Section I

10 marks
Attempt Questions 1 – 10
Use the multiple choice answer sheet located at the back of the paper.
Allow about 15 minutes for this section

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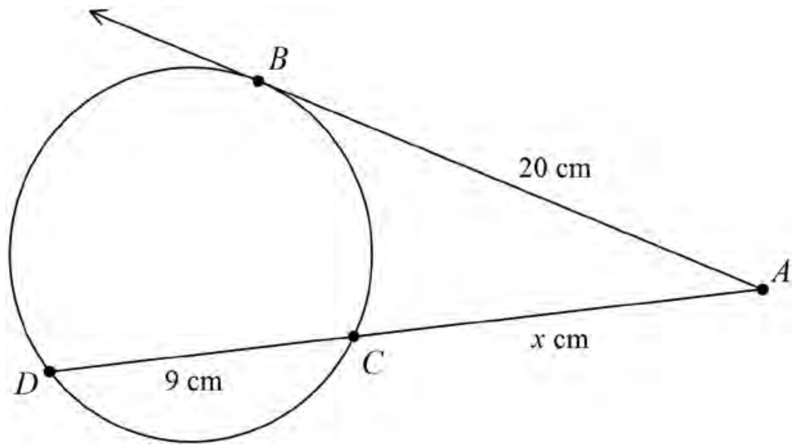
1. If A and B are the points $(2,-1)$ and $(-2,-4)$ respectively, the coordinates of the point M which divides AB externally in the ratio 1:3 is,

(A) $M\left(4,\frac{1}{2}\right)$ (B) $M\left(-2,-\frac{11}{2}\right)$ (C) $M\left(-2,-\frac{7}{4}\right)$ (D) $M\left(2,-\frac{13}{4}\right)$

2.



$AB = 20\,\text{cm}, CD = 9\,\text{cm}$ and $AC = x\,\text{cm}$. Find the value of x.

(A) 5 (B) 9 (C) 11 (D) 16

3. The derivative of $3\sin^{-1}\frac{x}{2}$ is,

(A) $\dfrac{3}{\sqrt{4-x^2}}$ (B) $\dfrac{3}{\sqrt{1-4x^2}}$

(C) $\dfrac{3}{\sqrt{\frac{1}{4}-x^2}}$ (D) $\dfrac{3}{\sqrt{1-\frac{x^2}{4}}}$

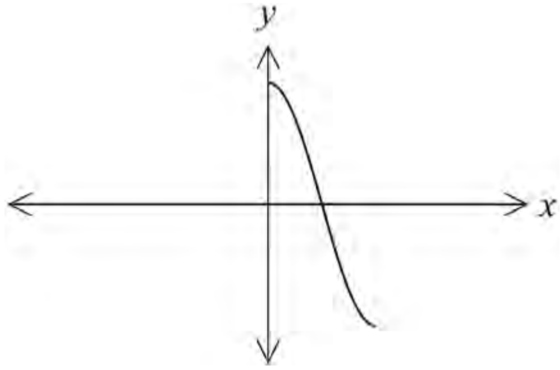
4. The exact value of $\cos 105°$ is,

(A) $\dfrac{\sqrt{3}-1}{2\sqrt{2}}$ (B) $\dfrac{1-\sqrt{3}}{2\sqrt{2}}$ (C) $\dfrac{\sqrt{3}+1}{2\sqrt{2}}$ (D) $\dfrac{1+\sqrt{3}}{2\sqrt{2}}$

5. When the polynomial $P(x)$ is divided by $x^2 - 4$, the remainder is $2x - 3$. What is the remainder when $P(x)$ is divided by $x + 2$?

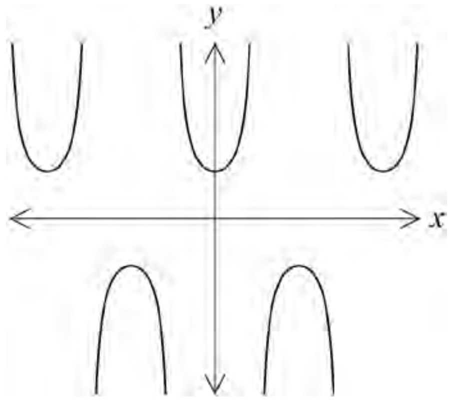
(A) -7 (B) 1 (C) 5 (D) 7

6. The diagram shows the graph $y = f(x)$.

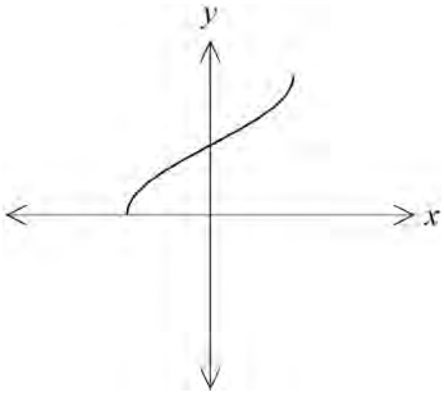


Which diagram shows the graph of $y = f^{-1}(x)$

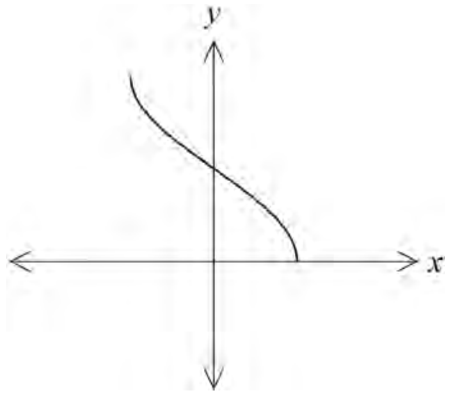
(A)



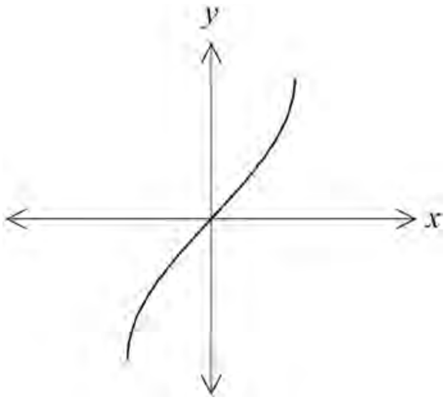
(B)



(C)



(D)



7. A curve is defined by the parameters $x = 2p + \frac{2}{p}$, $y = p^2 + \frac{1}{p^2}$. Which of the following represents this curve in Cartesian form?

