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NAME_______________

CLASS 12MT___

CHERRYBROOK TECHNOLOGY HIGH SCHOOL

2017

YEAR 12

AP4

MATHEMATICS EXTENSION 1

Time allowed – 2 HOURS

+ 5 Minutes Reading Time

DIRECTIONS TO CANDIDATES:

- Attempt all sections.
- Write your name on the paper. Questions 1 - 10 are to be answered on the multiple choice answer sheet.
- Write your answers to questions 11 – 14 in separate writing booklets
- All necessary working should be shown in every question. Full marks may not be awarded for careless or badly arranged work.
- Approved calculators may be used. Reference Sheets are provided.

Section I

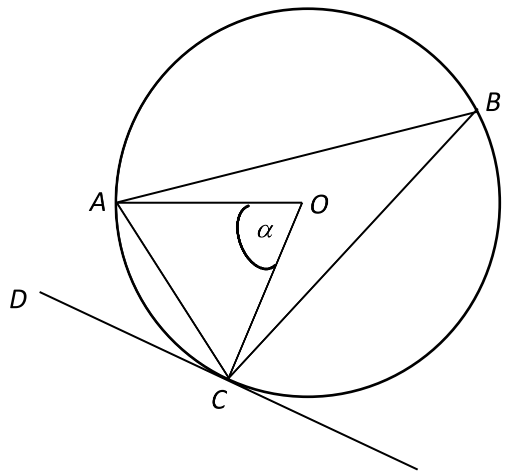
10 marks

Attempt Questions 1 – 10.

Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for Questions 1 – 10.

1. *A, B* and *C* are points on a circle with centre *O*. The line *DC* is tangent to the circle at *C*. $\angle AOC = \alpha$.



What is $\angle DCA$ in terms of α ?

(A) 2α

(B) $\frac{\alpha}{2}$

(C) $180° - 2\alpha$

(D) $\frac{360° - \alpha}{2}$

2. The equation of the line l_1 is $y = (\tan\theta)x + b$.

The equation of the line l_2 is $y = (-\tan\theta)x - b$.

What are the correct expressions for α, the angle between the two lines?

(A) $\alpha = 90°, 270°$

(B) $\alpha = \pm 2\theta$

(C) $\alpha = \theta \pm 90°$

(D) $\alpha = 180° \pm \theta$

3. One approximate solution of the equation $f(x) = 4x^3 - 15x^2 + 22x - 12$ is $x = 1.3$.

What is another approximation to this solution using one application of Newton's method?

(A) $x = 1.2884$

(B) $x = 1.2885$

(C) $x = 1.2886$

(D) $x = 1.2887$

4. What are the domain and the range of the function $y = 2\tan^{-1} 3x$?

(A) domain: all real x, range: $-\pi \leq y \leq \pi$

(B) domain: all real x, range: $-2\pi \leq y \leq 2\pi$

(C) domain: $-\frac{1}{3} \leq x \leq \frac{1}{3}$, range: $-\pi \leq y \leq \pi$

(D) domain: $-\frac{1}{3} \leq x \leq \frac{1}{3}$,, range: $-2\pi \leq y \leq 2\pi$

5. An object is projected from the ground with an initial velocity of 35 m/s at an angle of 30° to the horizontal. Acceleration due to gravity is $g = 10m/s^2$.

What is the time of flight of the object?

(A) 0.68 s

(B) 6.8 s

(C) 1.7 s

(D) 3.5 s

6. A particle is moving with simple harmonic motion.

Its velocity is given by $v^2 = 30 + 9x - 3x^2$.

What are the endpoints of its motion?

(A) $x = -5,2$

(B) $x = -10,10$

(C) $x = -2,5$

(D) $x = -30,30$

7. How many solutions exist for the equation $(1 + \sin 2\theta)(1 - \sin 2\theta) = 0.5$ in the domain $0 \leq \theta \leq 2\pi$?

(A) 1

(B) 2

(C) 4

(D) 8

8. The polynomial $P(x) = 2(x-3)^2(x+3)^2$.

What is the coefficient of the x^2 term?

(A) 0

(B) -9

(C) -18

(D) -36

9. Evaluate $\int_{-5}^{-4} x(x+5)^3\, dx$ using the substitution $x = u - 5$.

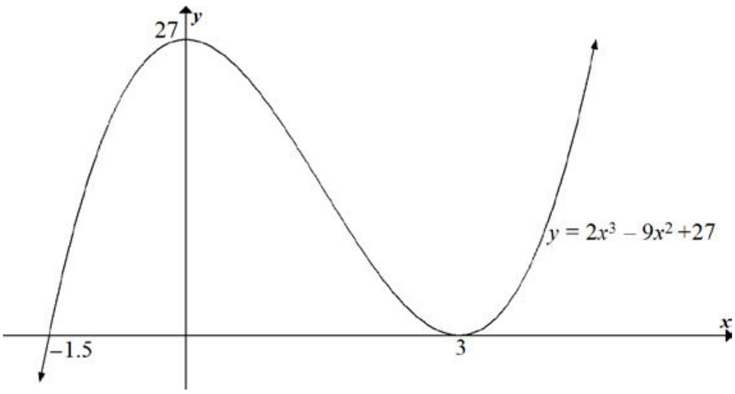
(A) $-\frac{21}{20}$

(B) -4

(C) $\frac{68}{5}$

(D) 14

10. The diagram shows the function $y = 2x^3 - 9x^2 + 27$. It crosses the x-axis at x = –1.5. It has turning points at $(0,27)$ and $(3,0)$.



