \angle AOC + \angle CPD = 180^\circ$ (1)

$\therefore$ OAPC is a cyclic Quad. (opp $\angle$'s supplementary)

## Question 5.

a) $x^{\log_2 x} = 8x^2$

take $\log_2$ on both sides.

$\log_2(x^{\log_2 x}) = \log_2 8x^2$ (1)

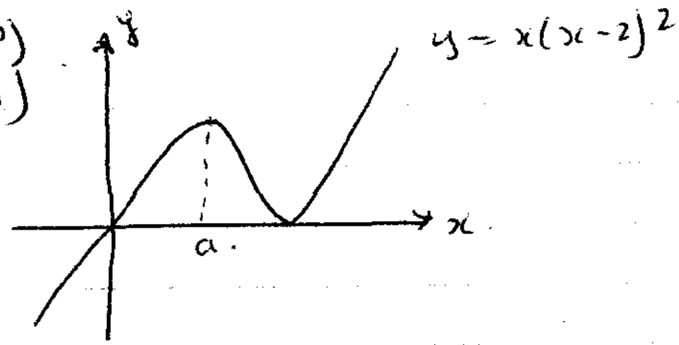
$\log_2 x \times \log_2 x = \log_2 8 + \log_2 x^2$

$(\log_2 x)^2 - 2\log_2 x - 3 = 0$

$(\log_2 x - 3)(\log_2 x + 1) = 0$ (1)

$\therefore \log_2 x = 3$ or $\log_2 x = -1$

$\therefore x = 8$ or $x = \frac{1}{2}$ (1)

b)
i) 


The inverse will exist if $x \le a$ where a is the turning pt.

$y' = 3x^2 - 8x + 4$ (1)

$= (3x-2)(x-2)$

$\therefore y' = 0$ if

$3x - 2 = 0$ or $x - 2 = 0$

$x = \frac{2}{3}$ $\quad$ $x = 2$

$\therefore f^{-1}(x)$ exists if $a \le \frac{2}{3}$ (1)

ii) Domain of $f^{-1}(x)$ is

$x \le (\frac{2}{3} - 2)^2(\frac{2}{3})$

$\therefore x \le \frac{32}{27}$ (1)

Question 5 (continued)

c) $\alpha+\beta+\gamma = -\frac{b}{a} = 4.$

$\alpha\beta+\beta\gamma+\gamma\alpha = \frac{c}{a} = 3$ 1

$\alpha\beta\gamma = -\frac{d}{a} = -2$

$\therefore \alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha)$

$= 16-2(3)$

$= 10.$ 1

d) let $f(m) = m^3-3m+2$

$f(1) = 1-3+2 = 0$

$\therefore m-1$ is a factor 1

$$\begin{array}{r} m^2+m-2 \\ m-1\overline{)m^3 \quad -3m+2} \\ \underline{m^3-m^2} \\ m^2-3m \\ \underline{m^2-m} \\ -2m+2 \\ \underline{-2m+2} \end{array}$$

$\therefore$ Factors are $(m-1)(m^2+m-2)$

i.e. $(m-1)(m-1)(m+2)$ 1

$(3x-4)^3-3(2x-5)-(3x+1))$

$= (3x-4)^3-6x+15-3x-1$

$= (3x-4)^3-9x+14.$

$= (3x-4)^3-3(3x-4)+2$

$\therefore (3x-4)^3-3(3x-4)+2=0.$ 1

let $m = 3x-4$

$\therefore (3x-4-1)^2(3x-4+2)=0.$

$\therefore (3x-5)^2=0$ or $3x-2=0$

$\therefore x=\frac{5}{3}$ or $x=\frac{2}{3}$ 1

## Question 6

a)(i) $\frac{d^2x}{dt^2} = 9(x-2).$

$\frac{d\,\frac{1}{2}v^2}{dx} = 9(x-2)$

$\therefore \frac{1}{2}v^2 = 9\int(x-2)\,dx$

$= 9\left(\frac{x^2}{2}-2x\right)+c.$

initially $x=4$, $v=-6$.

$\therefore 18 = 9(8-8)+c$

$\therefore c = 18.$ 1

$\therefore \frac{1}{2}v^2 = 9\left(\frac{x^2}{2}-2x\right)+18$

$v^2 = 9x^2-36x+36$

$= 9(x^2-4x+4)$

$= 9(x-2)^2$ 1

as required.

Question 6 (continued)

ii) $v^2 = 9(x-2)^2$

$\therefore v = \pm 3(x-2)$

$t=0, \ x=4 \quad v=-6$

$\therefore v = -3(x-2)$. 1

So $\frac{dx}{dt} = -3(x-2)$.