(A) $y = \frac{x^2}{4} - 2$ (B) $y = \frac{x^2}{2}$ (C) $x + y^2 = 2$ (D) $y = x^2 - 2$

8. Which of the following is an expression for $1 + \sec x$ in terms of $t = \tan\frac{x}{2}$?

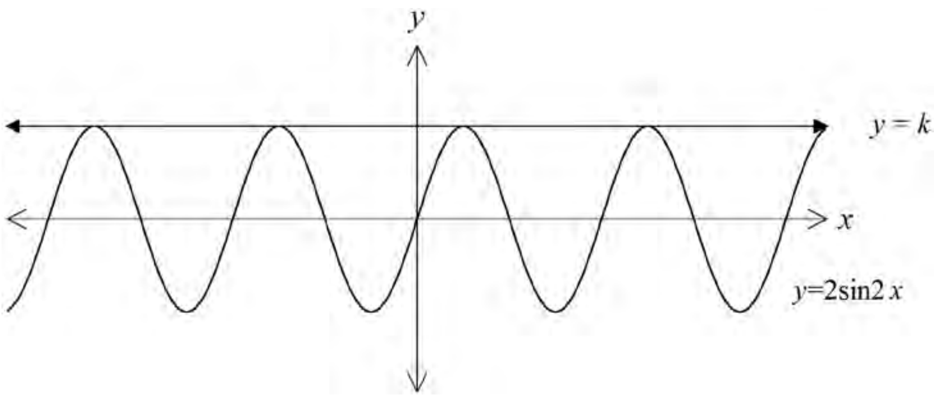
(A) $\frac{2}{1+t^2}$ (B) $\frac{2}{1-t^2}$ (C) $\frac{2t^2}{1+t^2}$ (D) $\frac{2t^2}{1-t^2}$

9. Which integral is obtained when the substitution $u = x + 2$ is applied to $\int \frac{x}{3}\sqrt{x+2}\ dx$?

(A) $\frac{1}{3}\int (u^{\frac{1}{2}} + 2u^{\frac{1}{3}})du$ (B) $\frac{1}{3}\int (u^{\frac{1}{2}} - 2u^{\frac{3}{2}})du$

(C) $\frac{1}{3}\int (u^{\frac{3}{2}} + 2u^{\frac{1}{2}})du$ (D) $\frac{1}{3}\int (u^{\frac{3}{2}} - 2u^{\frac{1}{2}})du$

10. Part of the graph of $y = 2\sin 2x$ is drawn below.



The horizontal line $y = k$ is also drawn to touch $y = 2\sin 2x$ as shown above.
Given that n is an integer, the general solution of the intersection of these two functions is,

(A) $\frac{n\pi}{2} + (-1)^n\frac{\pi}{4}$ (B) $2n\pi \pm \frac{\pi}{4}$ (C) $n\pi + (-1)^n\frac{\pi}{2}$ (D) $\frac{n\pi}{2} + (-1)^n\frac{\pi}{2}$

Section II
60 marks
Attempt Questions 11 – 14
Allow about 1 hour 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available upon request from the supervising teachers.
In questions 11-14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 Marks) Use a SEPARATE writing booklet. **Marks**

(a) Find the value of $\lim_{x \to 0} \frac{\tan 3x}{2x}$. **1**

(b) Find the gradient of the tangent to the curve $y = \cos(\ln x)$ when $x = 1$. **2**

(c) Find $\int \frac{1}{4 + 9x^2}\, dx$ **2**

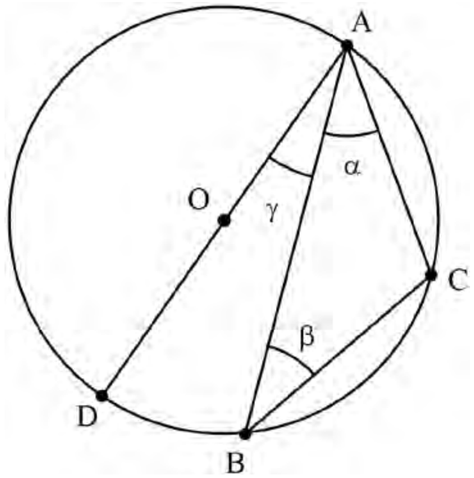
(d) Evaluate $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos^2 3x \, dx$ **3**

(e) Solve $\frac{x^2 + x - 6}{x} \geq 2$. **3**

(f) The curves $y = \sin x$ and $y = \cos x$ intersect at $P\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$. If α is the acute angle between these curves at P, then show that $\tan \alpha = 2\sqrt{2}$ **2**

Question 11 continues

(g) **2**



The diagram shows points B and C on a circle with centre O and diameter AD, as shown in the diagram.

Let $\angle BAC = \alpha$, $\angle ABC = \beta$ and $\angle OAB = \gamma$.

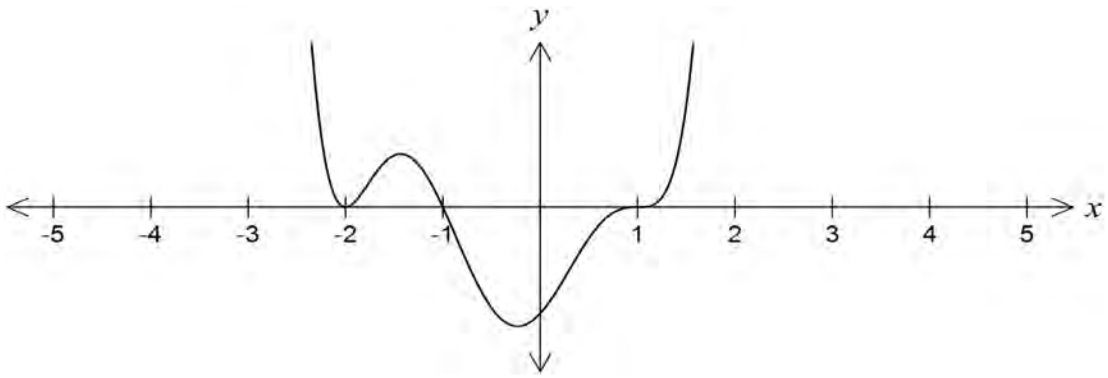
Copy or trace this diagram into your writing booklet.

Find the value of $\alpha + \beta + \gamma$, giving reasons for your answer.

End of Question 11

Question 12 (15 Marks) Use a SEPARATE writing booklet. **Marks**

(a) Write a polynomial equation for the graph $y = P(x)$ shown below. **1**



(b) Prove by mathematical induction, that for $n = 1, 2, 3 \dots$. **3**

$1 + (1+2) + \dots + (1 + 2 + \dots + 2^{n-1}) = 2^{n+1} - n - 2$

(c) The equation $x^3 + x^2 - 4x - k^2 = 0$, where $k > 0$ has only positive roots. If one of the roots is the product of the other two roots.

