AW
AT
JG

Name: ____________________
Class: 12MTX________
Teacher: ____________________

CHERRYBROOK TECHNOLOGY HIGH SCHOOL

2009 AP4

YEAR 12 TRIAL HSC EXAMINATION

MATHEMATICS EXTENSION 1

Time allowed - 2 HOURS
(Plus 5 minutes' reading time)

DIRECTIONS TO CANDIDATES:

- Attempt all questions.
- All questions are of equal value.
- Each question is to be commenced on a new page clearly marked Question 1, Question 2, etc on the top of the page. **
- All necessary working should be shown in every question. Full marks may not be awarded for careless or badly arranged work.
- Approved calculators may be used. Standard Integral Tables are provided
- Your solutions will be collected in one bundle stapled in the top left corner.

Please arrange them in order, Q1 to 7.

Each page must show your name and your class.* *

Question 1 (12 Marks) **Marks**

(a) P (– 2, 3) and Q (6, –1) are two points in the number plane.
Find the coordinates of the point R that divides the interval PQ internally in the ratio 3:2. **2**

(b) Find the limiting sum of the geometric series

$$\left(\frac{e}{e+1}\right)+\left(\frac{e}{e+1}\right)^2+\left(\frac{e}{e+1}\right)^3+\ldots\ldots\ldots\ldots\ldots\ldots$$ **2**

(c) Solve the inequality $\frac{4}{1-x}\leq 3$ and graph your solution on a number line **3**

(d) The equation $x^3+2x^2+3x+6=0$ has α, β and γ as its roots.

Find the value of $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$. **2**

(e) Find $\lim\limits_{x\to 0}\frac{\sin 3x}{2x}$ **1**

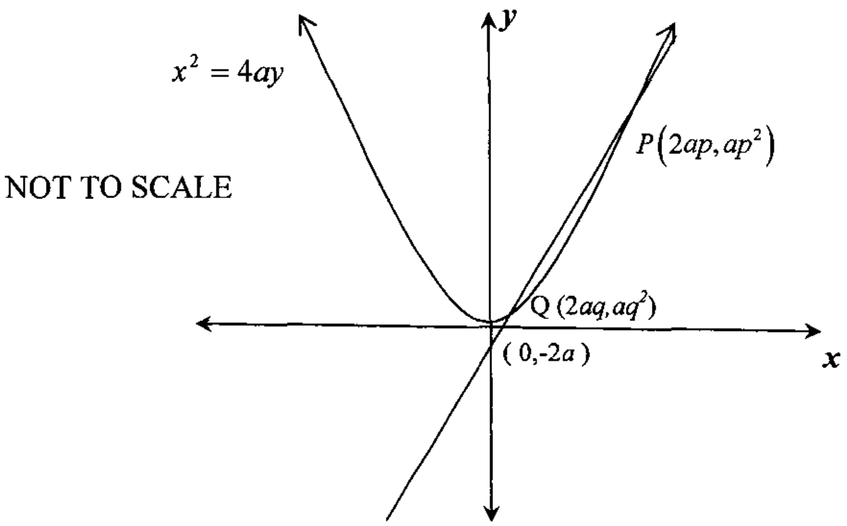
(f) Find the acute angle between the lines: $x-\sqrt{3}y+1=0$ **2**

and $y=x-4$.

Give your answer correct to the nearest degree.

Question 2 (12 Marks) START A NEW PAGE **Marks**

(a) The points $P\left(2ap, ap^2\right)$ and $Q\left(2aq, aq^2\right)$ lie on the parabola $x^2 = 4ay$



(i) Given the gradient of the chord PQ is $\dfrac{p+q}{2}$, show that the equation of PQ is $2y = (p+q)x - 2apq$. **1**

(ii) The point joining P and Q passes through the point $(0, -2a)$. Show that $pq = 2$. **1**

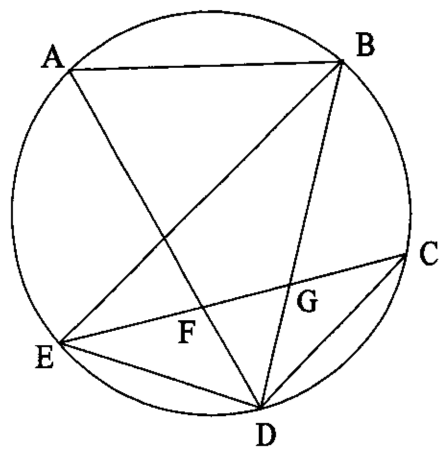
(iii) The normals to the parabola $x^2 = 4ay$ at points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ intersect at K. The coordinates of K are $\left(-apq(p+q),\ a(p^2 + q^2 + pq + 2)\right)$. **Do not prove this.**

Prove that the locus of K is the parabola $x^2 = 4ay$. **2**

Question 2 continues on the page 3............

Question 2 continued.......... **Marks**

(b) A, B, C, D and E are points on a circle such that $\angle DEC = \angle ECD$.



(i) Give a reason why $\angle CED = \angle EBD$. **1**

(ii) Show that ABGF is a cyclic quadrilateral. **3**

(c) A tower CX is observed at an angle of elevation of 14° from a point A on level ground. The same tower is observed from B, 1 km from A, with an angle of elevation of 17°. $\angle ACB = 120^\circ$. C is the base of the tower.

(i) Draw a diagram showing this information. **1**

(ii) Calculate h, the height of the tower CX. Give your answer correct to the nearest metre. **3**

Question 3 (12 Marks) START A NEW PAGE **Marks**

(a) Consider graph of the function $h(x) = \dfrac{3x}{1-x^2}$

(i) Find the equation of any asymptotes. **2**

(ii) Show why $h(x)$ has no turning points. **2**

(iii) Sketch $h(x)$ showing any asymptotes and intercepts. **2**

(b) A polynomial $P(x)$ of degree three, has zeros at $x = 1$, $x = -1$ and $x = 2$, and a remainder of 16 when divided by $(x-3)$.