$\frac{dt}{dx} = \frac{-1}{3(x-2)}$

$t = -\frac{1}{3}\int \frac{1}{x-2}\,dx$

$= -\frac{1}{3}\log_e(x-2) + c$.

$t=0 \ x=4$

$\therefore 0 = -\frac{1}{3}\log_e 2 + c$

$\therefore c = \frac{1}{3}\log_e 2$

$\therefore t = \frac{1}{3}\log_e \frac{2}{x-2}$

$\therefore 3t = \log_e \frac{2}{x-2}$

$e^{3t} = \frac{2}{x-2}$

$x-2 = 2e^{-3t}$

$x = 2(1+e^{-3t})$ 1

iii) $v = -3(x-2)$

if $v=0 \quad x=2$ 1

if $x=2 \quad e^{-3t}=0$ which has no solution 1.

$\therefore$ $v$ is never $0$.

b) i) $\ddot{x}=0$

$\therefore \dot{x} = \int 0\,dt$

$= c$

when $t=0$, $\dot{x} = V\cos\theta$

$\therefore \dot{x} = V\cos\theta$ 1.

$x = \int V\cos\theta\,dt$

$= Vt\cos\theta + c_1$

when $t=0, x=0$.

$\therefore c_1 = 0$. 1.

$\therefore x = Vt\cos\theta$ as required.

ii) Max. height $= 2$ it occurs when $\dot{y}=0$.

$y = Vt\sin\theta - 5t^2 + 1$

$\dot{y} = V\sin\theta - 10t$.

if $\dot{y}=0$

$\therefore 0 = V\sin\theta - 10t$

$t = \frac{V\sin\theta}{10}$. 1.

## Question 6 b (continued)

Now $y = Vt\sin\theta - 5t^2 + 1$

$\therefore \quad 2 = \dfrac{V^2\sin^2\theta}{10} - \dfrac{5V^2\sin^2\theta}{100} + 1$

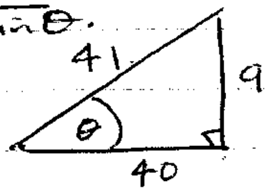
$1 = \dfrac{V^2\sin^2\theta}{20}$

$V^2 = \dfrac{20}{\sin^2\theta}$

$\therefore \quad V = \dfrac{\sqrt{20}}{\sin\theta}$ 1 $(V > 0)$

$= \dfrac{2\sqrt{5}}{\sin\theta}$

iii)




$V = \dfrac{\sqrt{20}}{\sin\theta}$

also $\theta = \tan^{-1}\dfrac{9}{40}$

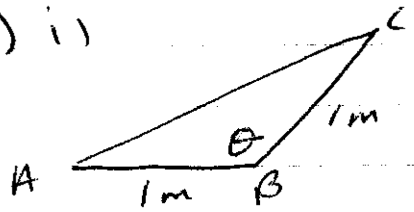
$\therefore \quad \sin\theta = \dfrac{9}{41}$ 1

$\therefore \quad V = \dfrac{2\sqrt{5} \times 41}{9}$

$= 20.37\ m/s^{-1}$ 1

## Question 7

a) i)



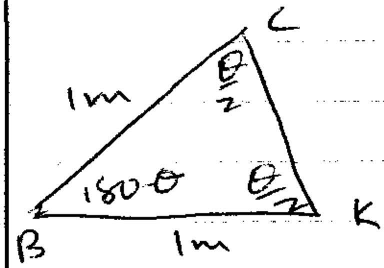

1

$\therefore \quad AC^2 = 1^2 + 1^2 - 2 \times 1 \times 1\cos\theta$

$= 2 - 2\cos\theta$

$\therefore \quad AC = \sqrt{2 - 2\cos\theta}$ as required

ii) In $\triangle BCK$



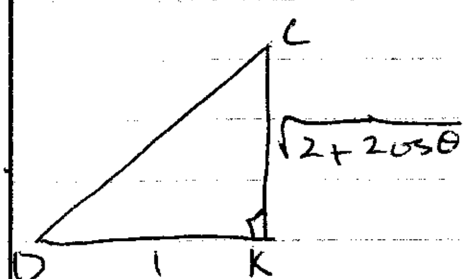

$\therefore \quad CK^2 = 1^2 + 1^2 - 2 \times 1 \times 1\cos(180 - \theta)$

$= 2 - 2\cos(180 - \theta)$

$= 2 + 2\cos\theta$

$CK = \sqrt{2 + 2\cos\theta}$ 1

In $\triangle CDK$




$\therefore \quad CD^2 = 1^2 + \left(\sqrt{2 + 2\cos\theta}\right)^2$ 1

$= 1 + 2 + 2\cos\theta$

$= 3 + 2\cos\theta$

$CD = \sqrt{3 + 2\cos\theta}$ as req'd

## Question 7 (continued)

iii) $l = AC + CD$

$= \sqrt{2-2\cos\theta} + \sqrt{3+2\cos\theta}$

so $\frac{dl}{d\theta} = \frac{1}{2}(2-2\cos\theta)^{-\frac{1}{2}} \times 2\sin\theta$