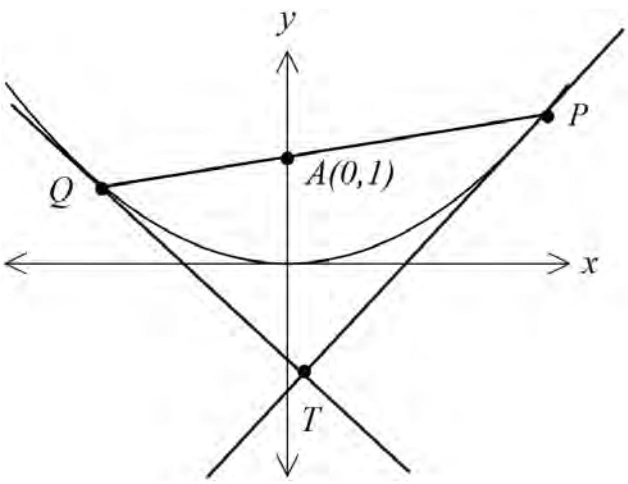
(i) Show that $x = k$ is a root of the equation. **2**

(ii) Hence or otherwise find the value of k. **2**

(d) The equation $x^2 - \sqrt{x} - 2 = 0$ has a root near $x = 2$. Use Newton's Method once to obtain a better approximation of x. Answer to 2 decimal places. **2**

Question 12 continues

(e)



The points $P\left(4p, 2p^2\right)$ and $Q\left(4q, 2q^2\right)$ lie on the parabola $x^2 = 8y$ The tangents to the parabola at *P* and *Q* intersect at T. The chord *PQ* passes through the point $A\left(0,1\right)$.

(i) Write down the equation of the tangent at *P*. **1**

(ii) Hence or otherwise find the coordinates of *T*. **1**

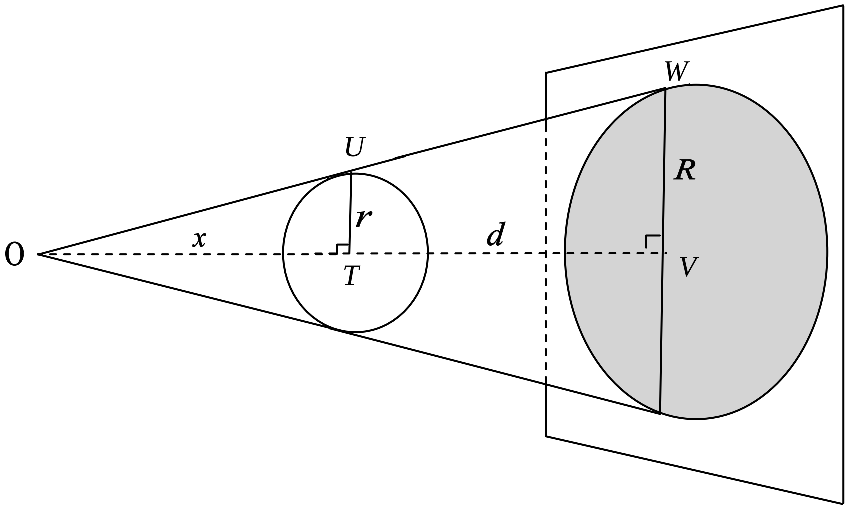
(iii) Show that the equation of the chord *PQ* is given by $2y = \left(p+q\right)x - 4pq$ **1**

(iv) Show that $pq = -\frac{1}{2}$ **2**

End of Question 12

Question 13 (15 Marks) Use a SEPARATE writing booklet. **Marks**

(a)



A coin of radius r cm is placed *x* cm from the light source *O*, such that its horizontal axis of symmetry passes through *O*.

A coin is placed d cm away from a screen.

The light source O is moving horizontally towards the coin at a speed of 4cm per second and casts a circular shadow of radius R cm on the screen as shown.

(i) Show that the area of the shadow is $A = \dfrac{\pi r^2 (x+d)^2}{x^2}$. **2**

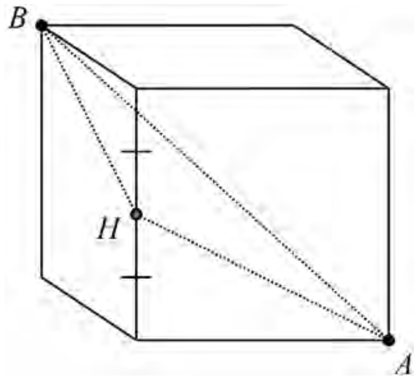
(ii) Find the rate of increase, $\dfrac{dA}{dt}$ of the area of the shadow when $x = d$. **2**

(b) Prove that $\tan^{-1}\dfrac{2}{3} + \cos^{-1}\dfrac{2}{\sqrt{5}} = \tan^{-1}\dfrac{7}{4}$. **3**

Question 13 continues

Question 13 continued **Marks**

(c)



The diagram shows a cube with edge length 2 units. H is the midpoint of the edge as shown in the diagram.

Using triangle AHB or otherwise, find the size of $\angle AHB$ **2**

(d) A particle moves in a straight line and its position at time t seconds is given by,

$$x = 4 + \frac{\sin 4t}{\sqrt{3}} - \cos 4t$$

(i) Express $\frac{\sin 4t}{\sqrt{3}} - \cos 4t$ in the form $R\sin(4t - \alpha)$, where α is in radians. **2**

(ii) The particle is undergoing Simple Harmonic Motion, show that the equation for acceleration is, **2**

$$\ddot{x} = -16(x - 4)$$

(iii) When does the particle first reach its maximum speed? **2**

End of Question 13

Question 14 (15 Marks) Use a SEPARATE writing booklet. **Marks**

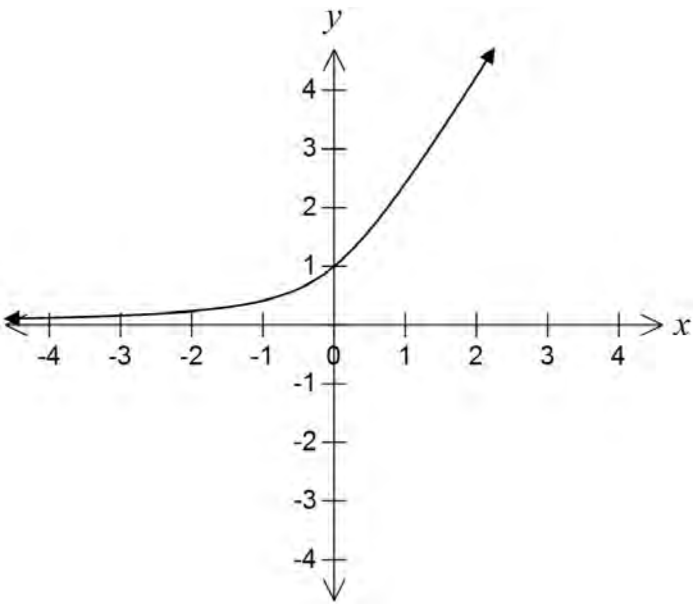
(a) By expanding both sides of the identity **2**