For what values of k does the function $y = 2x^3 - 9x^2 + k$ have one root?

(A) $k < -1.5, k > 3$

(B) $-1.5 < k < 3$

(C) $k < 0, k > 27$

(D) $0 < k < 27$

Section II

60 marks

Attempt Questions 11 – 14.

Allow about 1 hour and 45 minutes for this section.

Answer each <u>question</u> in a new writing booklet. Extra writing booklets are available.

In Questions 11 – 14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a new writing booklet. **Marks**

(a) Find the equation of the directrix of the parabola. $x = t, y = \dfrac{t^2}{4}$ **1**

(b) Find $\lim\limits_{x \to 0} \dfrac{\sin 3x}{9x}$ **1**

(c) The points A and B have coordinates $(-1,6)$ and $(14,-9)$ respectively. The point P (x, y) divides the interval AB externally in the ratio $2:3$. Find the coordinates of P. **2**

(d) Differentiate $\sin^{-1}\dfrac{5x}{2}$ **1**

(e) Find the general solution of the equation $\cot\theta\sec^2\theta - \cot\theta - \sqrt{3}\sec^2\theta + \sqrt{3} = 0.$ **3**

(f) A chemical is absorbed into a person's blood through a patch placed on the person's skin.

The rate at which the chemical is absorbed is given by the equation

$\frac{dA}{dt} = -k(A - 0.9)$ where A is the amount in milligrams and t is time in minutes.

i) Show that $A = 0.9 - 0.7e^{-kt}$ satisfies this equation.

ii) How much chemical was initially in the person's blood? 1

iii) After 1 hour, the amount of chemical in the person's blood is 0.193 mg. 1

Show that $k = -\frac{\ln(1.01)}{60}$ 2

iv) Find the amount of chemical in the person's blood after 16 hours, correct to two significant figures.

v) After 16 hours, the patch is removed. The rate at which the chemical is absorbed into the blood is then given by $\frac{dA}{dt} = -\frac{\ln(1.01)}{60}A$ mg/min. 1

Find the amount of chemical in the blood 24 hours after the patch was applied to the skin. 2

Question 12 (15 marks) Use a new writing booklet.

(a) i) Simplify $\frac{r({}^nC_r)}{({}^nC_{r-1})}$ **2**

ii) Hence, or otherwise, show that $\frac{{}^nC_1}{{}^nC_0} + \frac{2\,{}^nC_2}{{}^nC_1} + \frac{3\,{}^nC_3}{{}^nC_2} + \cdots . + \frac{n\,{}^nC_n}{{}^nC_{n-1}} = \frac{n(n+1)}{2}$ **2**

(b) i) Show that $\frac{x^3+x+2}{1+x^2} = x + \frac{2}{1+x^2}$. **2**

ii) Hence find $\int \frac{x^3+x+2}{1+x^2}\,dx$. **2**

(c) Consider the function $f(x) = \log_e x + 1$.

i) Find an expression for $g(x)$, the inverse function of $f(x)$. **1**

ii) State the domain and range of $g(x)$. **1**

(d) The point $P(2ap, ap^2)$ lies on the parabola $x^2 = 4ay$. $S(0, a)$ is the focus of the parabola. The interval PQ is perpendicular to the x-axis and meets the x-axis at Q. **2**

Find the equation of the locus of the midpoint of SQ.

(e) Use mathematical induction to prove that $7^n - 3^n$ is divisible by 4 for $n \geq 1$. **3**

Question 13 (15 marks) Use a new writing booklet

(a) The acceleration of a particle is given by $\ddot{x} = \cos^2 x$. The velocity $v \geq 0$ for all x and the particle is initially at rest at the origin. **3**

Find an expression for the velocity in terms of x.

(b) The circumference of a circular drop of liquid with radius r and area A increases at the rate of 5mm/s.

i) Show that $\frac{dr}{dt} = \frac{5}{2\pi}$. **2**

ii) Show that $\frac{dA}{dt} = 5r$. **1**

(c) A particle moves in a straight line so that its acceleration is given by $a = x + 1.5$ ms^{-2}. Initially, the particle is 5 metres to the right of O and moving towards O with a velocity of 6 ms^{-1}.

i) Is the particle speeding up or slowing down? Give a reason. **1**

ii) Show that $v^2 = x^2 + 3x - 4$. **2**

iii) Where does the particle first change direction? **1**

(d)

i) Show that $y = \frac{x^2+1}{x^2-1}$ is an even function. **1**

ii) Sketch the graph of $y = \frac{x^2+1}{x^2-1}$, clearly labelling any asymptotes and any intercepts with the x- and y-axes. **4**

Question 14 (15 marks) Use a new writing booklet **Marks**

(a) A particle moves with simple harmonic motion. Its amplitude is 3. Its acceleration is given by $\ddot{x} = -4(x-3)$ m/s^2.

i) State the centre of motion of the particle. **1**

ii) Explain why the speed of the particle is 0 m/s at the origin. **1**

iii) Find the maximum acceleration of the particle. **1**

(b) A particle is moving in a straight line under SHM. At any time (t seconds) its displacement (x metres) from a fixed point O is given by:

$$x = A\cos\left(\frac{\pi}{4}t + \alpha\right) \text{ where } A > 0 \text{ and } 0 < \alpha < \frac{\pi}{2}$$

After 1 second the particle is 2 metres to the right of O and after 3 seconds the particle is 4 metres to the left of O.

