JG
AW
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Name:

Class: 12MTX

Teacher:

CHERRYBROOK TECHNOLOGY HIGH SCHOOL

2008 AP4

YEAR 12 TRIAL HSC EXAMINATION

MATHEMATICS EXTENSION 1

Time allowed - 2 HOURS
(Plus 5 minutes' reading time)

DIRECTIONS TO CANDIDATES:

- Attempt all questions.
- All questions are of equal value.
- Each question is to be commenced on a new page clearly marked Question 1, Question 2, etc on the top of the page. **
- All necessary working should be shown in every question. Full marks may not be awarded for careless or badly arranged work.
- Approved calculators may be used. Standard Integral Tables are provided
- Your solutions will be collected in one bundle stapled in the top left corner.

Please arrange them in order, Q1 to 7.

***Each page must show your name and your class. ***

Marks

QUESTION ONE

a) Find the integral $\int \sec^2 2x\, dx$. **1**

b) Evaluate the limit $\lim_{x \to 0} \dfrac{\sin\left(\frac{5x}{2}\right)}{3x}$ **2**

c) Find the exact value of $\sin 15^\circ$. **2**

d) Find the values of x for which $\dfrac{x}{x+1} \geq 2$. **3**

e) Find the point that divides the line joining A (2, 8) and B (-5, 7) in the ratio 2 : 3 externally. **2**

f) The line L makes an angle of 45° with the line $x - 2y + 3 = 0$. Find the gradient, m, of line L given that $m > 0$. **2**

QUESTION TWO (START A NEW PAGE)

a) (i) Prove that $\cot\left(\dfrac{\alpha}{2}\right) = \dfrac{\sin\alpha}{1 - \cos\alpha}$. **2**

(ii) Hence find the value of $\cot\left(\dfrac{\alpha}{2}\right)$ when $\sin\alpha = \dfrac{3}{5}$ and $\dfrac{\pi}{2} < \alpha < 2\pi$. **2**

b) (i) Show that there is a real root of the equation $2\tan x + 2x - \pi = 0$ between $x = 0.6$ and $x = 0.75$. **1**

(ii) Start with $x = 0.6$, and use one application of Newton's method to approximate the root of $2\tan x + 2x - \pi = 0$ in part (i). Give your answer correct to 2 decimal places. **2**

c) $(2 + 5x)^n$ is expanded in ascending powers of x.

(i) Write down the coefficient of the 8th term in the expansion. **1**

(ii) Show that $\dfrac{\textit{the coefficient of the 8th term}}{\textit{the coefficient of the 10th term}} = \dfrac{288}{25(n-7)(n-8)}$. **2**

(iii) Hence determine the value of n if $\dfrac{\textit{the coefficient of the 8th term}}{\textit{the coefficient of the 10th term}} = \dfrac{36}{175}$. **2**

Marks

QUESTION THREE (START A NEW PAGE)

a) Find the exact value of $\displaystyle\int_0^{4\sqrt{3}} \frac{dx}{16 + x^2}$. 2

b) Find the general solution to the equation $6\sin^2 x + 5\sin x - 4 = 0$. 2

c) (i) Use the substitution $u = 1 - x^2$ to find the integral $\displaystyle\int \frac{x}{\sqrt{1 - x^2}}\, dx$. 2

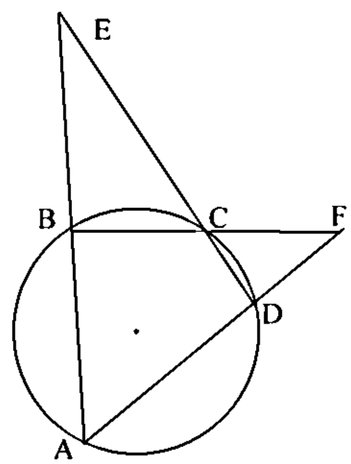
(ii) Find the derivative of $x\sin^{-1}x$. 1

(iii) Hence find the integral $\displaystyle\int \sin^{-1}x\, dx$. 2

d) Given $f(x) = 3\sin^{-1}(4x - 1)$,

(i) find the domain and range of $f(x)$. 2

(ii) Sketch the graph of $y = f(x)$. 1



QUESTION FOUR (START A NEW PAGE)

a) In the diagram shown on the right, ABE, BCF, ADF and ECD are all straight lines and $\angle AED = \angle BFD$.

(i) Explain why $\angle ABC = \angle ADC$. 2

(ii) Hence prove that AC is a diameter. 1

b) Prove, by mathematical induction, that

$$11 \times 2! + 19 \times 3! + 29 \times 4! + \ldots + (n^2 + 5n + 5)(n + 1)! = (n + 4)[(n + 2)!] - 8$$ 3

c) Show that if $x = \alpha$ is a double root of the equation $P(x) = 0$, then $x = \alpha$ is also a root of the equation $P'(x) = 0$. 2

d) $P(x) = kx^4 - (2k + 5)x^3 + (2k + 10)x^2 - (2k + 5)x + k$, where k is an integer.

(i) Show that $x = 1$ is a double root of $P(x) = 0$. [You can assume the result of part (c)] 1

(ii) Show that if $x = \alpha$ $(\alpha \neq 1)$ is a root of $P(x)=0$, then $x = \dfrac{1}{\alpha}$ is another root of $P(x)=0$. 2

(iv) Hence, show that $\alpha^2 + \dfrac{1}{\alpha^2} = \dfrac{25}{k^2} - 2$. 1

QUESTION FIVE (START A NEW PAGE) **Marks**

a) The velocity, $v\ ms^{-1}$, of a particle travelling along a straight line is given by the expression $v^2 = 48 + 16x - 4x^2$.