Find $P(x)$, expressing it in the form $P(x) = P_0x^3 + P_1x^2 + P_2x + P_3$ **2**

(c) The area bounded by the x axis and the part of the curve $y = \sin x$ from $x = 0$ to $x = \pi$ is rotated about the x axis to form a solid.

Find the exact volume of the solid. **2**

(d) Using the substitution $u = x^4$, find $\int \dfrac{x^3}{1+x^8}\,dx$. **2**

Question 4 (12 Marks) START A NEW PAGE

(a) Consider the function $f(x) = (x+2)^2 - 9,\ -2 \le x \le 2.$

(i) Find the equation of the inverse function $f^{-1}(x)$. **1**

(ii) On the same diagram, sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$, showing clearly the coordinates of the endpoints and the intercepts on the axes. **3**

(b) (i) State the domain and range of $y = \cos^{-1}\left(\dfrac{5x}{3}\right)$ **2**

(ii) Hence sketch the graph of $y = \cos^{-1}\left(\dfrac{5x}{3}\right)$ **1**

(c) Evaluate $\displaystyle\int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9-4x^2}}$ in exact form. **2**

(d) Consider the function $f(x) = 2\tan^{-1}x + \sin^{-1}(\log_e x)$ where $x \ge 0$.

Find $f'(x)$. **2**

(e) Find the general solution of the equation $\cos x = \dfrac{\sqrt{3}}{2}$.

Express your answer in terms of π. **1**

Question 5 (12 Marks) START A NEW PAGE **Marks**

(a) (i) Show that $\ddot{x} = \frac{d}{dx}\left(\frac{v^2}{2}\right)$. **1**

(ii) The velocity, $v\,ms^{-1}$ of a particle moving in a straight line is given by $v = \sqrt{25 - x^2}$, where x is the displacement in metres from O. Show that the acceleration is $\ddot{x} = -x$. **1**

(b) When x *cm* from the origin, the acceleration of a particle moving in a straight line is given by:

$$\frac{d^2x}{dt^2} = \frac{-5}{(x+2)^3}$$

It has an initial velocity of *2 cm/s* at $x = 0$. If the velocity is *V cm/s*, find *V* in terms of *x*. **2**

(c) A particle is moving in simple harmonic motion in a straight line. At time t seconds, it has displacement x metres from a fixed point O in the line, velocity $\dot{x}\,ms^{-1}$ given by $\dot{x} = -12\sin(2t + \frac{\pi}{3})$ and acceleration $\ddot{x}\,ms^{-2}$. Initially the particle is 5 metres to the right of O.

(i) Show that $\ddot{x} = -4(x - 2)$. **1**

(ii) Find the period and the extremities of the motion. **2**

(iii) Find the time taken by the particle to return to its starting point for the first time. **1**

(d) A rock is hurled from the top of a $15m$ cliff with an initial velocity of $26ms^{-1}$ at an angle of projection equal to $\tan^{-1}\left(\frac{5}{12}\right)$ above the horizontal.

The cliff overlooks a flat paddock.

The equations of motion of the stone are $\ddot{x} = 0$ and $\ddot{y} = -10$

(i) Taking the origin as the base of the cliff, show the components of the rock's displacement are $x = 24t$ and $y = -5t^2 + 10t + 15$. **2**

(ii) Calculate the time until impact with the paddock, and the distance of the point of impact from the base of the cliff. **2**

Question 6 (12 Marks) START A NEW PAGE **Marks**

(a) Prove by Mathematical Induction that,

$(n)^3+(n+1)^3+(n+2)^3$ is divisible by 9 for all positive whole numbers n. **3**

(b) Three consecutive coefficients in the expansion of $(1+x)^n$ are in the ratio 6:3:1.

(i) Find the value of n. **4**

(ii) State which terms have their coefficients in the ratio 6:3:1. **1**

(c) Let n and m be positive integers with $m \le n \le 2m+1$.

(i) Show that $(1+x)^{n-m}\ (1+\frac{1}{x})^m = \dfrac{(1+\frac{1}{x})^n}{x^{m-n}}$ **1**

(ii) By applying the binomial theorem to part (i) and equating the coefficient of x, find a simpler expression for

$${}^{n-m}C_1\ {}^{m}C_0 + {}^{n-m}C_2\ {}^{m}C_1 + {}^{n-m}C_3\ {}^{m}C_2 + \dots\dots + {}^{n-m}C_{n-m}\ {}^{m}C_{n-m-1}$$ **3**

Question 7 (12 Marks) START A NEW PAGE

(a) The equation $f(x)=\sqrt{x}+\sqrt{x+1}+\sqrt{x+2}-5$ has a root α between $x=1\cdot5$ and $x=2$.
Find the interval in which α lies by applying halving the interval twice. **1**

(b) A spherical bubble is expanding so that its volume is increasing at the constant rate of 10 mm^3/s. What is the rate of increase of the radius when the surface area is 500 mm^2? **2**

(c) After t hours, the number of individuals in a population is given by $N=500-400e^{-0.1t}$.