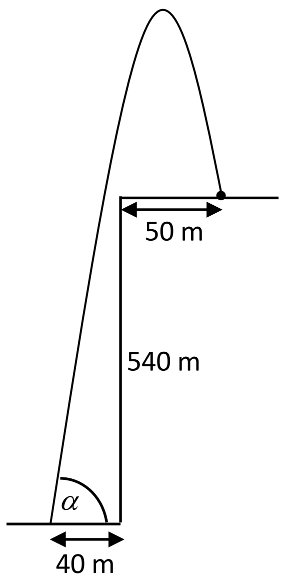
i) Show that $A\sin\alpha - A\cos\alpha = -2\sqrt{2}$ and $A\sin\alpha + A\cos\alpha = 4\sqrt{2}$. **2**

ii) Show that $A = 2\sqrt{5}$ and $\alpha = \tan^{-1}\frac{1}{3}$. **2**

iii) When does the particle first pass through O? **1**

(c) The motion of a projectile can be represented by the equations **3**

$x = 150t\cos\alpha$ and $y = -5t^2 + 150t\sin\alpha$ where α is the angle of projection.

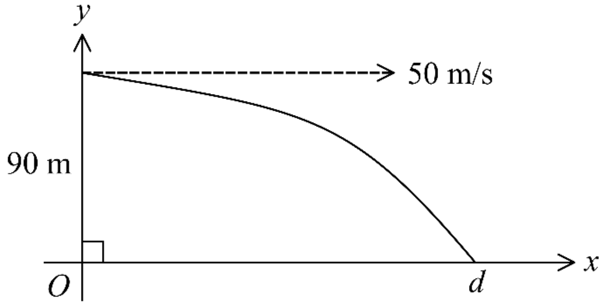


NOT TO SCALE

The projectile is propelled from ground level 40 metres from the base of a cliff that is 540 metres high. The projectile lands on top of the cliff, 50 metres from the edge.

Find the size of the angle of projection, to the nearest degree.

(d) The diagram below shows the trajectory of a ball thrown horizontally, at a speed of 50 ms^{-1}, from the top of a tower 90 metres above ground level.



2

The ball strikes the ground d metres from the base of the tower.

i) Show that the equations describing the trajectory of the ball are:

$$x = 50t \text{ and } y = 90 - \frac{1}{2}gt^2$$

where g is the acceleration due to gravity and t is the time in seconds. **1**

ii) Prove that the ball strikes the ground at time $t = 6\sqrt{\frac{5}{g}}$ seconds. **1**

iii) How far from the base of the tower does the ball strike the ground?

End of Exam.

CTHS Trial AP4 2017

Mathematics Extension 1 Course

Name ______________________________ Teacher ______________________

Section I – Multiple Choice Answer Sheet

Allow about 15 minutes for this section
Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: 2 + 4 = (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ● C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word **correct** and drawing an arrow as follows.

correct

A ● B ● C ○ D ○

1.	A ○	B ○	C ○	D ○
2.	A ○	B ○	C ○	D ○
3.	A ○	B ○	C ○	D ○
4.	A ○	B ○	C ○	D ○
5.	A ○	B ○	C ○	D ○
6.	A ○	B ○	C ○	D ○
7.	A ○	B ○	C ○	D ○
8.	A ○	B ○	C ○	D ○
9.	A ○	B ○	C ○	D ○
10.	A ○	B ○	C ○	D ○

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NAME_____SOLUTIONS__________

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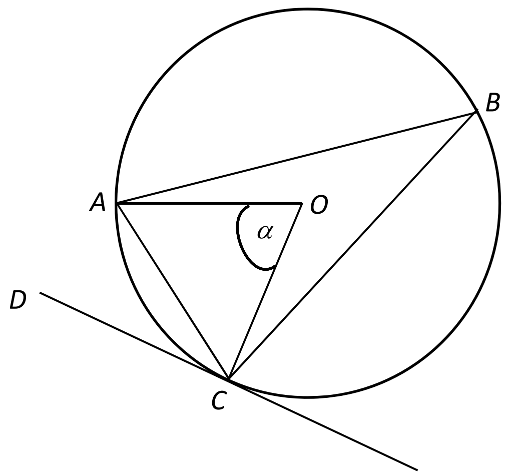

What is $\angle DCA$ in terms of α?

(A) 2α

(B) $\dfrac{\alpha}{2}$ *

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The equation of the line l_2 is $y = (-\tan\theta)x - b$.

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Its velocity is given by $v^2 = 30 + 9x - 3x^2$.

What are the endpoints of its motion?

(A) $x = -5, 2$

(B) $x = -10, 10$

(C) $x = -2, 5$ *

(D) $x = -30, 30$

7. How many solutions exist for the equation $(1 + \sin 2\theta)(1 - \sin 2\theta) = 0.5$ in the domain $0 \le \theta \le 2\pi$?