(i) Show that the particle executes simple harmonic motion. **2**

(ii) Find the amplitude of the motion. **1**

(iii) If the particle starts from the point furthest to the right, find the expression for its position x, in terms of time t. **2**

b) Robin Hood shot an arrow with a speed of $V\ ms^{-1}$ at an angle of 60° to the horizontal.

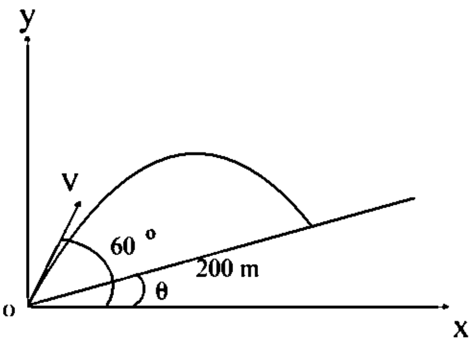
(i) Derive the expression for the vertical component of the displacement of the arrow at time t. [Ignore air resistance and you may assume the result $\dot{} = - \quad + \frac{\sqrt{}}{}$] **1**

(ii) Show that the Cartesian equation of the path of the arrow is given by $y = \sqrt{3}\ x - \frac{2g x^2}{V^2}$. **1**

[You may assume $x = \frac{Vt}{2}$.]

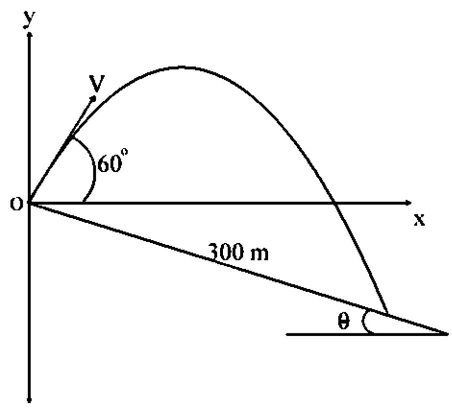
(iii) Robin Hood stood at the bottom of a hill inclined at an angle θ to the horizontal and shot an arrow at an angle of 60° to the horizontal at a speed of $V\ ms^{-1}$. He could shoot 200m up the hill. Use the result of part (ii) to show that **2**

$$\tan\theta = \sqrt{3} - \frac{400g\cos\theta}{V^2}.$$



(iv) If he was standing on the hill and shot an arrow at the same speed of $V\ ms^{-1}$ and the same angle of projection of 60°, but down the hill, he could shoot 300m down the hill. Show that **1**

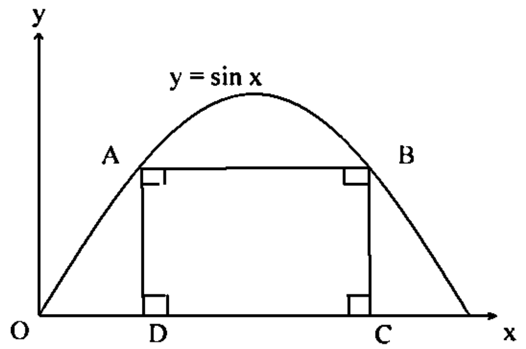
$$\tan\theta = \frac{600g\cos\theta}{V^2} - \sqrt{3}.$$



v) Hence, show that $\theta = \tan^{-1}\left(\frac{\sqrt{3}}{5}\right)$. **2**

QUESTIONS SIX (START A NEW PAGE) **Marks**

a)



The diagram shows a rectangle inscribed under the curve $y = \sin x$ in $0 \le 0 \le \pi$.

(i) The coordinates of point A are $(x, \sin x)$. Explain why the coordinates of B are $(\pi - x\ ,\ \sin x)$. **1**

(ii) Show that the area $A(x)$ of the rectangle ABCD is given by $A(x) = (\pi - 2x)\sin x$. **1**

(iii) Hence determine the dimensions of the rectangle with the largest area that can be inscribed under the graph $y = \sin x\ ,\ 0 \le x \le \pi$. [You may assume the result in part (b) (ii) of Question Two.]. **3**

b) Given $f(x) = \sin^{-1}\left(\frac{1}{x}\right) + \cos^{-1}\left(\frac{1}{x}\right)$.

(i) Find the derivative of $f(x)$. **2**

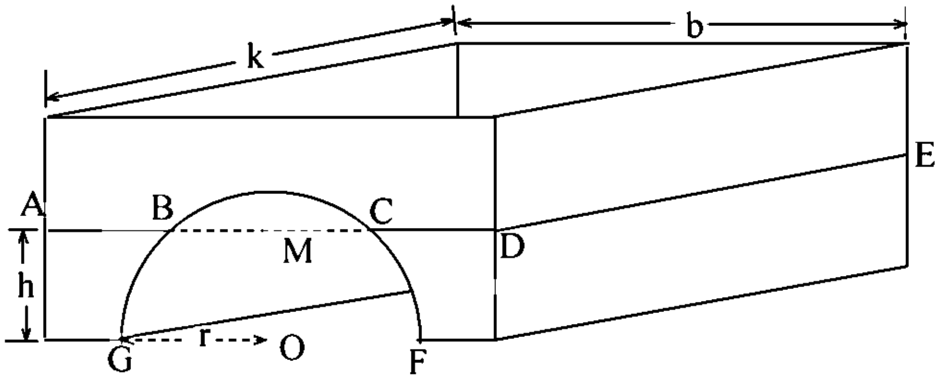
(ii) Find the domain of $f(x)$ and sketch the graph of $y = f(x)$. **3**

(iii) Find the coordinates of the point(s) of intersection between $y = f(x)$ and $y = f^{-1}(x)$. **2**

Question Seven is on the next page....