(i) Sketch the graph of N as a function of t, showing clearly the initial population size and the limiting population size. **2**

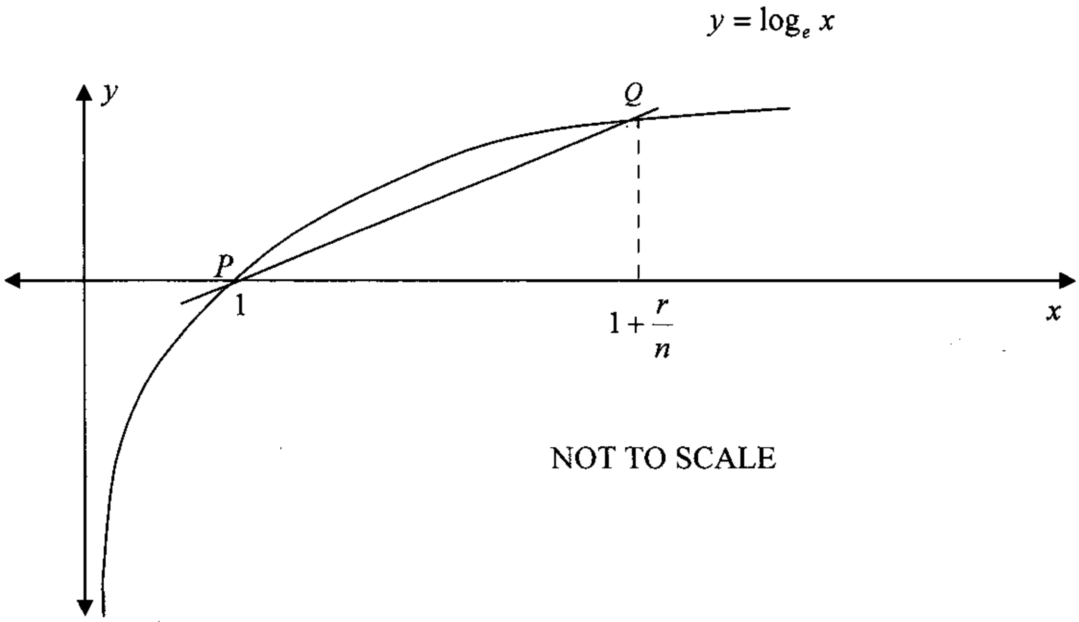
(ii) Show that $\dfrac{dN}{dt}=0.1(500-N)$. **1**

(iii) Find the population size for which the rate of growth of the population is half the initial rate of growth. **1**

Question 7 continues on the page 7.........

Question 7 continued.........

(d) The diagram below shows the graph of $y = \log_e x$ and the secant joining points P and Q on the curve. P is at $x = 1$ and Q is at $x = 1 + \frac{r}{n}$.



(i) Show that the gradient of the secant is $\frac{1}{r}\log_e\left(1+\frac{r}{n}\right)^n$. **1**

(ii) Use $\frac{d}{dx}\log_e x = \frac{1}{x}$ to show that $\lim_{n \to \infty}\left(1+\frac{r}{n}\right)^n = e^r$. **3**

(iii) Use part (ii) to determine an expression for the effective annual rate of interest when an annual rate of 6% p.a. is compounded continually, that is, compounded an infinite number of times per year. **1**

END OF EXAMINATION

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, n \neq -1;\ x \neq 0,\ \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ \ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ \ a \neq 0$$

$$\int \cos ax dx = \frac{1}{a}\sin ax,\ a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax dx, = \frac{1}{a}\tan ax,\ \ a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln(x + \sqrt{x^2 - a^2}),\ \ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln(x + \sqrt{x^2 + a^2})$$

NOTE : $\ln x = \log_e x, \quad\quad x > 0$

STHS Mathematics Extension 1 Trial Exam 2009

SOLUTIONS

1. a) $P(-2,3)$ $Q(6,-1)$

$3:2$

$R = \left(\frac{-2\times 2 + 6\times 3}{3+2}, \frac{2\times 3 + 3\times -1}{3+2}\right)$ 1

$= \left(\frac{14}{5}, \frac{3}{5}\right)$ 1

b) $r = \frac{e}{e+1}$, $a = \frac{e}{e+1}$

Limiting Sum

$= \frac{e}{1+e} \times \frac{e+1}{e+1-e}$ 1

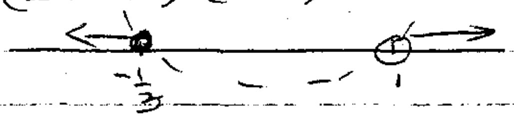
$= \frac{e\cancel{(e+1)}}{\cancel{(1+e)}}$

$= e$ 1

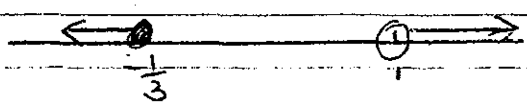
c) $(1-x)^2 \times \frac{4}{1-x} \leq 3(1-x)^2, x \neq 1$

$\left.\begin{array}{l} 4(1-x) \leq 3(1-2x+x^2) \\ 4-4x \leq 3-6x+3x^2 \\ 3x^2-2x-1 \geq 0 \end{array}\right\}$ 1

$(3x+1)(x-1) \geq 0$



$x \leq -\frac{1}{3}$ or $x > 1$ 1



1

d) $\alpha+\beta+\gamma = -2$

$\alpha\beta + \beta\gamma + \alpha\gamma = 3$

$\alpha\beta\gamma = -6$

$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

$\frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma} = \frac{3}{-6}$

1 $= -\frac{1}{2}$ 1

e) $\lim_{x\to 0} \frac{\sin 3x}{2x}$

$\lim_{x\to 0} \frac{\sin 3x}{3x} \times \frac{3}{2}$

$= \frac{3}{2}$ 1

f) $y = \frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}}$

$y = x - 4$

$m_1 = \frac{1}{\sqrt{3}}$, $m_2 = 1$

$\tan\theta = \left|\frac{\frac{1}{\sqrt{3}} - 1}{1 + \frac{1}{\sqrt{3}}}\right|$ 1