$+ \frac{1}{2}(3+2\cos\theta)^{-\frac{1}{2}} \times -2\sin\theta$

$= \frac{\sin\theta}{\sqrt{2-2\cos\theta}} - \frac{\sin\theta}{\sqrt{3+2\cos\theta}}$ 1

we require

$\frac{dl}{d\theta} = 0.$

$\frac{dl}{d\theta} = \frac{\sin\theta\sqrt{3+2\cos\theta} - \sin\theta\sqrt{2-2\cos\theta}}{(\sqrt{2-2\cos\theta})(\sqrt{3+2\cos\theta})}$

$\frac{dl}{d\theta} = 0$ when

$\sin\theta(\sqrt{3+2\cos\theta} - \sqrt{2-2\cos\theta}) = 0$

$\therefore \sin\theta = 0$ or $\sqrt{3+2\cos\theta} = \sqrt{2-2\cos\theta}$.

$\theta = 0°$ or $180°$ or $3+2\cos\theta = 2-2\cos\theta$

$1 = -4\cos\theta$

1 $\cos\theta = -\frac{1}{4}$

$\theta = \cos^{-1}(-\frac{1}{4})$

When $\theta = 0°$ trap door fully open $\therefore$ distance $= \sqrt{5}$.

if $\theta = 180°$ trap door is closed.

and $l = AK + KD$

$= 2 + 1$

$= 3m$.

These two are end points of function.

when $\theta = \cos^{-1}(-\frac{1}{4})$

$l = \sqrt{2 - 2\cos(\cos^{-1}(-\frac{1}{4}))} + \sqrt{3 + 2\cos(\cos^{-1}(-\frac{1}{4}))}$

$= \sqrt{2 - 2 \times -\frac{1}{4}} + \sqrt{3 + 2 \times -\frac{1}{4}}$

$= \sqrt{\frac{5}{2}} + \sqrt{\frac{5}{2}}$

$= 2\sqrt{\frac{5}{2}}$

$= \sqrt{10}$. 1

$\therefore$ max length of $l$ is $\sqrt{10}$.

check for maximum

$\cos^{-1}(-\frac{1}{4}) \approx 1.8234$ rad

$\theta = 1$ $\frac{dl}{d\theta} \approx 0.46$ $> 0$

$\theta = 2$ $\frac{dl}{d\theta} \approx -0.08$ $< 0$

1

$\therefore$ change of sign $\therefore$ maximum.

Question 7 (continued)

a) (i) $T = 20 + Ae^{-kt}$

$$\frac{dT}{dt} = -kAe^{-kt}$$

$$= -k(T-20) \quad 1$$

(ii) when $t=0$, $T=95$

$\therefore 95 = 20 + A$

$\therefore A = 75$

$\therefore T = 20 + 75e^{-kt}$.

if $t = 10$, $T = 70$

$\therefore 70 = 20 + 75e^{-10k}$

$$e^{-10k} = \frac{2}{3}$$

$$\therefore k = -\frac{1}{10}\ln\frac{2}{3} \quad 1$$

$\therefore$ if $t = 15$

$$T = 20 + 75e^{\frac{1}{10}\ln\frac{2}{3}(15)}$$

$$= 60.82\ldots \quad 1$$

$$= 61\ ^\circ C.$$

iii) $35 = 20 + 75e^{\frac{1}{10}\ln\frac{2}{3}t}$

$$t = \frac{\ln\frac{1}{5}}{\frac{1}{10}\ln\frac{2}{3}}$$

$$= 39.69\ldots \quad 1$$

$$= 40 \text{ min}$$

iv) $\dfrac{dT}{dt} = 75 \times \frac{1}{10}\ln\frac{2}{3}e^{\frac{1}{10}\ln\frac{2}{3}}$

$t = 40$. $\therefore \dfrac{dT}{dt} = -0.600\ldots$

$$= -0.6 \text{ deg/min} \quad 1$$

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