$(1+x)^{n+4} = (1+x)^n (1+x)^4$ prove that

$$\binom{n+4}{r} = \binom{n}{r} + 4\binom{n}{r-1} + 6\binom{n}{r-2} + 4\binom{n}{r-3} + \binom{n}{r-4}, \quad \text{for } r = 4, 5, \ldots, n$$

(b) Using the substitution $u = \tan x$, find $\int \left(\tan^3 x \sec^2 x\right) dx$ **2**

(c) Consider the function $f(x) = x + \sqrt{x^2+1}$



(i) State the range of $f(x)$ **1**

(ii) Show that $f'(x) = \dfrac{f(x)}{\sqrt{x^2+1}}$ and hence show that $f'(x) > 0$ for all x in the domain. **2**

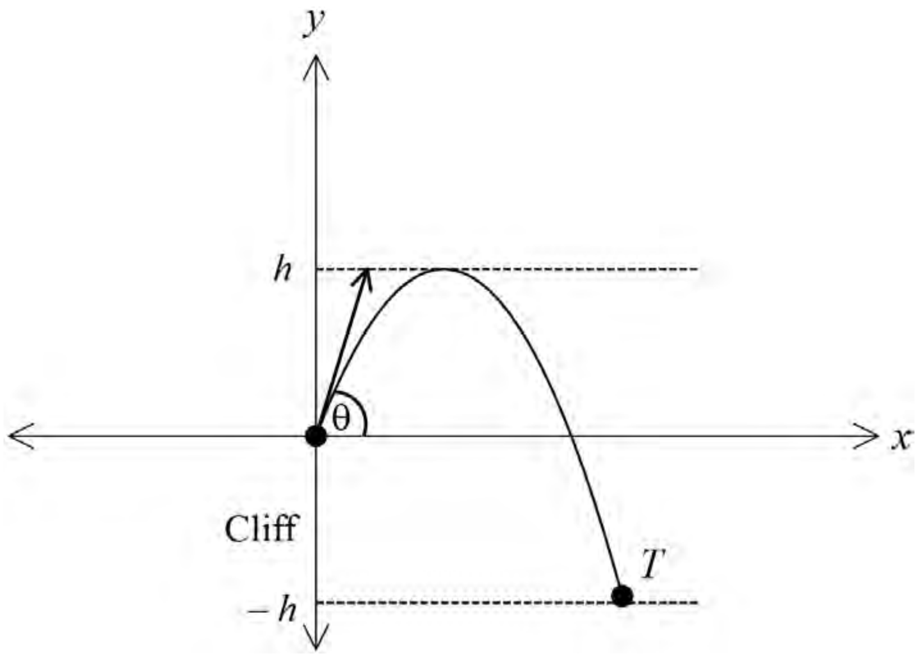
(iii) State the range of the inverse function of $f(x)$. **1**

(iv) Show that the inverse function is $f^{-1}(x) = \dfrac{1}{2}\left(x - \dfrac{1}{x}\right)$. **2**

Question 14 continues

Marks

(d)



The path of a projectile fired from the top *O* of a cliff is shown in the diagram above.

Its initial velocity is *V* m/s at an angle θ to the horizontal plane at *O*. It rises to a maximum height *h* metres above *O* and strikes a target *T* on the ground *h* metres below *O*.

Assuming the usual horizontal and vertical components of displacement in metres at time *t* seconds, are $x = Vt\cos\theta$ and $y = Vt\sin\theta - \frac{1}{2}gt^2$ respectively.

(i) Prove that $h = \frac{V^2\sin^2\theta}{2g}$. **2**

(i) Prove that the time taken for the projectile to reach its target is, **2**

$$\frac{V\sin\theta\left(1+\sqrt{2}\right)}{g} \text{ seconds}$$

(ii) Hence, show that the distance from the base of the cliff to the target is, **1**

$$\frac{V^2\left(1+\sqrt{2}\right)\sin 2\theta}{2g}$$

End of paper

2014 CTHS Mathematics Extension 1 AP4 Solutions

SECTION 1 MULTIPLE CHOICE

	Working	**Answer**
1.	$(2,-1) \quad (-2,-4)$ $-1:3$ $x=\frac{-1\times -2+3\times 2}{-1+3} = 4 \qquad y=\frac{-1\times -4+3\times -1}{-1+3} = \frac{1}{2}$	**A**
2.	$x(x+9)=400$ $x^2+9x=400$ $x^2+9x-400=0$ $(x+25)(x-16)=0$ $x=-25$ or $x=16$ x is a length $\therefore x=16$	**D**
3.	$\frac{d}{dx}\left(3\sin^{-1}\frac{x}{2}\right)=\frac{3}{\sqrt{2^2-x^2}} = \frac{3}{\sqrt{4-x^2}}$	**A**
4.	$\cos 105° = \cos(60°+45°)$ $= \cos 60° \cos 45° - \sin 60° \sin 45°$ $= \frac{1}{2}\times\frac{1}{\sqrt{2}}-\frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}$ $= \frac{1-\sqrt{3}}{2\sqrt{2}}$	**B**
5.	$P(x)=Q(x)(x^2-4)+2x-3$ $P(-2)=0+2\times -2-3$ $=-7$	**A**
6.	Inverse cos graph	**C**

	Working	**Answer**
7.	$x = 2p + \frac{2}{p} \qquad y = p^2 + \frac{1}{p^2}$	**A**

$$x = 2\left(p + \frac{1}{p}\right) \quad ......(1)$$

$$y = \left(p + \frac{1}{p}\right)^2 - 2p \times \frac{1}{p} \quad ......(2)$$

$$= \left(p + \frac{1}{p}\right)^2 - 2$$

$$= \left(\frac{x}{2}\right)^2 - 2 \quad ....(1)\,\text{in}\,(2)$$

$$y = \frac{x^2}{4} - 2$$

8. **B**

$$\cos\theta = \frac{1-t^2}{1+t^2}$$

$$1 + \sec\theta = 1 + \frac{1+t^2}{1-t^2}$$

$$= \frac{1-t^2+1+t^2}{1-t^2}$$

$$= \frac{2}{1-t^2}$$

9. **D**

$$\int \frac{x}{3}\sqrt{x+2}\ dx$$

$$u = x + 2$$

$$x = u - 2$$

$$\frac{du}{dx} = 1$$

$$du = dx$$

$$\therefore \int \frac{x}{3}\sqrt{x+2}\ dx = \int\left(\frac{u-2}{3}\sqrt{u}\right)du$$

$$= \frac{1}{3}\int\left((u-2)\sqrt{u}\right)\,du$$

$$= \frac{1}{3}\int\left(u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right)\,du$$