$\theta = 15°$. 1

Q2 (a) (i) $m = \frac{p+q}{2}$

Equation of PQ

$y - aq^2 = \frac{p+q}{2}(x - 2aq)$

$2y - 2aq^2 = (p+q)x - 2apq - 2aq^2$ ①

$2y = (p+q)x - 2apq$ given

(ii) $x = 0,\ y = -2a$

$-4a = (p+q) \times 0 - 2apq$ ①

$2apq = 4a$

$pq = \frac{4a}{2a}$

$\therefore pq = 2$

(iii) $K(-apq(p+q),\ a(p^2+q^2+pq+2))$

Since $pq = 2$

$K(-2a(p+q),\ a(p^2+q^2+4))$

$x = -2a(p+q)$, $y = a(p^2+q^2+4)$

$= a[(p+q)^2 - 2pq + 4]$

$x^2 = 4a^2(p+q)^2$ ①, $y = a(p+q)^2$ ①

$\therefore x^2 = 4ay$

b) (i) given $\angle DEC = \angle ECD$

$\angle ECD = \angle EBD$ ($\angle$'s at circumference equal subtended by same arc ED) ①

$\therefore \angle CED = \angle EBD$

3

Question 2

(ii) $\angle EDA = \angle ABE$ ($\angle$'s at circumference subtended by arc AE equal) ①

$\angle GFD = \angle CED + \angle EDA$ (exterior $\angle$ of $\triangle EFD$ equal sum of opposite interior $\angle$'s) ①

$\therefore \angle ABG = \angle ABE + \angle EBD$ (adjacent $\angle$'s)

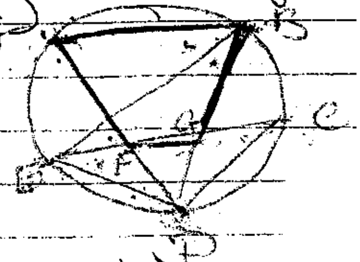
$= \angle EDA + \angle ECD$ (from part (i))

$= \angle GFD$ ①

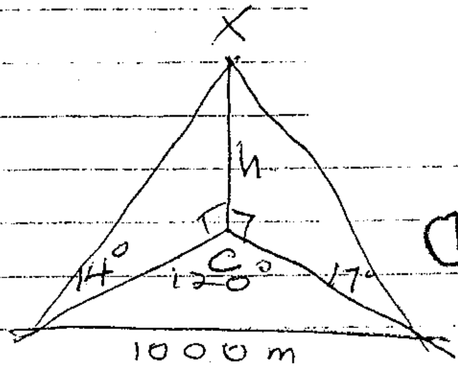
$\angle GFD$ is exterior $\angle$ of quadrilateral ABGF

$\angle ABG$ is opposite interior $\angle$ to $\angle GED$ in quadrilateral ABGE

$\therefore$ ABGF is a cyclic quadrilateral (exterior $\angle$ of cyclic quadrilateral equal opposite interior $\angle$)



(b) (i)



① (must have information)

(ii) $\tan 14° = \dfrac{h}{AC}$

$AC = h\cot 14°$

$\tan 17° = \dfrac{h}{BC}$

$BC = h\cot 17°$ ①

$1000^2 = h^2\cot^2 14° + h^2\cot^2 17° - 2 \times h^2 \cot 14° \cot 17° \cos 120°$

$= h^2(\cot^2 14° + \cot^2 17° - 2\cot 14° \cot 17° \cos 120°)$

$h^2 = \dfrac{1000^2}{\cot^2 14 + \cot^2 17° + \cot 14° \cot 17°}$ ①

$= 25060.44965$

$h = 158.3049262$

$= 158\text{ m}$ (nearest m)

Question 3

(4)

a) (i) Vertical asymptotes

$1 - x^2 = 0$

$(1-x)(1+x) = 0$

$\therefore x = 1, \ x = -1$ are asymptotes (1) for both

Other asymptote

$y = \lim_{x \to \infty} \frac{3x}{1-x^2}$

$= \lim_{x \to \infty} \frac{\frac{3}{x}}{\frac{1}{x^2} - 1}$

$= 0$

$\therefore y = 0$ is an asymptote (1)

(ii) $h'(x) = \frac{3(1-x^2) + 3x \times 2x}{(1-x^2)^2}$

$= \frac{3 - 3x^2 + 6x^2}{(1-x^2)^2}$

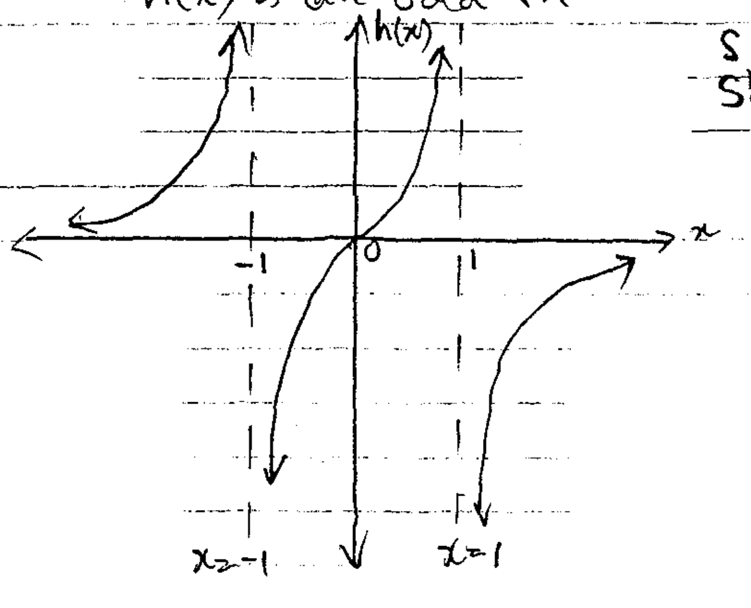
$= \frac{3 + 3x^2}{(1-x^2)^2}$ (1)