QUESTIONS SEVEN (START A NEW PAGE) **Marks**

a)



The diagram above shows a water trough in the shape of a rectangular prism with a half cylindrical cavity, of radius r, at the bottom. The length and width of the trough are k and b respectively. It is partly filled with water to a depth of h ($h < r$). Let O be the centre of the semi-circle BCFG and M is the mid-point of BC. AB, CD and DE show the water surface inside the trough.

(i) Express BM in terms of r and h. **1**

(ii) Hence show that the area A of the water surface is given by $A = k\left[b - 2\sqrt{r^2 - h^2}\,\right]$. **1**

(iii) The water in the trough is evaporating in such a way that h is decreasing at a constant rate. The trough measurements are k = 3 m, b =2 m, r = 50 cm, and the water surface is descending at a constant rate of 0.6 cm/day. Find the rate at which the surface area is decreasing when the depth of the water in the trough is 30 cm. **3**

b) The variable point P has coordinates $(a\cos 2\theta,\ a\cos\theta)$.

(i) Show that P lies on the curve $y^2 = \frac{a}{2}(x + a)$. **2**

(ii) Sketch the locus of P as θ varies, taking account of any restriction on x and y. Label the focus and the vertex. **3**

(iii) Find the equation of the tangent to the curve at the point where $\theta = \frac{\pi}{3}$. **2**

E N D

Solution to Ext 1 AP4 2008

Question 1

a) $\int \sec^2 2x\,dx = \frac{1}{2}\tan 2x + C$ $\boxed{1\checkmark}$

b) $\lim_{x\to 0} \frac{\sin\left(\frac{5x}{2}\right)}{3x} = \lim_{x\to 0} \frac{5}{6}\cdot\frac{\sin\frac{5x}{2}}{\frac{5x}{2}}$ $\boxed{1\checkmark}$

$= \frac{5}{6}$ $\boxed{\checkmark}$

c) $\sin 15^\circ = \sin(45^\circ - 30^\circ)$

$= \sin 45^\circ \cos 30^\circ - \sin 30^\circ \cos 45^\circ$ $\boxed{1\checkmark}$

$= \frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2} - \frac{1}{2}\times\frac{1}{\sqrt{2}}$

$= \frac{\sqrt{3}-1}{2\sqrt{2}}$ $\boxed{1\checkmark}$

Alt 1

$\sin 15^\circ = \sin(60^\circ - 45^\circ)$

$= \sin 60^\circ \cos 45^\circ - \sin 45^\circ \cos 60^\circ$ $\boxed{1\checkmark}$

$= \frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\times\frac{1}{2}$

$= \frac{\sqrt{3}-1}{2\sqrt{2}}$ $\boxed{1\checkmark}$

Alt 2

$2\sin^2 15^\circ = 1 - \cos 30^\circ$

$= 1 - \frac{\sqrt{3}}{2}$

$\therefore \sin^2 15^\circ = \frac{2-\sqrt{3}}{4}$ $\boxed{1\checkmark}$

$\sin 15^\circ = \sqrt{\frac{2-\sqrt{3}}{4}}$, $-\sqrt{\frac{2-\sqrt{3}}{4}}$ (rejected, since $\sin 15^\circ > 0$)

$= \frac{\sqrt{3}-1}{2\sqrt{2}}$ $\boxed{1\checkmark}$

d) $\frac{x}{x+1} \geq 2$

$x(x+1) \geq 2(x+1)^2$ $\boxed{1\checkmark}$

$2(x+1)^2 - x(x+1) \leq 0$

$(x+1)[2(x+1) - x] \leq 0$

$(x+1)(x+2) \leq 0$ $\boxed{1\checkmark}$

$-2 \leq x < -1$ $\boxed{1\checkmark}$

[Both inequality signs must be correct]

e) At the point of division,

$x = \frac{(-2)(-5)+(3)(2)}{-2+3}$

$= 16$ $\boxed{1\checkmark}$

$y = \frac{(-2)(7)+(3)(8)}{-2+3}$

$= 10$ $\boxed{1\checkmark}$

$\therefore$ The point is $(16, 10)$

f) $x - 2y + 3 = 0$

$y = \frac{1}{2}(x+3)$

$m_1 = \frac{1}{2}$

$\tan 45^\circ = \left|\frac{m - \frac{1}{2}}{1 + \frac{m}{2}}\right|$ $\boxed{1\checkmark}$

$$\therefore \frac{m-\frac{1}{2}}{1+\frac{m}{2}} = 1 \quad \text{or} \quad \frac{m-\frac{1}{2}}{1+\frac{m}{2}} = -1$$

$$\frac{2m-1}{2+m} = 1 \qquad \frac{2m-1}{2+m} = -1$$

$$2m-1 = 2+m \qquad 2m-1 = -2-m$$

$$m = 3 \ \boxed{\checkmark} \qquad 3m = -1$$

$$m = -\tfrac{1}{3} \ \text{(rejected, } m>0\text{)}$$

Question 2

(a) (i) Let $t = \tan\frac{\alpha}{2}$

then $$\frac{\sin\alpha}{1-\cos\alpha} = \frac{\frac{2t}{1+t^2}}{1-\frac{1-t^2}{1+t^2}} \quad \boxed{\checkmark}$$

$$= \frac{2t}{1+t^2-(1-t^2)}$$

$$= \frac{2t}{2t^2}$$

$$= \frac{1}{t} \quad \boxed{\checkmark}$$

$$= \frac{1}{\tan\frac{\alpha}{2}}$$

$$= \cot\frac{\alpha}{2}$$

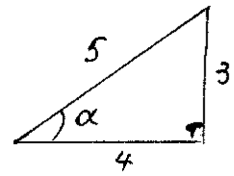
(ii) since $\sin\alpha > 0$ and $\frac{\pi}{2} < \alpha < 2\pi$