	Working	**Answer**
10.	$2\sin 2\theta = 2$	**A**

$$\sin 2\theta = 1$$

$$2\theta = \pi n + (-1)\sin^{-1}(1)$$

$$\theta = \frac{\pi n}{2} + \frac{(-1)}{2} \times \frac{\pi}{2}$$

$$\theta = \frac{\pi n}{2} + \frac{(-1)\pi}{4}$$

SECTION 2 SHORT ANSWER

11.	Solution	Marks
(a)	$\lim_{x\to 0}\frac{\tan 3x}{2x}=\frac{1}{2}\lim_{x\to 0}\frac{\tan 3x}{x}$ $=\frac{1}{2}\times\frac{3}{3}\lim_{x\to 0}\frac{\tan 3x}{x}$ $=\frac{3}{2}\lim_{x\to 0}\frac{\tan 3x}{3x}$ $=\frac{3}{2}\times 1$ $=\frac{3}{2}$	1
(b)	$y=\cos(\ln x)$ $\frac{dy}{dx}=-\sin(\ln x)\times\frac{1}{x}$ When $x=1$ $\frac{dy}{dx}=-\sin(\ln 1)\times\frac{1}{1}$ $=0$ $\therefore$ the gradient of the tangent to the curve $y=\cos(\ln x)$ is 0.	2
(c)	$\int\frac{1}{4+9x^2}\,dx$ $=\int\frac{1}{9\left(\frac{4}{9}+x^2\right)}\,dx$ $=\frac{1}{9}\int\frac{1}{\frac{4}{9}+x^2}\,dx$ $=\frac{1}{9}\times\frac{3}{2}\tan^{-1}\left(\frac{3}{2}\times x\right)+C$ $=\frac{1}{6}\tan^{-1}\left(\frac{3x}{2}\right)+C$	2

11.	Solution	Marks
(d)		3

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos^2 3x\, dx$$

$$\cos 6x = 2\cos^2 3x - 1$$

$$2\cos^2 3x = \cos 6x + 1$$

$$\cos^2 3x = \frac{\cos 6x}{2} + \frac{1}{2}$$

$$\therefore \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos^2 3x\, dx = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left(\frac{\cos 6x}{2} + \frac{1}{2}\right) dx$$

$$= \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (\cos 6x + 1)\, dx$$

$$= \frac{1}{2}\left[\frac{\sin 6x}{6} + x\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[\left(\frac{\sin 6\left(\frac{\pi}{2}\right)}{6} + \frac{\pi}{2}\right) - \left(\frac{\sin 6\left(\frac{\pi}{3}\right)}{6} + \frac{\pi}{3}\right)\right]$$

$$= \frac{1}{2}\left[\left(0 + \frac{\pi}{2}\right) - \left(0 + \frac{\pi}{3}\right)\right]$$

$$= \frac{1}{2} \times \left(\frac{\pi}{2} - \frac{\pi}{3}\right)$$

$$= \frac{1}{2} \times \frac{\pi}{6}$$

$$= \frac{\pi}{12}$$

(e) **3**

$$\frac{x^2 + x - 6}{x} \geq 2$$

$$x^2 \times \left(\frac{x^2 + x - 6}{x}\right) \geq 2 \times x^2$$

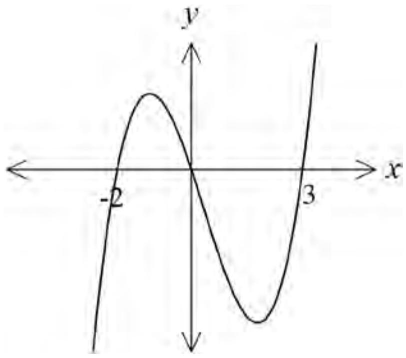
$$x\left(x^2 + x - 6\right) \geq 2x^2$$

$$x\left(x^2 + x - 6\right) - 2x^2 \geq 0$$

$$x\left(x^2 + x - 6 - 2x\right) \geq 0$$

$$x\left(x^2 - x - 6\right) \geq 0$$

$$x(x-3)(x+2) \geq 0$$



$$-2 \leq x < 0 \quad \text{or} \quad x \geq 3$$

11.	Solution	Marks
(f)	$y_1 = \sin x$ and $y_2 = \cos x$	2

$\frac{dy}{dx_1} = \cos x$ and $\frac{dy}{dx_2} = -\sin x$

Curves intersect at $P\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$

At $x = \frac{\pi}{4}$

$\frac{dy}{dx_1} = \cos\left(\frac{\pi}{4}\right)$ and $\frac{dy}{dx_2} = -\sin\left(\frac{\pi}{4}\right)$

$m_1 = \frac{1}{\sqrt{2}}$ and $m_2 = -\frac{1}{\sqrt{2}}$

$$\tan\alpha = \left|\frac{\frac{1}{\sqrt{2}} - -\frac{1}{\sqrt{2}}}{1 + \frac{1}{\sqrt{2}} \times -\frac{1}{\sqrt{2}}}\right|$$

$$= \left|\frac{\frac{2}{\sqrt{2}}}{1 - \frac{1}{2}}\right|$$

$$= \left|\frac{2}{\sqrt{2}} \div \frac{1}{2}\right|$$

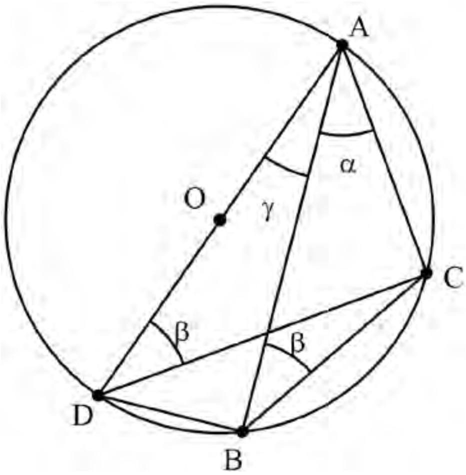
$$= \left|\frac{2}{\sqrt{2}} \times \frac{2}{1}\right|$$

$$= \frac{4}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{4\sqrt{2}}{2}$$

$$= 2\sqrt{2}$$

11. (g) **Solution** **Marks 2**



$\angle ABC = \beta$ (given)

$\angle ABC = \angle ADC$ (angles subtended by the same arc are equal)

$\therefore \angle ADC = \beta$

$\angle ACD = 90°$ (angle at the centre is twice the angle at the circumference, diameter is 180°)

$\angle ADC + \angle DCA + \angle CAD = 180°$ (angle sum of a triangle is 180°)

$\angle DAB = \gamma$ and $\angle BAC = \alpha$ (given)

$\angle DAC = \alpha + \gamma$ (adjacent angles)

$\therefore \beta + 90° + (\alpha + \gamma) = 180°$

$\therefore \alpha + \beta + \gamma = 90°$

OR

$ABCD$ is a cyclic quadrilateral

$\therefore \angle DAC + \angle DBC = 180°$ (opposite angles in a cyclic quadrilateral are supplementary)