Since $3 + 3x^2 > 0$ for all values of x
$h'(x) \neq 0$ for all values of x
$\therefore h(x)$ does not have any turning points. } (1)

(iii) $h(-x) = \frac{-3x}{1-x^2}$

$= -h(-x)$

$\therefore h(x)$ is an odd fn



Show asymptote.
Shape (1) & intercepts (1)

Question 3

b) $P(x) = k(x-1)(x+1)(x-2)$

$= 0$

$P(3) = 16$

$\therefore 16 = k(3-1)(3+1)(3-2)$ 1

$\therefore k = 2$

$P(x) = 2(x-1)(x+1)(x-2)$

$= 2(x^2-1)(x-2)$

$= 2(x^3 - x - 2x^2 + 2)$

$= 2x^3 - 4x^2 - 2x + 4$ 1

Alternative method

Roots are $1, -1, 2$

$\alpha + \beta + \gamma = -\frac{b}{a}$

$1 - 1 + 2 = -\frac{b}{a}$

$\therefore b = -2a$

$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$

$1 \times -1 + 1 \times 2 + -1 \times 2 = \frac{c}{a}$

$-1 + 2 - 2 = \frac{c}{a}$

$\therefore c = -a$

$\alpha\beta\gamma = -\frac{d}{a}$

$1 \times -1 \times 2 = -\frac{d}{a}$

$\therefore d = 2a$

$P(3) = 16$

$a(3^3) + b(3^2) + 3c + d = 16$

$27a + 9b + 3c + d = 16$

$27a - 18a - 3a + 2a = 16$

$8a = 16$

$\therefore a = 2$ 1

$b = -4, c = -2, d = 4$

$P(x) = 2x^3 - 4x^2 - 2x + 4$ 1

c) $V = \pi \int_0^{\pi} \sin^2 x \, dx$

$= \frac{\pi}{2} \int_0^{\pi} (1 - \cos 2x) \, dx$ 1

$= \frac{\pi}{2} \left[x - \frac{1}{2} \sin 2x \right]_0^{\pi}$

$= \frac{\pi}{2} \left[\left(\pi - \frac{1}{2}\sin 2\pi\right) - \left(0 - \frac{1}{2}\sin 0\right) \right]$

$= \frac{\pi^2}{2}$ units^2 1

d) $u = x^4, \; du = 4x^3 \, dx$

$\int \frac{x^3}{1+x^8} \, dx = \frac{1}{4} \int \frac{4x^3}{1+(x^4)^2} \, dx$

$= \frac{1}{4} \int \frac{du}{1+u^2}$ 1

$= \frac{1}{4} \tan^{-1} u + c$

$= \frac{1}{4} \tan^{-1}(x^4) + c$ 1

Question 4

a) (i) $f(x) = (x+2)^2 - 9$, $-2 \le x \le 2$

$0 \le x+2 \le 4$

$(x+2)^2 - 9 = y$

$(x+2)^2 = y+9$

$x+2 = \sqrt{y+9}$

$x = -2 + \sqrt{y+9}$, $-9 \le y \le 7$

$\therefore f^{-1}(x) = -2 + \sqrt{x+9}$, $-9 \le x \le 7$ (1)

when $x+2 = 0$

$\sqrt{y+9} = 0$

$y = -9$

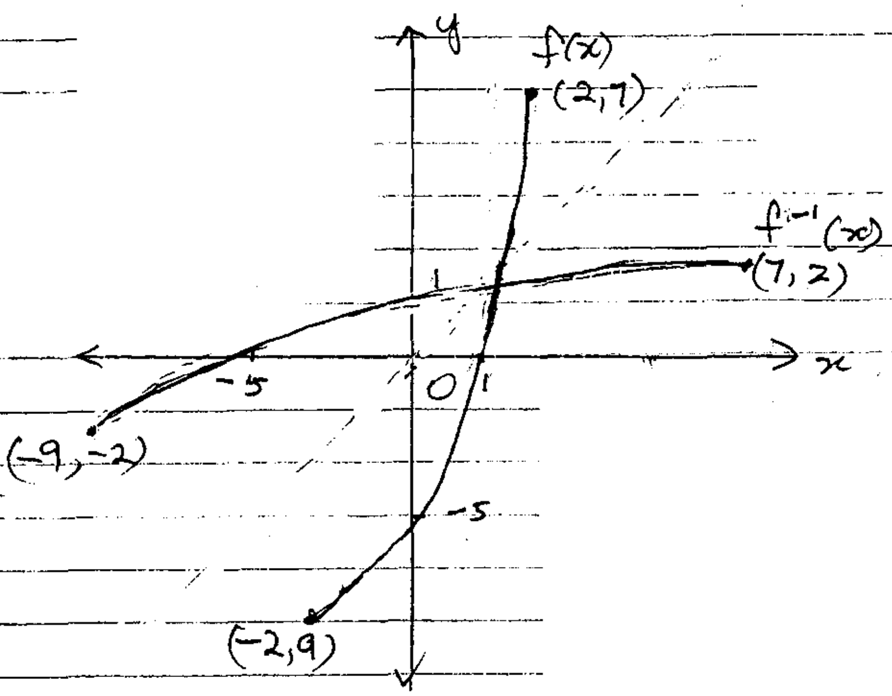
when $x+2 = 4$

$\sqrt{y+9} = 4$

$y+9 = 16$

$y = 7$

(ii)