$\therefore \frac{\pi}{2} < \alpha < \pi$

hence $\frac{\pi}{4} < \frac{\alpha}{2} < \frac{\pi}{2}$

then $\cos\alpha < 0$ and $\cot\frac{\alpha}{2} > 0$

$$\cos\alpha = -\frac{4}{5} \quad \boxed{\checkmark}$$



[or $\cos\alpha = \sqrt{1-\sin^2\alpha}$

$= \sqrt{1-\left(\frac{3}{5}\right)^2}$

$= -\frac{4}{5}$ since $\cos\alpha < 0$]

By result of (i)

$$\cot\frac{\alpha}{2} = \frac{\sin\alpha}{1-\cos\alpha}$$

$$= \frac{\frac{3}{5}}{1-\left(-\frac{4}{5}\right)}$$

$$= \frac{3}{5+4}$$

$$= \frac{1}{3} \quad \boxed{\checkmark}$$

b) (i) Let $f(x) = 2\tan x + 2x - \pi$

$f(0.6) = -0.5733 < 0$

$f(0.75) = 0.2216 > 0$

since $f(0.6)$ and $f(0.75)$ are opposite in sign, $\boxed{\checkmark}$ there must be a real of $f(x)=0$ between $x=0.6$ and $x=0.75$

(ii) $f'(x) = 2\sec^2 x + 2$

With one application of Newton's method, the next approximation is given by

$$x = 0.6 - \frac{f(0.6)}{f'(0.6)}$$ ☑

$$= 0.6 - \frac{-0.5733}{4.9361}$$

$$= 0.71614$$

$$= 0.72 \quad (2 \text{ d.pl.})$$ ☑

c) (i) $(2+5x)^n = \sum_{r=0}^{n} \binom{n}{r} 2^{n-r} (5x)^r$

$\therefore$ coeff. of the 8^{th} term $= \binom{n}{7} 2^{n-7} 5^7$ ☑

(ii) coeff of the 10^{th} term $= \binom{n}{9} 2^{n-9} 5^9$

$$\frac{\text{coeff. of the } 8^{th} \text{ term}}{\text{coeff of the } 10^{th} \text{ term}} = \frac{\binom{n}{7} 2^{n-7} 5^7}{\binom{n}{9} 2^{n-9} 5^9}$$ ☑

$$= \frac{n!}{7!(n-7)!} \times \frac{9!(n-9)!}{n!} \frac{2^2}{5^2}$$

$$= \frac{9 \times 8}{(n-7)(n-8)} \times \frac{4}{25}$$ ☑

$$= \frac{288}{25(n-7)(n-8)}$$

(iii) $$\frac{\text{coeff. of the } 8^{th} \text{ term}}{\text{coeff. of the } 10^{th} \text{ term}} = \frac{36}{175}$$

$$\left.\begin{aligned} \frac{288}{25(n-7)(n-8)} &= \frac{36}{175} \\ \frac{8}{(n-7)(n-8)} &= \frac{1}{7} \\ 56 &= (n-7)(n-8) \\ 56 &= n^2 - 15n + 56 \end{aligned}\right\}$$ ☑

$$n^2 - 15n = 0$$

$$n(n-15) = 0$$

$\therefore n = 0$ (rejected), $n = 15$. ☑

Question 3

a) $$\int_0^{4\sqrt{3}} \frac{dx}{16+x^2} = \frac{1}{4}\left[\tan^{-1}\frac{x}{4}\right]_0^{4\sqrt{3}}$$ ☑

$$= \frac{1}{4}\left[\tan^{-1}\sqrt{3} - 0\right]$$

$$= \frac{1}{4} \cdot \frac{\pi}{3}$$

$$= \frac{\pi}{12}$$ ☑

b) $6\sin^2 x + 5\sin x - 4 = 0$

$(2\sin x - 1)(3\sin x + 4) = 0$

$\therefore \sin x = \frac{1}{2}$ or $\sin x = -\frac{4}{3}$ (rejected) $\boxed{\checkmark}$

$x = \frac{\pi}{6}, \pi - \frac{\pi}{6}, 2\pi + \frac{\pi}{6}, 3\pi - \frac{\pi}{6}, \ldots$

$= n\pi + (-1)^n \frac{\pi}{6}$ $\boxed{\checkmark}$

(c) (i)

$$\left.\begin{aligned} u &= 1 - x^2 \\ \frac{du}{dx} &= -2x \end{aligned}\right\} \boxed{\checkmark}$$

$\therefore x\,dx = -\frac{1}{2}du$

$$\int \frac{x\,dx}{\sqrt{1-x^2}} = -\int \frac{\frac{1}{2}du}{\sqrt{u}}$$

$$= -\sqrt{u} + C$$

$$= -\sqrt{1-x^2} + C \quad \boxed{\checkmark}$$

(ii) $\frac{d}{dx} x\sin^{-1}x = \sin^{-1}x + \frac{x}{\sqrt{1-x^2}}$ $\boxed{\checkmark}$