(A) 1

(B) 2

(C) 4

(D) 8 *

8. The polynomial $P(x) = 2(x-3)^2(x+3)^2$.

What is the coefficient of the x^2 term?

(A) 0

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(C) -18

(D) -36 *

9. Evaluate $\int_{-5}^{-4} x(x+5)^3 \, dx$ using the substitution $x = u - 5$.

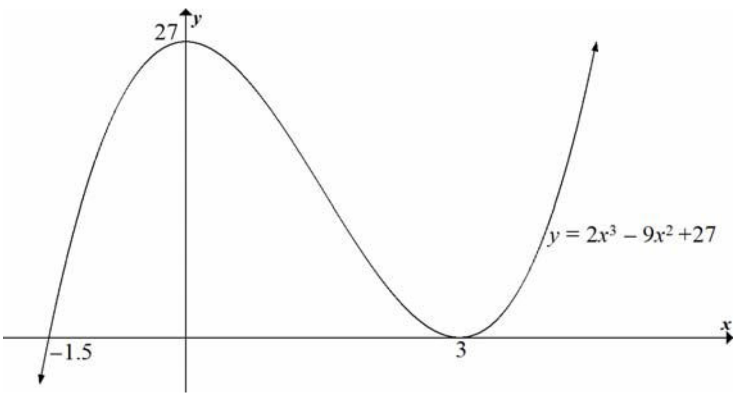
(**A**) $-\frac{21}{20}$ *

(B) -4

(C) $\frac{68}{5}$

(D) 14

10. The diagram shows the function $y = 2x^3 - 9x^2 + 27$. It crosses the x-axis at x = -1.5. It has turning points at $(0,27)$ and $(3,0)$.



For what values of k does the function $y = 2x^3 - 9x^2 + k$ have one root?

(A) $k < -1.5, k > 3$

(B) $-1.5 < k < 3$

(C) $k < 0, k > 27$ *

(D) $0 < k < 27$

Section II

60 marks

Attempt Questions 11 – 14.

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Question 11 (15 marks) Use a new writing booklet. **Marks**

(a) Find the equation of the directrix of the parabola $x = t, y = \frac{t^2}{4}$. **1**

$$x = t$$
$$y = \frac{t^2}{4}$$
$$y = \frac{x^2}{4}$$
$$x^2 = 4y$$
$$\therefore a = 1$$
$$\therefore directrix\ is\ y = -1$$

(b) Find $\lim\limits_{x\to 0}\frac{\sin 3x}{9x}$. **1**

$$\lim_{x\to 0}\frac{\sin 3x}{9x} = \frac{1}{3}\lim_{x\to 0}\frac{\sin 3x}{3x}$$
$$= \frac{1}{3}\times 1$$
$$= \frac{1}{3}$$

(c) The points A and B have coordinates $(-1,\ 6)$ and $(14, -9)$ respectively. The point P (x, y) divides the interval AB externally in the ratio $2:3$. Find the coordinates of P. **2**

$$m:n = -2:3$$
$$P(x,y) = \left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$$
$$= \left(\frac{(-2)\times 14+3\times(-1)}{-2+3}, \frac{(-2)\times(-9)+3\times 6}{-2+3}\right)$$
$$= (-31, 36)$$

(d) Differentiate $\sin^{-1}\left(\frac{5x}{2}\right)$. **1**

$$\frac{d}{dx}\left[\sin^{-1}\left(\frac{5x}{2}\right)\right] = \frac{1}{\sqrt{2^2-(5x)^2}}\times 5$$
$$= \frac{5}{\sqrt{4-25x^2}}$$

(e) Find the general solution of the equation $\cot\theta\sec^2\theta-\cot\theta-\sqrt{3}\sec^2\theta+\sqrt{3}=0$. **3**

$$\cot\theta\sec^2\theta-\cot\theta-\sqrt{3}\sec^2\theta+\sqrt{3}=0$$
$$\cot\theta\left(\sec^2\theta-1\right)-\sqrt{3}\left(\sec^2\theta-1\right)=0$$
$$\tan^2\theta\left(\cot\theta-\sqrt{3}\right)=0$$

$$\tan\theta=0$$
$$\theta=\ldots,-\pi,0,\pi,\ldots$$
$$\theta=\pm n\pi \text{ where } n=0,1,2,\ldots$$

$$\cot\theta=\sqrt{3}$$
$$\tan\theta=\frac{1}{\sqrt{3}}=\tan\left(\frac{\pi}{6}\right)$$
$$\theta=\ldots,-\pi+\frac{\pi}{6},\frac{\pi}{6},\pi+\frac{\pi}{6},2\pi+\frac{\pi}{6},\ldots$$
$$\theta=\pm n\pi+\frac{\pi}{6} \text{ where } n=0,1,2,\ldots$$

$$\therefore \theta=\pm n\pi,\pm n\pi+\frac{\pi}{6} \text{ where } n=0,1,2,\ldots$$

(f) A chemical is absorbed into a person's blood through a patch placed on the person's skin. The rate at which the chemical is absorbed is given by the equation

$\frac{dA}{dt}=-k\left(A-0.9\right)$ where A is the amount in milligrams and t is time in minutes.

i) Show that $A=0.9-0.7e^{-kt}$ satisfies this equation. 1

ii) How much chemical was initially in the person's blood? 1

iii) After 1 hour, the amount of chemical in the person's blood is 0.193 mg. 2

Show that $k=-\frac{\ln\left(1.01\right)}{60}$

iv) Find the amount of chemical in the person's blood after 16 hours, correct to two significant figures. 1

v) After 16 hours, the patch is removed. The rate at which the chemical is absorbed into the blood is then given by $\frac{dA}{dt}=-\frac{\ln\left(0.1\right)}{60}A$ mg/min . 2

Find the amount of chemical in the blood 24 hours after the patch was applied to the skin.

i)

$A = 0.9 - 0.7e^{-kt}$ 1.

