CRANBROOK SCHOOL

Form VI/Year 12 MATHEMATICS - 3 Unit (Second Paper), 4 Unit (First Paper)

Term 3 1997

Time: 2 hrs (CGH / GPP)

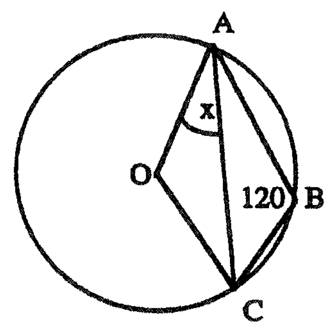
All questions may be attempted.
All questions are of equal value.
All necessary working should be shown in every question.
Full marks may not be awarded if work is careless or badly arranged.
Standard integrals are provided at the end of the paper.
Approved silent calculators may be used.
Begin each question on a new page. Submit your work in two booklets
(i) qq. 1 - 4
(ii) qq. 5 - 7

1.

(a) Solve for x: $\dfrac{2x-5}{x+3} \geq 1$

(b) Find the point P which divides the interval AB externally in the ratio 3 : 2, given that A is the point (5, 3) and B is the point (1, -3).

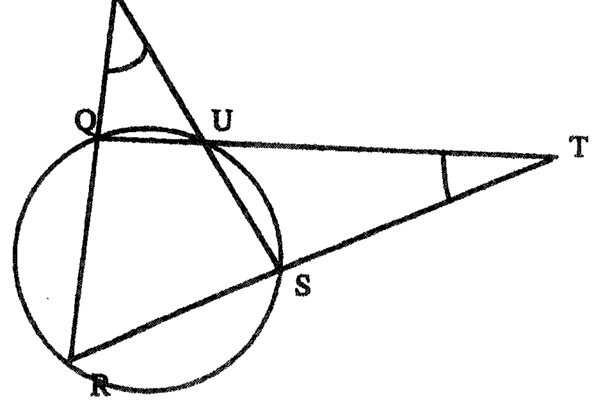
(c) In the diagram opposite find x, giving full reasons at each step.



(d) In the diagram opposite $\angle RPS = \angle QTR$, and PQR, TUQ and TSR are straight lines.

(i) Prove that $\angle UQR = \angle USR$.

(ii) Hence explain why UR is a diameter of the circle.



5. **(new page please)**

(a) (i) Show that there is a root of the equation $3\sin 2x = x$ between 1.3 and 1.4.
(ii) Use Newton's method to find this root correct to two decimal places.

(b) Given that $f(x) = ax^3 + bx^2 + cx + d$ is a function with a double zero at $x = 1$ and with a minimum value of -4 when $x = -1$, find the value of each coefficient in $f(x)$.

(c) Prove by the principle of mathematical induction that $n^3 + 2n$ is divisible by 3 for all positive integers n.

6. **(new page please)**

(a) In the expansion of $\left(x^2 + \dfrac{2}{x}\right)^{10}$ find the coefficient of x^2.

(b) (i) Assuming that, for all real numbers x and for all positive integers n

$$(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r \qquad \left[\text{where } \binom{n}{r} = {}^nC_r\right]$$

show that :

$$\sum_{r=0}^{n} (-1)^r \binom{n}{r} = 0$$

(ii) Similarly, evaluate :

$$\sum_{r=0}^{n} 2^r \binom{n}{r} \qquad \text{and} \qquad \sum_{r=0}^{n} r\binom{n}{r}$$

(c) Using the substitution $u = x^4$, or otherwise, show that $\displaystyle\int_0^1 \frac{x^3}{1+x^8}\,dx = \frac{\pi}{16}$

7. **(new page please)**

The distinct points P, Q on the parabola $x = 2t$, $y = t^2$ have parameters equal to p and q respectively.

(i) Write down the equation of the tangent to the parabola at P.
(ii) Show that the equation of the chord PQ is $2y - (p+q)x + 2pq = 0$
(iii) Show that M, the point of intersection of the tangents to the parabola at P and at Q has the co-ordinates $(p+q \,,\, pq)$.
(iv) Prove that for any value of p , except $p = 0$, there are exactly two values of q for which M lies on the parabola $x^2 = -4y$ and find these two values in terms of p. Find also the co-ordinates of the corresponding point M.
(v) Show that, for these values of q, the chord PQ is a tangent to the parabola $x^2 = -4y$.

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HMX1

CRANBROOK 1997 3U TRIAL

SOLUTIONS

1. (a) $\dfrac{2x-5}{x+3} \geq 1$ $\quad x \neq -3$

$(x+3)^2 \times \dfrac{2x-5}{x+3} \geq (x+3)^2$

$(x+3)(2x-5) \geq (x+3)^2$

$2x^2 + x - 15 \geq x^2 + 6x + 9$

$x^2 - 5x - 24 \geq 0$ $\quad -8, 3$

$(x-8)(x+3) \geq 0$

$x < -3$ or $x \geq 8$

(3)

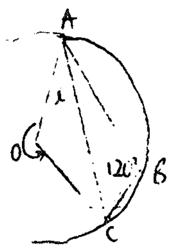
(b) $\begin{matrix}(5,3) \\ (1,-3)\end{matrix}$ $\quad -3 : 2$

$P = \left(\dfrac{2\times5 - 3\times1}{2-3}, \dfrac{2\times3 - 3\times(-3)}{2-3}\right)$

$= (-7, -15)$

(2)

(c)



reflex $\angle AOC = 2\times120°$ (angle at centre = twice angle at circumference standing on same arc)