(iii) $\int \sin^{-1}x + \frac{x}{\sqrt{1-x^2}}\,dx = x\sin^{-1}x + C$

$\int \sin^{-1}x\,dx + \int \frac{x}{\sqrt{1-x^2}}\,dx = x\sin^{-1}x + C$ $\boxed{\checkmark}$

$\int \sin^{-1}x\,dx - \sqrt{1-x^2} = x\sin^{-1}x + C$

$\int \sin^{-1}x\,dx = x\sin^{-1}x + \sqrt{1-x^2} + C$ $\boxed{\checkmark}$

d) (i) domain: $-1 \le 4x - 1 \le 1$

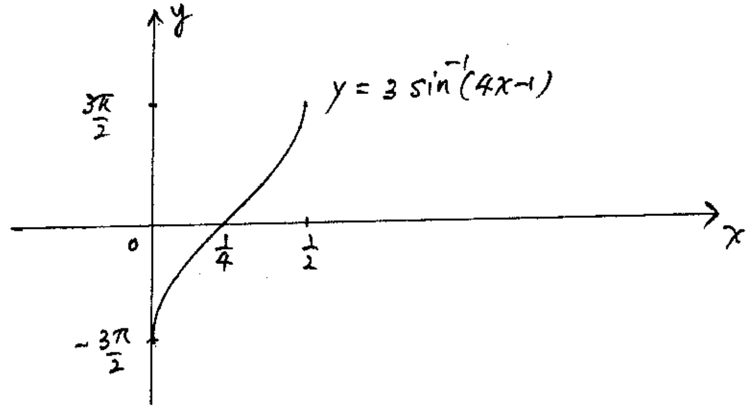
$0 \le 4x \le 2$

$0 \le x \le \frac{1}{2}$ $\boxed{\checkmark}$

range: $-\frac{\pi}{2} \le \frac{y}{3} \le \frac{\pi}{2}$

$-\frac{3\pi}{2} \le y \le \frac{3\pi}{2}$ $\boxed{\checkmark}$

(ii)



$\boxed{\checkmark}$

Question 4

a) In $\triangle ABF$ and $\triangle ADE$

$\angle BAF = \angle DAE$ (common)

$\angle BFA = \angle DEA$ (given)

$\therefore \triangle ABF \,|||\, \triangle ADE$ $\boxed{\checkmark}$ (equiangular $\triangle$'s)

$\therefore \angle ABC = \angle ADC$ $\boxed{\checkmark}$ (corresponding $\angle$'s of similar $\triangle$'s)

Alt

$\angle BCE = \angle DCE$ (vert. opp. $\angle$'s)

$\angle BEC = \angle CED$ (given)

$\angle ABC = \angle BEC + \angle BCE$ $\boxed{\checkmark}$ (ext $\angle$ of $\triangle$ equals int. opp $\angle$'s sum)

$= \angle CED + \angle DCE$ $\boxed{\checkmark}$ (proved)

$= \angle ADC$ (ext $\angle$ of $\triangle$)

1.4

(ii) $\angle ABC + \angle ADC = 180^\circ$ (opp $\angle$s of cyclic quad are supplementary)

$\therefore\ 2\angle ABC = 180^\circ$ ($\angle ABC = \angle ADC$, proven)

$\angle ABC = 90^\circ$ ✓

$\therefore$ AC is a diameter ($\angle$ in semi-circle is a rt $\angle$.)

b) When $n=1$,

$11 \times 2! = 22$

$(n+4)(n+2)! - 8 = 5 \times 3! - 8$
$= 22$

$\therefore$ It is true for $n=1$ ✓

Assume it is true for k, where k is an integer,

ie $11\times 2! + 19\times 3! + 29\times 4! + \cdots + (k^2+5k+5)(k+1)!$
$= (k+4)(k+2)! - 8$

then $11\times 2! + 19\times 3! + 29\times 4! + \cdots + (k^2+5k+5)(k+1)!$
$+ [(k+1)^2 + 5(k+1) + 5](k+2)!$

$= (k+4)(k+2)! - 8 + (k^2+7k+11)(k+2)!$ by assumption. ✓

$= (k+2)!\,[(k+4) + (k^2+7k+11)] - 8$

$= (k+2)!\,(k^2+8k+15) - 8$

$= (k+2)!\,(k+3)(k+5)$

$= (k+5)(k+3)! - 8$

$= [(k+1)+4][(k+1)+2]! - 8$ ✓

$\therefore$ It will be true for $n = k+1$ if it is true for $n=k$. Since it is proved true for $n=1$, $\therefore$ it will be true for $n = 2, 3, 4, \ldots$ all integers n.

(c) Since $x=\alpha$ is a double root of $P(x)=0$

$\therefore\ P(x) = (x-\alpha)^2 Q(x)$ ✓

where $Q(x)$ is a polynomial in x.