$\frac{dA}{dt} = 0.7ke^{-kt}$

$= -k\left(-0.7e^{-kt}\right)$

$= -k\left(A - 0.9\right)$ from 1.

ii)

$A = 0.9 - 0.7e^{-kt}$

Sub $t = 0$

$A = 0.9 - 0.7 = 0.2$ mg

iii)

$A = 0.9 - 0.7e^{-kt}$

$t = 1$ h $= 60$ min

$A = 0.193$ mg

$0.193 = 0.9 - 0.7e^{-k \times 60}$

$e^{-60k} = \frac{0.707}{0.7}$

$k = -\frac{\ln(1.01)}{60}$

iv)

$A = 0.9 - 0.7e^{-\left(-\frac{\ln(1.01)}{60}\right) \times 16 \times 60}$

$= 0.9 - 0.7e^{16\ln(1.01)}$

$= 0.079194949$

$= 0.079$ mg (2sf)

v)

$\frac{dA}{dt} = -\frac{\ln(1.01)}{60}A$

$A = Be^{\frac{-\ln(1.01)}{60}t}$

$t = 0, A = 0.08$ [from (iv)]

$\therefore B = 0.08$

$t = 8h = 480min$

$A = 0.08e^{\frac{-480\ln(1.01)}{50}}$

$= 0.073$ (2sf)

Question 12 (15 marks) Use a new writing booklet.

(a) i) Simplify $\frac{r({}^nC_r)}{({}^nC_{r-1})}$ **2**

ii) Hence, or otherwise, show that $\frac{{}^nC_1}{{}^nC_0} + \frac{2\,{}^nC_2}{{}^nC_1} + \frac{3\,{}^nC_3}{{}^nC_2} + \cdots + \frac{n\,{}^nC_n}{{}^nC_{n-1}} = \frac{n(n+1)}{2}$ **2**

(b)(i)

$$\frac{r({}^nC_r)}{{}^nC_{r-1}} = \frac{r\frac{n!}{r!(n-r)!}}{\frac{n!}{(r-1)!(n-r+1)!}}$$
$$= \frac{rn!}{r!(n-r)!} \times \frac{(r-1)!(n-r+1)!}{n!}$$
$$= \frac{(n-r+1)!}{(n-r)!}$$
$$= \frac{(n-r+1)(n-r)!}{(n-r)!}$$
$$= n-r+1$$

(ii)

$$\frac{{}^nC_1}{{}^nC_0} + \frac{2\,{}^nC_2}{{}^nC_1} + \frac{3\,{}^nC_3}{{}^nC_2} + \ldots\ldots + \frac{n\,{}^nC_n}{{}^nC_{n-1}}$$
$$= \sum_{r=1}^{n} \frac{r\,{}^nCr}{{}^nCr-1}$$
$$= \sum_{r=1}^{n} (n-r+1)$$
$$= n + (n-1) + (n-2) + \ldots\ldots 3 + 2 + 1 \quad \therefore \text{AP}$$

$Sn = \frac{n(a+1)}{2}$

$$= \frac{n(1+n)}{2}$$

(b) i) Show that $\frac{x^3+x+2}{1+x^2} = x + \frac{2}{1+x^2}$. **2**

ii) Hence find $\int \frac{x^3+x+2}{1+x^2}dx$. **2**

i) RHS
$$= \frac{x+x^3}{1+x^2} + \frac{2}{1+x^2}$$
$$= LHS$$

ii)
$$\int \frac{x^3+x+2}{1+x^2}dx = \int x + \frac{2}{1+x^2}dx$$
$$= \frac{x^2}{2} + 2\tan^{-1}x + C$$

(c) Consider the function $f(x) = \log_e x + 1$.

i) Find an expression for $g(x)$, the inverse function of $f(x)$. **1**

ii) State the domain and range of $g(x)$. **1**

$y = \log_e x + 1$

Let $x = \log_e y + 1$

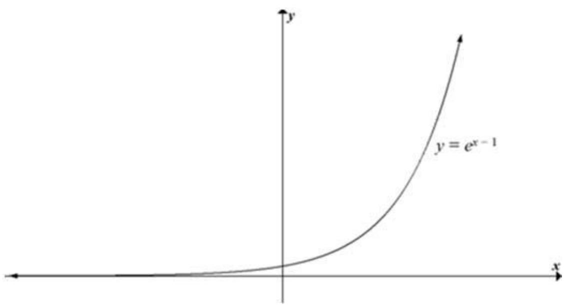
$\log_e y = x - 1$

$y = e^{x-1}$

For $y = e^{x-1}$:

Domain: all real x

Range: y > 0



(d) The point $P\left(2ap, ap^2\right)$ lies on the parabola $x^2 = 4ay$. $S(0, a)$ is the focus of the parabola. The interval PQ is perpendicular to the x-axis and meets the x-axis at Q.