$\angle ABD = 90°$ (angle in a semicircle)

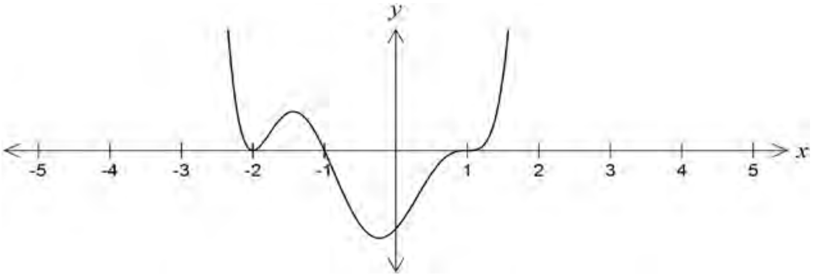
$\angle DAB = \gamma, \angle BAC = \alpha, \angle ABC = \beta$ (given)

$\angle DAC = \angle DAB + \angle BAC$ (adjacent angles)

$\therefore \gamma + \alpha + \beta + 90° = 180°$

$\therefore \alpha + \beta + \gamma = 90°$

12.	Solution	Marks
(a)		1



Any equation of the form

$y = k(x+2)^2(x+1)(x-1)^3$

where $k > 0$

(b) **3**

Prove that,

$1+(1+2)+...+(1+2+...+2^{n-1}) = 2^{n+1}-n-2$

Prove true for $n=1$

$LHS = 1$

$RHS = 2^{1+1}-1-2$

$= 4-1-2$

$= 1$

$\therefore$ it is true for $n=1$

Assume true for $n=k$

$1+(1+2)+...+(1+2+...+2^{k-1}) = 2^{k+1}-k-2$

Prove true for $n=k+1$

i.e. Prove that

$1+(1+2)+...+(1+2+...+2^{k-1})+(1+2+...+2^{k-1}+2^{k-1+1}) = 2^{(k+1)+1}-(k+1)-2$

$1+(1+2)+...+(1+2+...+2^{k-1})+(1+2+...+2^{k-1}+2^{k}) = 2^{k+2}-(k+1)-2$

$LHS = 1+(1+2)+...+(1+2+...+2^{k-1})+(1+2+...+2^{k-1}+2^{k})$

$= 2^{k+1}-k-2+(1+2+...+2^{k-1}+2^{k})$ by assumption

Now, $1+2+...+2^{k-1}+2^{k}$ is a GP

$a=1,\ r=2,\ n=k+1$

$S_k = \dfrac{1\times(2^{k+1}-1)}{2-1}$

$= 2^{k+1}-1$

$LHS = 2^{k+1}-k-2+2^{k+1}-1$

$= 2\times 2^{k+1}-k-3$

$= 2^1\times 2^{k+1}-(k+1)-2$

$= 2^{k+2}-(k+1)-2$

$= RHS$

$\therefore$ Proved true by mathematical induction

12.		Solution	Marks
(c)	(i)	$x^3 + x^2 - 4x - k^2 = 0$	**2**

Let the roots of the equation be α, β and γ

$\gamma = \alpha\beta$

$a = 1, b = 1, c = -4$ and $d = -k^2$

Product of the roots

$\alpha\beta\gamma = -\dfrac{d}{a}$

$\gamma\gamma = -\dfrac{-k^2}{1}$

$\gamma^2 = k^2$

Since $k > 0$ and all $\alpha, \beta, \gamma > 0$

$\gamma = k$

$\therefore x = k$ is a root of the equation

(ii) Sum of the roots 1 at a time **2**

$\alpha + \beta + k = -\dfrac{b}{a}$

$\alpha + \beta + k = -1$

$\alpha + \beta = -k - 1$

Sum of the roots 2 at a time

$k + \alpha k + \beta k = \dfrac{c}{a}$

$k\left(1 + \alpha + \beta\right) = \dfrac{-4}{1}$

$k\left(1 + -k - 1\right) = -4$

$-k^2 = -4$

$\therefore k^2 = 4$

$\therefore k = 2 \ \left(\text{since } k > 0\right)$

OR

$k^3 + k^2 - 4k - k^2 = 0$

$k^3 - 4k = 0$

$k(k+2)(k-2) = 0$

$\therefore k = 2$ only as $k > 0$

12.		Solution	Marks
(d)		$x^2-\sqrt{x}-2=0$	2

Let $f(x)=x^2-\sqrt{x}-2$

$=x^2-x^{\frac{1}{2}}-2$

$f'(x)=2x-\frac{1}{2}x^{-\frac{1}{2}}$

$f'(2)=2\times 2-\frac{1}{2\sqrt{2}}$

$f(2)=2^2-\sqrt{2}-2$

$a_2=a_1-\frac{f(x)}{f'(x)}$

$=2-\frac{2-\sqrt{2}}{4-\frac{1}{2\sqrt{2}}}$

≈ 1.84

(e) (i) $x^2=8y$ **1**

$y=\frac{x^2}{8}$

$y'=\frac{2x}{8}$

$=\frac{x}{4}$

at $x=4p$

$y'=\frac{4p}{4}$

$=p$

$y-2p^2=p(x-4p)$

$y-2p^2=px-4p^2$

$\therefore$ tangent at P is

$y=px-2p^2$

12.	Solution	Marks
(ii)	Tangent at Q is	1

$y = qx - 2q^2$......(2)

$y = px - 2p^2$.......(1)

$(1)-(2)\quad 0 = (p-q)x - 2(p^2 - q^2)$

$(p-q)x = 2(p-q)(p+q)$

$x = 2(p+q)$

sub into (1) $y = p \times 2(p+q) - 2p^2$

$y = 2p^2 + 2pq - 2p^2$

$y = 2pq$

$\therefore T$ is $\left(2(p+q), 2pq\right)$

(iii) $P(4p, 2p^2)$ and $Q(4q, 2q^2)$ **1**

$$m = \frac{2p^2 - 2q^2}{4p - 4q}$$

$$= \frac{2(p+q)(p-q)}{4(p-q)}$$

$$= \frac{p+q}{2}$$

$y - 2p^2 = \frac{p+q}{2}(x - 4p)$

$2y - 4p^2 = (p+q)(x - 4p)$

$2y - 4p^2 = px - 4p^2 + qx - 4pq$

$2y = (p+q)x - 4pq$

(iv) Since PQ passes through $A(0,1)$ **2**

$2 \times 1 = (p+q) \times 0 - 4pq$

$2 = -4pq$

$pq = -\frac{1}{2}$

13.		Solution	Marks
(a)	(i)	$\Delta OWX \parallel \Delta OYZ$ (equiangular triangles are similar)	2

$\therefore \frac{x}{x+d} = \frac{r}{R}$ $\left(\begin{array}{l}\text{corresponding sides of similar triangles} \\ \text{are in proportion}\end{array}\right)$