$P'(x) = 2(x-\alpha)Q(x) + (x-\alpha)^2 Q'(x)$

$= (x-\alpha)[2Q(x) + (x-\alpha)Q'(x)]$ ✓

$\therefore\ x-\alpha$ is a factor of $P'(x)$

ie $x=\alpha$ is a root of $P'(x)=0$

d) (i) $P(x) = kx^4 - (2k+5)x^3 + (2k+10)x^2 - (2k+5)x + k$

$P'(x) = 4kx^3 - 3(2k+5)x^2 + 2(2k+10)x - (2k+5)$

$P'(1) = 4k - 3(2k+5) + 2(2k+10) - (2k+5)$

$= 0$

$\therefore\ x=1$ is a root of $P'(x)=0$, hence $x=1$ is a double root of $P(x)=0$ by part (c). ✓

(ii) Since $x=1$ is a double root and $x=\alpha$ is a root, $\therefore$ let the 4^{th} root be β.

product of roots $(1)(1)\alpha\beta = \frac{k}{k}$ ✓

$\alpha\beta = 1$

$$\beta = \frac{1}{\alpha}$$

$\therefore x = \frac{1}{\alpha}$ is another root of $P(x)=0$ $\boxed{\checkmark}$

(iv) sum of squares of root

$$1^2+1^2+\alpha^2+\frac{1}{\alpha^2} = (\text{sum of root})^2 - 2\times\text{sum of roots taken two at a time}$$

$$= \left(\frac{2k-5}{k}\right)^2 - \frac{2(2k+10)}{k}$$

$$= \frac{4k^2-20k+25-4k^2-20k}{k^2}$$

$$= \frac{25}{k^2} \quad \boxed{\checkmark}$$

$$\therefore \alpha^2+\frac{1}{\alpha^2} = \frac{25}{k^2}-2$$

Question 5

a) (i) $\ddot{x} = \frac{d}{dx}\left(\frac{v^2}{2}\right)$

$$= \frac{d}{dx}(24+8x-2x^2) \quad \boxed{\checkmark}$$

$$= 8-4x$$

$$= -4(x-2) \quad \boxed{\checkmark}$$

$\therefore$ particle executes SHM centre at $x=2$

(ii) $v^2 \geq 0$

$$\therefore 48+16x-4x^2 \geq 0$$

$$x^2-4x-12 \leq 0$$

$$(x+2)(x-6) \leq 0$$

$$-2 \leq x \leq 6$$

$\therefore$ Amplitude $= 4$ $\boxed{\checkmark}$

(iii) Since $\ddot{x} = -4(x-2)$ from (i)

$\therefore n = 2$

Let the expression for displacement be

$$x = 4\sin(2t+\alpha)+2 \quad \boxed{\checkmark}$$

$x = 6$ when $t=0$

$$6 = 4\sin\alpha + 2$$

$$4 = 4\sin\alpha$$

$$\alpha = \frac{\pi}{2}$$

$$\therefore x = 4\sin(2t+\tfrac{\pi}{2})+2$$

$$= 4\cos 2t + 2 \quad \boxed{\checkmark}$$

Alt 1 Assume $x = 4\cos(2t+\theta)+2$ $\boxed{\checkmark}$

$x = 6$ when $t=0$

$$6 = 4\cos\theta + 2$$

$$4 = 4\cos\theta$$

$$\theta = 0$$

$$\therefore x = 4\cos 2t + 2 \quad \boxed{\checkmark}$$

Alt 2.

$$v^2 = 48 + 16x - 4x^2$$

$$v = 2\sqrt{12+4x-x^2}$$

$$\frac{dx}{dt} = 2\sqrt{12+4x-x^2}$$

$$\int \frac{dx}{\sqrt{12+4x-x^2}} = \int 2\,dt$$

$$\int \frac{dx}{\sqrt{16-(x-2)^2}} = 2t + c$$

$$\sin^{-1}\frac{x-2}{4} = 2t + c \qquad \boxed{\checkmark}$$

$$\frac{x-2}{4} = \sin(2t+c)$$

$\because x = 6$ when $t = 0$

$$\frac{6-2}{4} = \sin c$$

$$1 = \sin c$$

$$\therefore c = \frac{\pi}{2}$$

$$\therefore \frac{x-2}{4} = \sin\left(2t+\frac{\pi}{2}\right)$$

$$x = 4\sin\left(2t+\frac{\pi}{2}\right) + 2$$

$$= 4\cos 2t + 2 \qquad \boxed{\checkmark}$$

b) (i) $\dot{y} = -gt + \frac{\sqrt{3}V}{2}$

$\therefore y = -\frac{1}{2}gt^2 + \frac{\sqrt{3}Vt}{2} + c$

$y = 0$ when $t = 0$

$\therefore c = 0$ $\qquad \boxed{\checkmark}$

$\therefore y = -\frac{1}{2}gt^2 + \frac{\sqrt{3}Vt}{2}$ $\qquad (1)$

(ii) $x = \frac{Vt}{2}$

$\therefore t = \frac{2x}{V}$ $\qquad (2)$

Put (2) into (1)

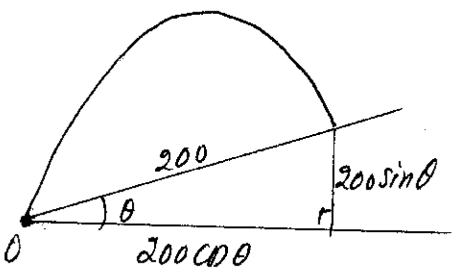
$y = -\frac{1}{2}g\left(\frac{2x}{V}\right)^2 + \frac{\sqrt{3}V}{2}\left(\frac{2x}{V}\right)$ $\qquad \boxed{\checkmark}$

$= -\frac{2gx^2}{V^2} + \sqrt{3}x$

$\therefore y = \sqrt{3}x - \frac{2gx^2}{V^2}$ $\qquad (3)$

iii) The point where the arrow hits the hillslope is

$(200\cos\theta, 200\sin\theta)$



Substitute into (3)