Find the equation of the locus of the midpoint of SQ. **2**

coords of $Q = (2ap, 0)$

$$\text{Midpoint of } SQ = M(x, y) = \left(\frac{2ap+0}{2}, \frac{0+a}{2}\right)$$

$$= \left(ap, \frac{a}{2}\right)$$

Locus: $y = \dfrac{a}{2}$ (independent of p)

(e) Use mathematical induction to prove that $7^n - 3^n$ is divisible by 4 for $n \geq 1$. **3**

$7^1 - 3^1 = 4$ which is divisible by 4

assume $7^k - 3^k = 4M$, where M a positive integer

$$\begin{aligned}
7^{k+1} - 3^{k+1} &= 7^{k+1} - 3\times 7^k + 3\times 7^k - 3^{k+1} \\
&= 7^k(7-3) + 3\left(7^k - 3^k\right) \\
&= 4\times 7^k + 3\times 4M \qquad \text{(assumption)} \\
&= 4\left(7^k + 3M\right) \\
&= 4N, \text{ where } N = 7^k + 3M, \text{ a positive integer}
\end{aligned}$$

If true for n=k, then true for n=k+1

As it is true for n=1, then it is true for n=2,3,…

By mathematical induction, $7^n - 3^n$ is divisible by 4 for n>=1

Question 13 (15 marks) Use a new writing booklet

(a) The acceleration of a particle is given by $\ddot{x} = \cos^2 x$. The velocity $v \geq 0$ for all x and the particle is initially at rest at the origin.

Find an expression for the velocity in terms of x. **3**

$$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \cos^2 x$$

$$\begin{aligned}
v^2 &= 2\int \cos^2 x\, dx \\
&= 2\int \frac{\cos 2x + 1}{2}\, dx \\
&= \frac{1}{2}\sin 2x + x + C
\end{aligned}$$

$x = 0, v = 0$

$\therefore C = 0$

$$v = \pm\sqrt{\frac{1}{2}\sin 2x + x}$$

$$\therefore v = \sqrt{\frac{1}{2}\sin 2x + x} \qquad (v \geq 0)$$

(b) The circumference of a circular drop of liquid with radius r and area A increases at the rate of 5mm/s.

i) Show that $\frac{dr}{dt} = \frac{5}{2\pi}$ **2**

ii) Show that $\frac{dA}{dt} = 5r$ **1**

i)
$$\frac{dC}{dt} = 5$$
$$C = 2\pi r$$
$$\frac{dC}{dr} = 2\pi$$
$$\frac{dC}{dt} = \frac{dC}{dr}\frac{dr}{dt}$$
$$5 = 2\pi\frac{dr}{dt}$$
$$\therefore \frac{dr}{dt} = \frac{5}{2\pi}$$

ii)
$$\frac{dr}{dt} = \frac{5}{2\pi}$$
$$A = \pi r^2$$
$$\frac{dA}{dr} = 2\pi r$$
$$\frac{dA}{dt} = \frac{dA}{dr}\frac{dr}{dt}$$
$$= 2\pi r \times \frac{5}{2\pi}$$
$$= 5r$$

(c) A particle moves in a straight line so that its acceleration is given by $a = x + 1.5$ ms^{-2}. Initially, the particle is 5 metres to the right of O and moving towards O with a velocity of 6 ms^{-1}.

i) Is the particle speeding up or slowing down? Give a reason. **1**

ii) Show that $v^2 = x^2 + 3x - 4$. **2**

iii) Where does the particle first change direction? **1**

i) Initially x = 5 and v = –6

Acceleration $a = x + 1.5 = 6.5$

Therefore a > 0 and v < 0 (different signs)

The particle is slowing down.

ii) $\frac{d}{dx}(\frac{1}{2}v^2) = x + 1.5$

$$\frac{1}{2}v^2 = \frac{1}{2}x^2 + 1.5x + c_1$$
$$v^2 = x^2 + 3x + c_2$$

When x=5, v=-6 then

$(-6)^2 = 5^2 + 3\times 5 + c_2$ or $c_2 = -4$

Therefore $v^2 = x^2 + 3x - 4$

iii)Particle changes direction when v=0

$$x^2 + 3x - 4 = 0$$
$$(x+4)(x-1) = 0$$

Particle starts at x=5 and is moving to the left (v=-6).