Endpoints + intercepts (1)

shape (1)

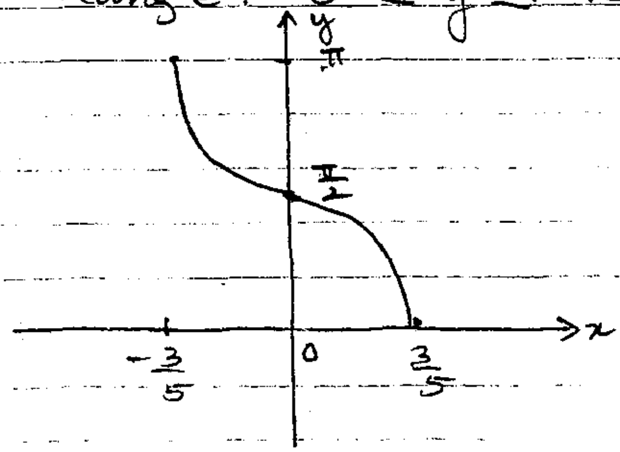
(1)

symmetry about $y=x$

(b) (i) $-1 \le \frac{5x}{3} \le 1$

Domain $\frac{-3}{5} \le x \le \frac{3}{5}$ (1)

Range: $0 \le y \le \pi$ (1)



(1) must show domain + range

Question 4

c) $$\int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9-4x^2}} = \int_0^{3/2} \frac{dx}{\sqrt{4\left(\frac{9}{4}-x^2\right)}}$$

$$= \frac{1}{2}\int_0^{3/2} \frac{dx}{\sqrt{\left(\frac{3}{2}\right)^2 - x^2}}$$

$$= \frac{1}{2}\left[\sin^{-1}\frac{2x}{3}\right]_0^{3/2} \quad 1$$

$$= \frac{1}{2}\left[\sin^{-1} 1 - \sin^{-1} 0\right]$$

$$= \frac{1}{2} \times \frac{\pi}{2}$$

$$= \frac{\pi}{4} \quad 1$$

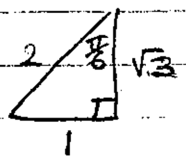
d) $f(x) = 2\tan^{-1} x + \sin^{-1}(\log_e x)$

$$f'(x) = \frac{2}{1+x^2} + \frac{\frac{1}{x}}{\sqrt{1-(\log_e x)^2}}$$

$$= \frac{2}{1+x^2} + \frac{1}{x\sqrt{1-(\log_e x)^2}}$$

1 1

e) $\cos x = \frac{\sqrt{3}}{2}$



$x = 2\pi n \pm \frac{\pi}{6}$ 1

estion 5

a) (i) $\ddot{x} = \frac{d^2x}{dt^2}$

$= \frac{dv}{dt}$

$= \frac{dv}{dx} \times \frac{dx}{dt}$

$= v.\frac{dv}{dx}$

$= \frac{d}{dv}\left(\frac{1}{2}v^2\right)\frac{dv}{dx}$

$= \frac{d}{dx}\left(\frac{v^2}{2}\right)$ given

1

(ii) $v = \sqrt{25 - x^2}$

$\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$= \frac{d}{dx}\left(\frac{1}{2}\left(\sqrt{25-x^2}\right)^2\right)$

$= \frac{1}{dx}\left[\frac{25-x^2}{2}\right]$

$= \frac{d}{dx}\left[\frac{25}{2} - \frac{x^2}{2}\right]$

$= -x$

$\therefore \ddot{x} = -x$ given

1

b) $\frac{d^2x}{dt^2} = \frac{-5}{(x+2)^3}$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -5(x+2)^{-3}$

$\frac{1}{2}v^2 = \frac{-5}{-2(x+2)^2} + c$ 1

$v = 2,\ x = 0$

$2 = \frac{5}{8} + C \quad \therefore c = \frac{11}{8}$

$\frac{1}{2}v^2 = \frac{5}{2(x+2)^2} + \frac{11}{8}$

$v^2 = \frac{5}{(x+2)^2} + \frac{11}{4} = \frac{20 + 11(x+2)^2}{4(x+2)^2}$

$v = \frac{\sqrt{20 + 11(x+2)^2}}{2(x+2)}$ since $v > 0$ when $x = 2$ 1

Question 5

i) $\dot{x} = -12\sin\left(2t + \frac{\pi}{3}\right)$

$x = 6\cos\left(2t + \frac{\pi}{3}\right) + c$

when $t = 0$, $x = 5$

$5 = 6\cos\frac{\pi}{3} + c$

$5 = 3 + c$

$\therefore c = 2$

$\therefore x = 2 + 6\cos\left(2t + \frac{\pi}{3}\right)$ 1

$6\cos\left(2t + \frac{\pi}{3}\right) = x - 2$

$\ddot{x} = -12 \times 2\cos\left(2t + \frac{\pi}{3}\right)$

$= -24\cos\left(2t + \frac{\pi}{3}\right)$

$= -4 \times 6\cos\left(2t + \frac{\pi}{3}\right)$

$= -4(x - 2)$ given

ii) Extremities: $x = 2 + 6 \times \pm 1$

$-4 \leq x \leq 8$ 1

Period: $\frac{2\pi}{2} = \pi$ seconds 1

ii) When $t = 0$, $x = 5$

$5 = 2 + 6\cos\left(2t + \frac{\pi}{3}\right)$

$3 = 6\cos\left(2t + \frac{\pi}{3}\right)$

$\cos\left(2t + \frac{\pi}{3}\right) = \frac{1}{2}$

$2t + \frac{\pi}{3} = \frac{\pi}{3}, 2\pi - \frac{\pi}{3}, \ldots$

$2t = 0, \frac{4\pi}{3}, \ldots$

$t = 0, \frac{2\pi}{3}$

First return after $\frac{2\pi}{3}$ seconds. 1

stion 5

a) If $\tan\theta = \frac{5}{12}$

$\cos\theta = \frac{12}{13}$

$\sin\theta = \frac{5}{13}$

(i) $\ddot{x} = 0, \quad \dot{x} = \int \ddot{x}\,dt$

$= C_1$

$\dot{x} = v\cos\theta$

$= 26 \times \frac{12}{13}$

$= 24$ when $t=0$, $C_1 = 0$.