$200\sin\theta = \sqrt{3}(200\cos\theta) - \frac{2g}{V^2}(200\cos\theta)^2$ $\qquad \boxed{\checkmark}$

$$200\sin\theta = 200\sqrt{3}\cos\theta - \frac{80000\,g\cos^2\theta}{V^2}$$

$$\left.\begin{aligned} \sin\theta &= \sqrt{3}\cos\theta - \frac{400\,g\cos^2\theta}{V^2} \\ \frac{\sin\theta}{\cos\theta} &= \sqrt{3} - \frac{400\,g\cos\theta}{V^2} \\ \text{ie } \tan\theta &= \sqrt{3} - \frac{400\,g\cos\theta}{V^2} \end{aligned}\right\} \boxed{\checkmark} \qquad (4)$$

(iv) Similarly, the pt where the arrow hits the downward slope is $(300\cos\theta, -300\sin\theta)$

$$\left.\begin{aligned} \therefore -300\sin\theta &= \sqrt{3}(300\cos\theta) - \frac{2g}{V^2}(300\cos\theta)^2 \\ -\frac{\sin\theta}{\cos\theta} &= \sqrt{3} - \frac{600\,g\cos\theta}{V^2} \\ \text{ie } \tan\theta &= \frac{600\,g\cos\theta}{V^2} - \sqrt{3} \end{aligned}\right\} \boxed{\checkmark} \qquad (5)$$

(V) $(4)\times 3 \quad 3\tan\theta = 3\sqrt{3} - \dfrac{1200\,g\cos\theta}{V^2} \quad (6)$

$(5)\times 2 \quad 2\tan\theta = \dfrac{1200\,g\cos\theta}{V^2} - 2\sqrt{3} \quad (7) \quad \boxed{\checkmark}$

$(6)+(7) \quad 5\tan\theta = \sqrt{3}$

$$\therefore \tan\theta = \frac{\sqrt{3}}{5} \qquad \boxed{\checkmark}$$

$$\theta = \tan^{-1}\frac{\sqrt{3}}{5}$$

Question 6

a) (i) y-coordinate of point B = y-coord. of pt A and given $\sin x = y$, the other solution to this equation is $\pi - x$ since $\sin(\pi - x) = \sin x$, $\boxed{\checkmark}$

$\therefore$ the coordinates of pt B are $(\pi - x, \sin x)$

(ii)
$$\begin{aligned} AB &= (\pi - x) - x \\ &= \pi - 2x \qquad \boxed{\checkmark} \\ AD &= \sin x \end{aligned}$$

$\therefore$ Area of $ABCD$ is $A(x) = (\pi - 2x)\sin x$

(iii)
$$\frac{dA}{dx} = -2\sin x + (\pi - 2x)\cos x$$

$$\left.\begin{aligned} \frac{d^2A}{dx^2} &= -2\cos x - 2\cos x - (\pi - 2x)\sin x \\ &= -4\cos x - (\pi - 2x)\sin x \\ &< 0 \qquad \text{for all } 0 < x < \frac{\pi}{2} \end{aligned}\right\} \boxed{\checkmark}$$

$\therefore$ A will be max when $\dfrac{dA}{dx} = 0$

ie
$$-2\sin x + (\pi - 2x)\cos x = 0$$

$$2\sin x = (\pi - 2x)\sin x$$

$$2\tan x = \pi - 2x$$

ie
$$2\tan x + 2x - \pi = 0 \qquad \boxed{\checkmark}$$

by result of (b) (i) of Q 2 $\quad x = 0.72$,

$$\begin{aligned} \sin x &= \sin 0.72 \\ &= 0.6594 \end{aligned}$$

$\therefore$ The dimensions of the largest rectangle are

$\pi - 2(0.72)$ by 0.66, ie 1.70×0.66 ☑

b) (i) $f(x) = \sin^{-1}\frac{1}{x} + \cos^{-1}\frac{1}{x}$

$$f'(x) = \frac{1}{\sqrt{1-\frac{1}{x^2}}} \times \left(-\frac{1}{x^2}\right) - \frac{1}{\sqrt{1-\frac{1}{x^2}}}\left(-\frac{1}{x^2}\right)$$ ☑

$$= 0$$ ☑

(ii) domain: $-1 \le \frac{1}{x} \le 1$

$x \le -1$ or $x \ge 1$ ☑

Since $f'(x) = 0$

$\therefore f(x) = C$ (a constant)

since $f(1) = \sin^{-1}1 + \cos^{-1}1$

$= \frac{\pi}{2}$

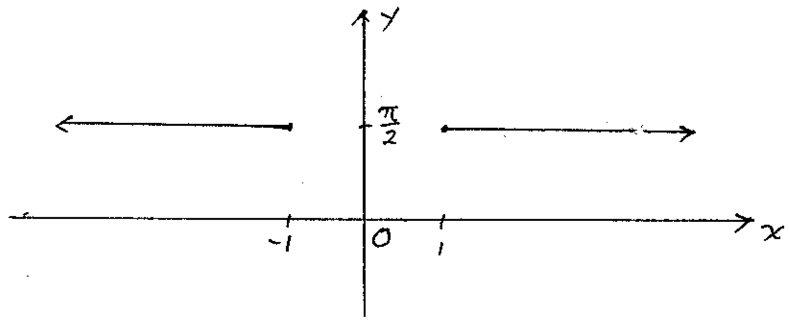
$f(-1) = \sin^{-1}(-1) + \cos^{-1}(-1)$

$= -\sin^{-1}1 + \pi - \cos^{-1}1$

$= -\frac{\pi}{2} + \pi$

$= \frac{\pi}{2}$ ☑

$\therefore f(x) = \frac{\pi}{2}$ $\quad x \le -1$ or $x \ge 1$



☑

(iii) $y = f(x)$ and $y = f^{-1}(x)$, if intersect, must intersect on the line $y = x$. ☑