At x=1 the particle is at rest v=0 and a=2.5>0

It then changes direction and moves to the right (v>0)

Therefore x=1m

(d)

i) Show that $y = \frac{x^2+1}{x^2-1}$ is an even function. **1**

ii) Sketch the graph of , clearly labelling any asymptotes and any intercepts with the x- and y-axes. **4**

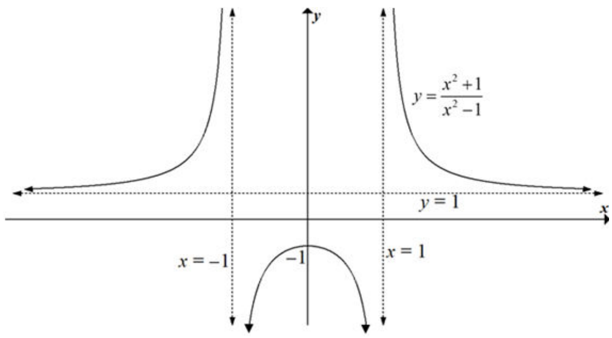
$$y = f(x) = \frac{x^2+1}{x^2-1}$$

$$f(-x) = \frac{(-x)^2+1}{(-x)^2-1}$$

$$= \frac{x^2+1}{x^2-1}$$

$$= f(x)$$

i) $\therefore$ even function



ii)

$$y = \frac{x^2+1}{x^2-1}$$

y-int $= y(0) = -1$

$x^2 - 1 = (x+1)(x-1) \neq 0$

$\therefore$ vertical asymptotes: $x = \pm 1$

$$\lim_{x \to \pm\infty} \frac{x^2+1}{x^2-1} = \lim_{x \to \pm\infty} \frac{\frac{x^2}{x^2} + \frac{1}{x^2}}{\frac{x^2}{x^2} - \frac{1}{x^2}}$$

$$= \frac{1+0}{1-0}$$

$\therefore$ horizontal asymptotes: $y = 1$

Question 14 (15 marks) Use a new writing booklet **Marks**

(a) A particle moves with simple harmonic motion. Its acceleration is given by $\ddot{x} = -4(x-3)\ \text{m}^2/\text{s}$. Its amplitude is 3.

i) State the centre of motion of the particle.

ii) Explain why the speed of the particle is 0 m/s at the origin.

iii) Find the maximum acceleration of the particle

$\ddot{x} = -4(x-3) = 0$ at centre of motion.

i) $\therefore$ centre of motion is $x = 3$

ii)amplitude=3 **1**

endpoints of motion are x=0,6 **1**

at endpoints v=0 **1**

speed=0 at x=0

iii)acc is max, min at endpoints

$$\ddot{x} = -4(6-3), -4(0-3)$$
$$= \pm 12$$

Max acc=12ms^{-2}

(b) A particle is moving in a straight line under SHM. At any time (t seconds) its displacement (x metres) from a fixed point O is given by:

$$x = A\cos\left(\frac{\pi}{4}t + \alpha\right) \text{ where } A > 0 \text{ and } 0 < \alpha < \frac{\pi}{2}$$

After 1 second the particle is 2 metres to the right of O and after 3 seconds the particle is 4 metres to the left of O.

i) Show that $A\sin\alpha - A\cos\alpha = -2\sqrt{2}$ and $A\sin\alpha + A\cos\alpha = 4\sqrt{2}$. **2**

ii) Show that $A = 2\sqrt{5}$ and $\alpha = \tan^{-1}\frac{1}{3}$. **2**

iii) When does the particle first pass through O? **1**

i) When t=1 then x=2

$$2 = A\cos\left(\frac{\pi}{4}\times 1 + \alpha\right)$$
$$= A\left(\cos\frac{\pi}{4}\cos\alpha - \sin\frac{\pi}{4}\sin\alpha\right)$$
$$= A\left(\frac{1}{\sqrt{2}}\cos\alpha - \frac{1}{\sqrt{2}}\sin\alpha\right)$$
$$2\sqrt{2} = A\cos\alpha - A\sin\alpha$$
$$A\sin\alpha - A\cos\alpha = -2\sqrt{2}$$

When t=3 then x=-4

$$-4 = A\cos\left(\frac{\pi}{4}\times 3 + \alpha\right)$$
$$= A\left(\cos\frac{3\pi}{4}\cos\alpha - \sin\frac{3\pi}{4}\sin\alpha\right)$$
$$= A\left(-\frac{1}{\sqrt{2}}\cos\alpha - \frac{1}{\sqrt{2}}\sin\alpha\right)$$
$$-4\sqrt{2} = -A\cos\alpha - A\sin\alpha$$
$$A\sin\alpha + A\cos\alpha = 4\sqrt{2}$$

ii) $A\sin\alpha - A\cos\alpha = -2\sqrt{2} \quad (1)$

$A\sin\alpha + A\cos\alpha = 4\sqrt{2} \quad (2)$

Adding equations (1) and (2) then $2A\sin\alpha = 2\sqrt{2}$

Subtracting equation (1) from (2) then $2A\cos\alpha = 6\sqrt{2}$

$$(2A\sin\alpha)^2 + (2A\cos\alpha)^2 = (2\sqrt{2})^2 + (6\sqrt{2})^2 \qquad \frac{2A\sin\alpha}{2A\cos\alpha} = \frac{2\sqrt{2}}{6\sqrt{2}}$$

$$4A^2(\sin^2\alpha + \cos^2\alpha) = 8 + 72$$

$$A^2 = 20 \quad \text{or} \quad A = 2\sqrt{5} \qquad \tan\alpha = \frac{1}{3} \text{ or } \alpha = \tan^{-1}\frac{1}{3}$$

iii) Particle passes through O when x=0

$$A\cos\left(\frac{\pi}{4}t + \alpha\right) = 0$$

$$\frac{\pi}{4}t + \alpha = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$$