$\frac{R}{r} = \frac{x+d}{x}$

$R = \frac{r(x+d)}{x}$

Area of the shadow is given by

$A = \pi R^2$

$= \pi\left[\frac{r(x+d)}{x}\right]^2$

$= \frac{\pi r^2 (x+d)^2}{x^2}$

(ii) **2**

$\frac{dx}{dt} = -4$

$A = \frac{\pi r^2 (x+d)^2}{x^2}$

$= \frac{\pi r^2 (x^2 + 2xd + d^2)}{x^2}$

$= \frac{\pi r^2 x^2 + 2\pi r^2 dx + \pi r^2 d^2}{x^2}$

$= \pi r^2 + 2\pi r^2 dx^{-1} + \pi r^2 d^2 x^{-2}$

$\frac{dA}{dx} = -2\pi r^2 dx^{-2} - 2\pi r^2 d^2 x^{-3}$

$\frac{dA}{dt} = \frac{dA}{dx} \times \frac{dx}{dt}$

$= \left(-\frac{2\pi r^2 d}{x^2} - \frac{2\pi r^2 d^2}{x^3}\right) \times -4$

When $x = d$

$\frac{dA}{dt} = \left(-\frac{2\pi r^2 d}{d^2} - \frac{2\pi r^2 d^2}{d^3}\right) \times -4$

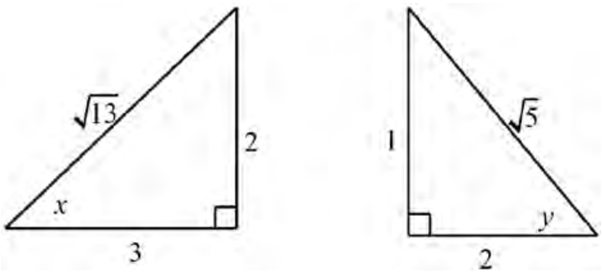
$= \frac{8\pi r^2}{d} + \frac{8\pi r^2}{d}$

$= \frac{16\pi r^2}{d}$

13. (b)	Solution	Marks
	$\tan^{-1}\frac{2}{3}+\cos^{-1}\frac{2}{\sqrt{5}}=\tan^{-1}\frac{7}{4}$	3

Let $x=\tan^{-1}\frac{2}{3}$ and $y=\cos^{-1}\frac{2}{\sqrt{5}}$

$\therefore \tan x=\frac{2}{3}$ and $\tan y=\frac{2}{\sqrt{5}}$



$$\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y}$$

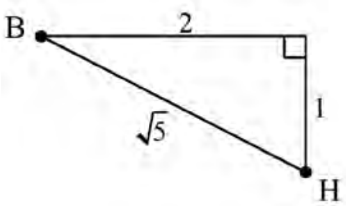
$$=\frac{\frac{2}{3}+\frac{1}{2}}{1-\frac{2}{3}\times\frac{1}{2}}$$

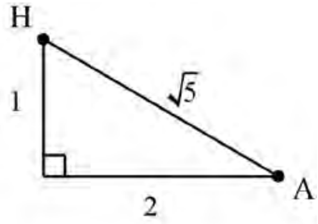
$$=\frac{7}{4}$$

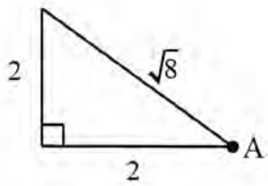
$\therefore\ x+y=\tan^{-1}\frac{7}{4}$

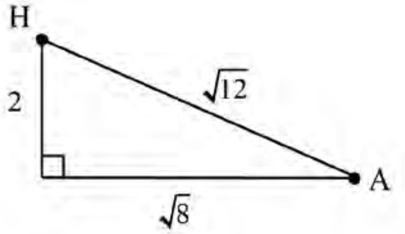
$\therefore \tan^{-1}\frac{2}{3}+\cos^{-1}\frac{2}{\sqrt{5}}=\tan^{-1}\frac{7}{4}$

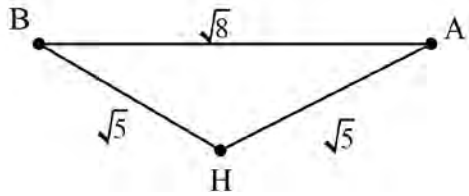
13. (c) **Solution** **Marks 2**











$$\cos(\angle AHB) = \frac{\left(\sqrt{5}\right)^2 + \left(\sqrt{5}\right)^2 - \left(\sqrt{12}\right)^2}{2 \times \sqrt{5} \times \sqrt{5}}$$

$$= \frac{5+5-12}{2 \times 5}$$

$$= -\frac{1}{5}$$

$$\therefore \angle AHB = 101 \cdot 5^\circ$$

13.	**Solution**	**Marks**

(d) (a) $\dfrac{\sin 4t}{\sqrt{3}} - \cos 4t = R\sin\left(4t - \alpha\right)$ **2**

$R\sin\left(4t-\alpha\right) = R\left(\sin 4t\cos\alpha - \cos 4t\sin\alpha\right)$

$R\sin 4t\cos\alpha - R\cos 4t\sin\alpha = \dfrac{\sin 4t}{\sqrt{3}} - \cos 4t$

Equating co-efficients

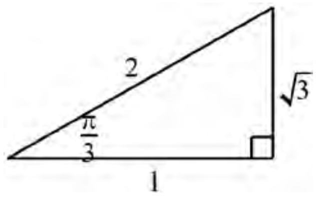
$R\sin\alpha = 1$.......(1)

$R\cos\alpha = \dfrac{1}{\sqrt{3}}$.....(2)

$(1)\div(2)\ldots\tan\alpha = \dfrac{1}{\frac{1}{\sqrt{3}}}$

$= \sqrt{3}$

$\alpha = \dfrac{\pi}{3}$



$R^2\sin^2\alpha = 1^2$.......$(1)^2$

$R^2\cos^2\alpha = \left(\dfrac{1}{\sqrt{3}}\right)^2$ .....$(2)^2$

$(1)^2+(2)^2 \ldots R^2\sin^2\alpha + R^2\cos^2\alpha = 1 + \dfrac{1}{3}$

$R^2\times 1 = \dfrac{4}{3}$

$R = \dfrac{2}{\sqrt{3}}$

$\therefore \dfrac{\sin 4t}{\sqrt{3}} - \cos 4t = \dfrac{2}{\sqrt{3}}\sin\left(4t - \dfrac{\pi}{3}\right)$