As $f(x) = \frac{\pi}{2}$,

$\therefore$ at pt of intersection $x = f(x) = \frac{\pi}{2}$

$\therefore$ pt of intersection is $(\frac{\pi}{2}, \frac{\pi}{2})$ ☑

Question 7

a) (i) $BM^2 = OB^2 - OM^2$

$= r^2 - h^2$

$\therefore BM = \sqrt{r^2 - h^2}$ ☑

(ii) $AB + CD = AD - BC$

$= b - 2BM$

$= b - 2\sqrt{r^2 - h^2}$ ☑

$\therefore$ Area $A = k \times (AB + CD)$

$= k\left[b - 2\sqrt{r^2 - h^2}\right]$

(iii) $$\frac{dA}{dt} = \frac{d}{dt} k\left[b - 2\sqrt{r^2-h^2}\right]$$

$$= \frac{d}{dh} k\left[b-2\sqrt{r^2-h^2}\right] \times \frac{dh}{dt}$$

$$= \frac{-2k}{2\sqrt{r^2-h^2}} \times -2h \times \frac{dh}{dt}$$

$$= \frac{2hk}{\sqrt{r^2-h^2}} \frac{dh}{dt}$$

when $k = 300$, $b = 200$, $r = 50$, $h = 30$

and $\frac{dh}{dt} = -0.6$ cm/day

$$\frac{dA}{dt} = \frac{2\times 30 \times 300}{\sqrt{50^2-30^2}} \times (-0.6) \text{ cm}^2/\text{day}$$

$$= -270 \text{ cm}^2/\text{day}$$

∴ surface area is decreasing at 90 cm²/day.

[also accept 0.009 m²/day]

b) (i) At P

$$x = a\cos 2\theta = a(2\cos^2\theta - 1) \quad (1)$$

$$y = a\cos\theta$$

$$\therefore \cos\theta = \frac{y}{a} \quad (2)$$

Put (2) into (1)

$$x = a\left(\frac{2y^2}{a^2} - 1\right) = \frac{2y^2}{a} - a$$

$$ax = 2y^2 - a^2$$

ie $$y^2 = \frac{ax}{2} + \frac{a^2}{2} = \frac{a}{2}(x+a)$$

(ii) The locus is a parabola with vertex $(-a, 0)$ and focal length $\frac{a}{8}$

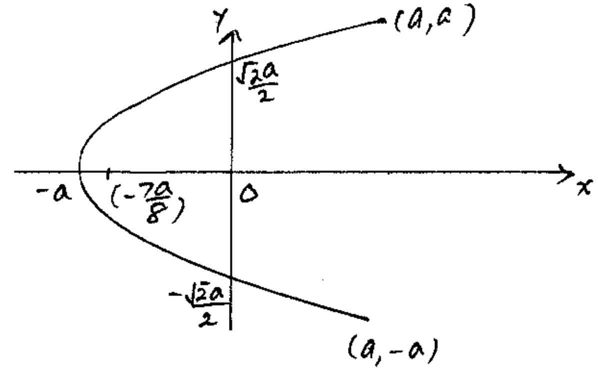
∴ Focus is $\left(-\frac{7a}{8}, 0\right)$

since $x = a\cos 2\theta$

and $-1 \le \cos 2\theta \le 1$

∴ $-a \le x \le a$

Similarly $-a \le y \le a$



for shape & some indication of scale

(iii)

$$x = a\cos 2\theta$$

$$\frac{dx}{d\theta} = -2a\sin 2\theta$$

$$y = a\cos\theta$$

$$\frac{dy}{d\theta} = -a\sin\theta$$

$$\therefore \frac{dy}{dx} = \frac{-a\sin\theta}{-2a\sin 2\theta}$$

$$= \frac{\sin\theta}{2\sin 2\theta}$$

$$= \frac{1}{4\cos\theta}$$

$\therefore$ at $\theta = \frac{\pi}{3}$

$$\frac{dy}{dx} = \frac{1}{4\cos\frac{\pi}{3}}$$

$$= \frac{1}{2}$$

$\checkmark$

Alternately :- $y^2 = \frac{a}{2}(x+a)$

$$y = \sqrt{\frac{a(x+a)}{2}}$$

$$\frac{dy}{dx} = \sqrt{\frac{a}{2}} \cdot \frac{1}{2\sqrt{x+a}}$$

$$= \frac{1}{2}\sqrt{\frac{a}{2(x+a)}}$$

when $\theta = \frac{\pi}{3}$, $x = a\cos\frac{2\pi}{3}$

$$= -\frac{a}{2}$$

$$y = a\cos\frac{\pi}{3}$$

$$= \frac{a}{2}$$

$$\therefore \frac{dy}{dx} = \frac{1}{2}\sqrt{\frac{a}{2(-\frac{a}{2}+a)}}$$

$$= \frac{1}{2}$$

$\checkmark$

At $\theta = \frac{\pi}{3}$, $x = a\cos\frac{2\pi}{3}$

$$= -\frac{a}{2}$$

$$y = a\cos\frac{\pi}{3}$$

$$= \frac{a}{2}$$

$\therefore$ Equation of the tangent is

$$y - \frac{a}{2} = \frac{1}{2}(x + \frac{a}{2})$$

$$2y - a = x + \frac{a}{2}$$

$$4y - 2a = 2x + a$$

or $2x - 4y + 3a = 0$

$\checkmark$