$x = \int \dot{x}\,dt$

$= \int 24\,dt$

$= 24t + C_2$

$x=0, \quad t=0 \quad \therefore C_2 = 0$

$x = 24t$ given

1

$\ddot{y} = -10$

$\dot{y} = \int \ddot{y}\,dt$

$= -10t + C_3$

$\dot{y} = v\sin\theta$

$= 26 \times \frac{5}{13}$

$= 10$

1

when $t=0$, $\dot{y}=10$, $C_3 = 10$

$\dot{y} = -10t + 10$

$y = \int(-10t+10)\,dt$

$= -5t^2 + 10t + C_4$

when $t=0$, $y=15$, $C_4 = 15$

$\therefore y = -5t^2 + 10t + 15$ given

(ii) Impact when $y=0$

$-5t^2 + 10t + 15 = 0$

$t^2 - 2t - 3 = 0$

$(t+1)(t-3) = 0$

$\therefore t = -1, 3$ But $t \geq 0$

$\therefore t = 3$

1

when $t = 3$

$x = 24 \times 3$

$= 72\text{ m}$. 1

Q6.

a) when $n=1$, $n^3+(n+1)^3+(n+2)^3 = 1+2^3+3^3$
$= 1+8+27$
$= 36$
$= 4\times 9$

$\therefore$ It is true for $n=1$. 1

Assume the statement is true for $n=k$, where k is a positive integer.
ie $k^3+(k+1)^3+(k+2)^3 = 9m$ for some positive integer m
$(k+1)^3+(k+2)^3 = 9m-k^3$ (1)

then $(k+1)^3+(k+2)^3+(k+3)^3$
$= 9m-k^3+(k+3)^3$ by assumption (1) 1
$= 9m+3[(k+3)^2+k(k+3)+k^2]$
$= 9m+3[k^2+6k+9+k^2+3k+k^2]$
$= 9m+3(3k^2+9k+9)$
$= 9m+9(k^2+3k+3)$
$= 9(m+k^2+3k+3)$ 1
which is divisible by 9.

$\therefore$ It will be true for $n=k+1$ if it is true for $n=k$.
Since it is proved true for $n=1$, $\therefore$ it will be true for $n=2,3,4,\ldots$, ie true for all positive integers n.

(b). $(1+x)^n = \sum_{r=0}^{n}\binom{n}{r}x^r$

(i) Let the 3 consecutive terms be $\binom{n}{k-1}x^{k-1}$, $\binom{n}{k}x^k$ & $\binom{n}{k+1}x^{k+1}$

$\therefore \binom{n}{k-1} : \binom{n}{k} : \binom{n}{k+1} = 6:3:1$

$\frac{n!}{(n-k+1)!(k-1)!} : \frac{n!}{(n-k)!\,k!} : \frac{n!}{(n-k-1)!(k+1)!} = 6:3:1$ 1

$$\frac{k}{n-k+1} = \frac{6}{3} \longrightarrow 1$$

$$\frac{k}{n-k+1} = 2$$

$$k = 2n-2k+2$$

$$3k = 2n+2 \qquad (1)$$

and $$\frac{k+1}{n-k} = \frac{3}{1} \longrightarrow 1$$

$$k+1 = 3n-3k$$

$$4k = 3n-1 \qquad (2)$$

(1) x 4 $\quad 12k = 8n+8 \qquad (3)$

(2) x 3 $\quad 12k = 9n-3 \qquad (4)$

(4) − (3) $\quad 0 = n-11$

$$n = 11 \longrightarrow 1$$

(ii) Put $n=11$ into (1)

$$3k = 22+2$$

$$k = 8$$

∴ The terms are the 8^{th}, 9^{th} and 10^{th} terms 1

(c) (i) $$(1+x)^{n-m} \cdot \left(1+\frac{1}{x}\right)^m = (1+x)^{n-m}\left(\frac{1+x}{x}\right)^m$$

$$= \frac{(1+x)^n}{x^m}$$

$$= \left(\frac{1+x}{x}\right)^n \cdot x^n \frac{1}{x^m}$$

$$= \left(1+\frac{1}{x}\right)^n \cdot \frac{x^n}{x^m}$$

1

$$= \frac{\left(1+\frac{1}{x}\right)^n}{x^{m-n}}$$

given

(ii) $$(1+x)^{n-m}\left(1+\frac{1}{x}\right)^m = \sum_{r=0}^{n-m}\binom{n-m}{r}x^r \cdot \sum_{j=0}^{m}\binom{m}{j}\frac{1}{x^j}$$