(b) $x = 4 + \dfrac{\sin 4t}{\sqrt{3}} - \cos 4t$ **2**

$\dot{x} = \dfrac{4\cos 4t}{\sqrt{3}} + 4\sin 4t$

$\ddot{x} = \dfrac{-16\sin 4t}{\sqrt{3}} + 16\cos 4t$

$= -16\left(\dfrac{\sin 4t}{\sqrt{3}} - \cos 4t\right)$

$= -16\left(4 + \dfrac{\sin 4t}{\sqrt{3}} - \cos 4t - 4\right)$

$= -16\left(x-4\right)$

13.	**Solution**	**Marks**
(c)	$x=\frac{2}{\sqrt{3}}\sin\left(4t-\frac{\pi}{3}\right)$	**2**
	$\dot{x}=4\times\frac{2}{\sqrt{3}}\cos\left(4t-\frac{\pi}{3}\right)$	
	Maximum speed occurs when $\cos\left(4t-\frac{\pi}{3}\right)=1$	
	$\therefore$ maximum speed is $\frac{8}{\sqrt{3}}$ or $\frac{8\sqrt{3}}{3}$	

14.	**Solution**	**Marks**
(a)	$(1+x)^{n+4}=(1+x)^n(1+x)^4$	**2**

$LHS=(1+x)^{n+4}$

$$=\binom{n+4}{0}+\binom{n+4}{1}x+\binom{n+4}{2}x^2+\ldots+\ldots+\binom{n+4}{r}x^r+\ldots+\binom{n+4}{n+4}x^{n+4}$$

$RHS=(1+x)^n(1+x)^4$

$$=\left[\binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+\ldots+\binom{n}{r}x^r+\ldots+\binom{n}{n}x^n\right]$$

$$\left[\binom{4}{0}+\binom{4}{1}x+\binom{4}{2}x^2+\binom{4}{3}x^3+\binom{4}{4}x^4\right]$$

Equating co-efficients of x^r

$$\binom{n+4}{r}=\binom{4}{0}\binom{n}{r}+\binom{4}{1}\binom{n}{r-1}+\binom{4}{2}\binom{n}{r-2}+\binom{4}{3}\binom{n}{r-3}+\binom{4}{4}\binom{n}{r-4}$$

$$=\binom{n}{r}+4\binom{n}{r-1}+6\binom{n}{r-2}+4\binom{n}{r-3}+\binom{n}{r-4}$$

(b) $\int\left(\tan^3 x\sec^2 x\right)dx$ **2**

$u=\tan x$

$\frac{du}{dx}=\sec^2 x$

$du=\sec^2 x\,dx$

$$\therefore\int\left(\tan^3 x\sec^2 x\right)dx=\int u^3du$$

$$=\left[\frac{u^4}{4}\right]+C$$

$$=\frac{\tan^4 x}{4}+C$$

14.		Solution	Marks
(c)	(i)	Range is all real $y > 0$	1
	(ii)	$f(x) = x + \sqrt{x^2+1}$	2

$$f(x) = x + \left(x^2+1\right)^{\frac{1}{2}}$$

$$f'(x) = 1 + \frac{1}{2}\left(x^2+1\right)^{-\frac{1}{2}} \times 2x$$

$$= 1 + \frac{x}{\sqrt{x^2+1}}$$

$$= \frac{\sqrt{x^2+1}}{\sqrt{x^2+1}} + \frac{x}{\sqrt{x^2+1}}$$

$$= \frac{x+\sqrt{x^2+1}}{\sqrt{x^2+1}}$$

$$= \frac{f(x)}{\sqrt{x^2+1}}$$

	(iii)	Range is all real y	1
	(iv)	$f:\ y = x + \sqrt{x^2+1}$	2

$$f^{-1}:\ x = y + \sqrt{y^2+1}$$

$$x - y = \sqrt{y^2+1}$$

$$\left(x-y\right)^2 = y^2+1$$

$$x^2 - 2xy + y^2 = y^2 + 1$$

$$x^2 - 2xy = 1$$

$$2xy = x^2 - 1$$

$$y = \frac{x^2-1}{2x}$$

$$= \frac{1}{2}\left(x - \frac{1}{x}\right)$$

14.	**Solution**	**Marks**
(d) (i)	Maximum height occurs when $\dot{y}=0$	**2**

$$y=Vt\sin\theta-\frac{1}{2}gt^2$$

$$\dot{y}=V\sin\theta-gt$$

$$V\sin\theta-gt=0$$

$$gt=V\sin\theta$$

$$t=\frac{V\sin\theta}{g}$$

at $t=\dfrac{V\sin\theta}{g}$

$$y=V\left(\frac{V\sin\theta}{g}\right)\sin\theta-\frac{1}{2}g\left(\frac{V\sin\theta}{g}\right)^2$$

$$=\frac{V^2\sin^2\theta}{g}-\frac{V^2\sin^2\theta}{2g}$$

$$=\frac{2V^2\sin^2\theta}{2g}-\frac{V^2\sin^2\theta}{2g}$$

$$=\frac{V^2\sin^2\theta}{2g}$$

(ii) Projectile reaches it's target when $y=-h$ **2**

$$Vt\sin\theta-\frac{1}{2}gt^2=-h$$

$$Vt\sin\theta-\frac{1}{2}gt^2=-\frac{V^2\sin^2\theta}{2g}$$

$$\frac{2Vt\sin\theta g}{g}-\frac{g^2t^2}{g}+\frac{V^2\sin^2\theta}{g}=0$$

$$g^2t^2-2V\sin\theta gt-V^2\sin^2\theta=0$$

$$\therefore\ t=\frac{--2V\sin\theta g\pm\sqrt{(-2V\sin\theta g)^2-4\times g^2\times -V^2\sin^2\theta}}{2g^2}$$

$$=\frac{2V\sin\theta g\pm\sqrt{4V^2\sin^2\theta g^2+4g^2V^2\sin^2\theta}}{2g^2}$$

Since t cannot be negative

$$t=\frac{2V\sin\theta g+\sqrt{8V^2\sin^2\theta g^2}}{2g^2}$$

$$=\frac{2V\sin\theta g+2\sqrt{2}(Vg\sin\theta)}{2g^2}$$

$$=\frac{V\sin\theta+\sqrt{2}(V\sin\theta)}{g}$$

$$=\frac{V\sin\theta\left(1+\sqrt{2}\right)}{g}\text{ seconds}$$

(iii) **1**

When the projectile reaches the target, $t=\dfrac{V\sin\theta\left(1+\sqrt{2}\right)}{g}$ seconds

$$x=V\frac{V\sin\theta\left(1+\sqrt{2}\right)}{g}\cos\theta$$

$$=\frac{V^2\sin\theta\cos\theta\left(1+\sqrt{2}\right)}{g}\times\frac{2}{2}$$

$$=\frac{V^2 2\sin\theta\cos\theta\left(1+\sqrt{2}\right)}{2g}$$

$$=\frac{V^2\left(1+\sqrt{2}\right)\sin 2\theta}{g}$$