First passes through O

$$\frac{\pi}{4}t + \alpha = \frac{\pi}{2}$$

$$\frac{\pi}{4}t + \tan^{-1}\frac{1}{3} = \frac{\pi}{2}$$

$$\frac{\pi}{4}t = \frac{\pi}{2} - \tan^{-1}\frac{1}{3}$$

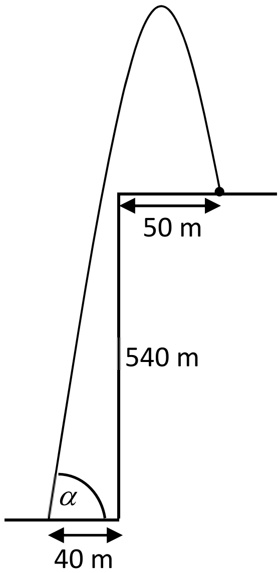
$$\frac{\pi}{4}t = \tan^{-1}3$$

$$t = \frac{4}{\pi}\tan^{-1}3$$

Accept $t = 2 - \frac{4}{\pi}\tan^{-1}\frac{1}{3}$ or 1.59 seconds

(c) The motion of a projectile can be represented by the equations **3**

$x = 150t\cos\alpha$
$y = -5t^2 + 150t\sin\alpha$ where α is the angle of projection.



The projectile is propelled from ground level 40 metres from the base of a cliff that is 540 metres high. The projectile lands on top of the cliff, 50 metres from the edge.
Find the size of the angle of projection, to the nearest degree.

$$x = 150t\cos\alpha = 40 + 50$$
$$\therefore t = \frac{90}{150\cos\alpha} = \frac{3}{5\cos\alpha}$$
$$y = -5t^2 + 150t\sin\alpha = 540$$
$$-5\left(\frac{3}{5\cos\alpha}\right)^2 + 150\left(\frac{3}{5\cos\alpha}\right)\sin\alpha = 540$$
$$-\frac{9}{5\cos^2\alpha} + \frac{90\sin\alpha}{\cos\alpha} = 540$$
$$450\tan\alpha - 9\sec^2\alpha - 2700 = 0$$
$$50\tan\alpha - \left(\tan^2\alpha + 1\right) - 300 = 0$$
$$-\tan^2\alpha + 50\tan\alpha - 301 = 0$$
$$\tan^2\alpha - 50\tan\alpha + 301 = 0$$
$$\tan\alpha = \frac{50 \pm \sqrt{50^2 - 4\times1\times301}}{2}$$
$$= 43,\ 7$$

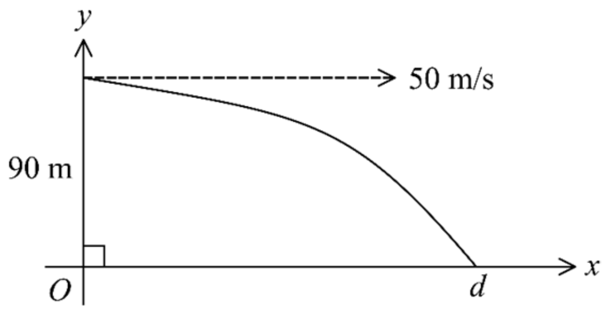
Projectile lands on cliff.

$$\therefore \alpha = \tan^{-1}(43)$$
$$= 88.66778015°$$
$$= 89° \text{ (nearest degree)}$$

$$If\ \tan\alpha = 7, \alpha = 82°$$
$$t = \frac{40}{150\cos 82°}$$
$$t = 1.916079076$$
$$y = -5t^2 + 150t\sin 82°$$
$$y = 266m$$
$$\therefore Projectile\ won't get\ over\ the\ cliff$$

(d)

The diagram below shows the trajectory of a ball thrown horizontally, at a speed of 50 ms^{-1}, from the top of a tower 90 metres above ground level.



The ball strikes the ground d metres from the base of the tower.

i) Show that the equations describing the trajectory of the ball are: **2**

$$x = 50t \text{ and } y = 90 - \frac{1}{2}gt^2$$

where g is the acceleration due to gravity and t is the time in seconds.

ii) Prove that the ball strikes the ground at time $t = 6\sqrt{\frac{5}{g}}$ seconds. **1**

iii) How far from the base of the tower does the ball strike the ground? **1**

i) Horizontal $\ddot{x} = 0$

$$\dot{x} = 50\cos 0^\circ = 50$$

$$x = 50t + c$$

When t=0, x=0 implies c=0

$x = 50t$

Vertical $\ddot{y} = -g$

$$\dot{y} = -gt + 50\sin 0^\circ = -gt$$

$$y = -\frac{1}{2}gt^2 + c$$

When t=0, y=90 implies c=90

$$y = 90 - \frac{1}{2}gt^2$$

ii) Ball strikes the ground y=0

$$90 - \tfrac{1}{2}gt^2 = 0$$

$$\tfrac{1}{2}gt^2 = 90$$

$$t^2 = \frac{180}{g}$$

$$t = \sqrt{\frac{180}{g}} = 6\sqrt{\frac{5}{g}} \quad \text{as } t > 0$$

iii) Ball strikes the ground when

$$t = 6\sqrt{\frac{5}{g}} \textbf{ \textit{seconds}}$$

Now $x = 50t$

$$d = 50 \times 6\sqrt{\frac{5}{g}} = 300\sqrt{\frac{5}{g}} \text{ metres}$$

End of Exam.