$\therefore$ coeff of x in the expansion

$$= \binom{n-m}{1}\binom{m}{0} + \binom{n-m}{2}\binom{m}{1} + \binom{n-m}{3}\binom{m}{2} + \cdots \binom{n-m}{n-m}\binom{m}{n-m-1} \quad 1$$

$$\frac{1}{x^{m-n}}\left(1+\frac{1}{x}\right)^n = \frac{1}{x^{m-n}}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{x^k}$$

$$= \sum_{k=0}^{n}\binom{n}{k}x^{n-m-k}$$

$\therefore$ coeff of x in the expansion $= \binom{n}{n-m-1}$ 1

Equating coeff of x on both sides of $(1+x)^{n-m}\left(1+\frac{1}{x}\right)^m = \dfrac{\left(1+\frac{1}{x}\right)^n}{x^{m-n}}$

$$\binom{n-m}{1}\binom{m}{0} + \binom{n-m}{2}\binom{m}{1} + \binom{n-m}{3}\binom{m}{2} + \cdots + \binom{n-m}{n-m}\binom{m}{n-m-1}$$

$$= \binom{n}{n-m-1}$$

ie $${}^{n-m}C_1\,{}^{m}C_0 + {}^{n-m}C_2\,{}^{m}C_1 + {}^{n-m}C_3\,{}^{m}C_2 + \cdots + {}^{n-m}C_{n-m}\,{}^{m}C_{n-m-1} = {}^{n}C_{n-m-1}$$

1

14

Question 7

(a) $f(x) = \sqrt{x} + \sqrt{x+1} + \sqrt{x+2} - 5$

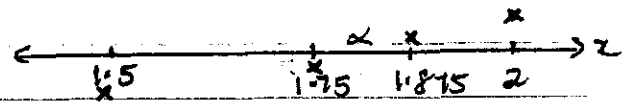
$f(1\cdot5) = -0\cdot323297605$

$f(2) = 0\cdot146254369$

Half the interval $x = \frac{1\cdot5+2}{2}$

$= 1\cdot75$

$f(1\cdot75) = -0\cdot082320276$



Half interval $x = \frac{1\cdot75+2}{2}$

$= 1\cdot875$

$f(1\cdot875) = 0\cdot0333390859$

$\therefore\ 1\cdot75 < \alpha < 1\cdot875$ 1

b) V = volume of sphere, r = radius

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$

$= 4\pi r^2 \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{1}{4\pi r^2} \times \frac{dV}{dt}$ 1

$\frac{dV}{dt} = 10$

$\therefore\ \frac{dr}{dt} = \frac{10}{4\pi r^2}$

When surface area $= 500\,mm^2$

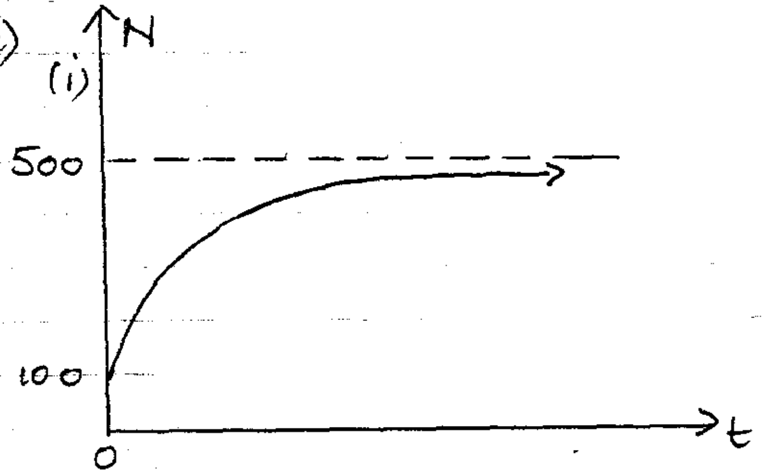
$4\pi r^2 = 500$

$\therefore\ \frac{dr}{dt} = \frac{10}{500}$ 1

$= 0\cdot02\ mm/s.$

Question 7

c) (i)



① Shape

① asymptote & intercept

(ii) $N = 500 - 400e^{-0.1t}$

$\frac{dN}{dt} = 0.1 \times 400e^{-0.1t}$ ①

$= 0.1(500 - N)$

(iii) Initial rate of growth is

$0.1(500 - 100)$

$= 0.1 \times 400$

When population is half

$= 0.1 \times 200$

$\therefore 0.1(500 - N) = 0.1 \times 200$

$500 - N = 200$

$N = 300$ ①

c) (i) At Q, $y = \log_e\left(1 + \frac{r}{n}\right)$

Gradient of PQ $= \dfrac{\log_e\left(1+\frac{r}{n}\right) - 0}{\left(1+\frac{r}{n}\right) - 1}$

$= \frac{n}{r}\log_e\left(1+\frac{r}{n}\right)$ ①

$= \frac{1}{r}\log_e\left(1+\frac{r}{n}\right)^n$

Question 7 (16)

d) ii) $\frac{d}{dx}(\log_e x) = \frac{1}{x}$

at $x = 1$, gradient at $P = 1$

as $\to \infty$, $PQ \to$ tangent at P (1)

$\therefore \lim_{n\to\infty} \frac{1}{r}\log_e\left(1+\frac{r}{n}\right)^n = 1$ (1)

$\lim_{n\to\infty} \log_e\left(1+\frac{r}{n}\right)^n = r$ (1)

$\lim_{n\to\infty} e^{\log_e(1+\frac{r}{n})^n} = e^r$

$\lim_{n\to\infty} \left(1+\frac{r}{n}\right)^n = e^r$

iii) If compound interest is paid n times per year at 6% p.a., then

$A = P\left(1+\frac{0.06}{n}\right)^n$

as $n \to \infty$, interest is compound continually

$A = \lim_{n\to\infty} P\left(1+\frac{0.06}{n}\right)^n$

$A = P \times e^{0.06}$

$1 + r = e^{0.06}$

$\therefore$ effective rate when compounded continuously is $r = e^{0.06} - 1$ (1)