$= 240°$

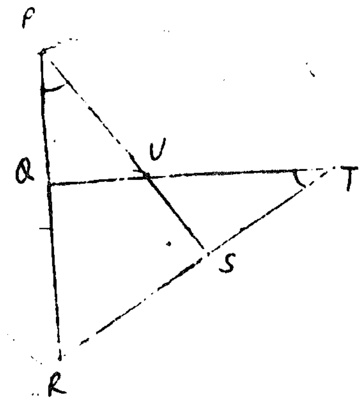
$\therefore \angle AOC = 120°$ (angles at a point = $360°$)

$2x + 120° = 180°$ (angle sum isos ΔAOC equal radii)

$\therefore x = 30°$

(3)

(d)



Data: $\angle RPS = \angle QTR$

(i) RTP: $\angle UQR = \angle USR$

Proof:

Considering Δs PRS, QRT

$\angle RPS = \angle QTR$ (given)

$\angle R$ common

$\therefore \angle PSR = \angle TQR$ (remaining angle in Δ)

QED

(2)

(ii) $\angle UQR + \angle USR = 180°$ (opp angles cyclic quad)

$\therefore \angle UQR = \angle USR = 90°$ since $\angle UQR = \angle USR$

Since angle in a semi-circle $= 90°$,

UR is a diameter.

(2)

→ alternative soln:

Produce AO to D on circumference

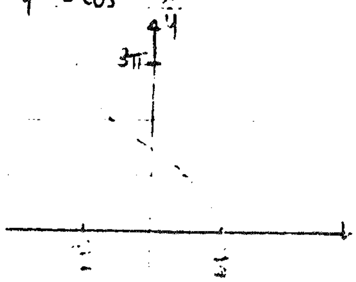
$\therefore$ AD diameter.

$\angle ABD = 90°$ (angle in a semi-circle)

$\therefore \angle CBD = 30°$ $\quad (120° - 90°)$

Now, $\angle OAC\ (x°) = 30°$ (angles at circumference standing on same arc)

2. (a)(i) $y = 3\cos^{-1} 2x$



(1)

(ii) domain: $-1 \le 2x \le 1$

$\{x : -\frac{1}{2} \le x \le \frac{1}{2}\}$ (2)

range: $0 \le y \le 3\pi$

(iii) $y = 3\cos^{-1} 2x$

$$\frac{dy}{dx} = 3 \times \frac{-2}{\sqrt{1-4x^2}}$$

$$= \frac{-6}{\sqrt{1-4x^2}}$$

$$= \frac{-6}{\sqrt{1-4(\frac{1}{4})^2}} \quad \text{when } x = \frac{1}{4}$$

$$= \frac{-6}{\frac{\sqrt{3}}{2}}$$

$$= \frac{-12}{\sqrt{3}}$$

$$= -4\sqrt{3}$$

eqn. $y - y_1 = m(x - x_1)$

$y = 3\cos^{-1}\frac{1}{2}$

$= \pi$ (3)

$y - \pi = -4\sqrt{3}(x - \frac{1}{4})$

$4\sqrt{3}x + y = \pi + \sqrt{3}$

(b) $p(\text{hits target}) = \frac{1}{3}$

(i) $P(3 \text{ successes}) = P(X=3)$

$= {}^5C_3 \left(\frac{1}{3}\right)^3 \times \left(\frac{2}{3}\right)^2$

$= 10 \times \frac{1}{27} \times \frac{4}{9}$

$= \frac{40}{243}$ (1)

(ii) $P(X \geq 1) = 1 - P(X=0)$

> 0.9

$1 - \left(\frac{2}{3}\right)^n > 0.9$ where n = no. of torpedoes fired

$0.1 > \left(\frac{2}{3}\right)^n$

$\ln 0.1 > n \ln \frac{2}{3}$

$n \quad \frac{\ln 0.1}{\ln \frac{2}{3}}$ since $\ln \frac{2}{3} < 0$

(3)

$n > 5.67\ldots$

∴ 6 or more must be fired.

(c) $2\sin 2x + \sqrt{3} = 0 \qquad 0 \le x \le 360°$

$\sin 2x = -\frac{\sqrt{3}}{2}$ Q3, 4 $\angle = 60°$

$2x = 240°, 300°, 600°, 660°$

$x = 120°, 150°, 300°, 330°$ (2)

3. $V = \pi \int_0^{3\sqrt{3}} \left(\frac{1}{\sqrt{x^2+9}}\right)^2 dx$

$= \pi \int_0^{3\sqrt{3}} \frac{dx}{x^2+9}$

$= \frac{\pi}{3} \int_0^{3\sqrt{3}} \frac{3}{x^2+9}\, dx$

$= \frac{\pi}{3}\left[\tan^{-1}\frac{x}{3}\right]_0^{3\sqrt{3}}$

$= \frac{\pi}{3}\left(\tan^{-1}\sqrt{3} - \tan^{-1} 0\right)$

$= \frac{\pi}{3}\left(\frac{\pi}{3} - 0\right)$

$= \frac{\pi^2}{9}$ units^3 (3)

(c) $a = -\frac{1}{2}e^{-x}$

when $x = 0$, $v = 1$ $t = 0$

(i) $\frac{1}{2}v^2 = \int -\frac{1}{2}e^{-x}\, dx$

$\frac{1}{2}v^2 = \frac{1}{2}e^{-x} + c_1$

$\frac{1}{2}\times 1^2 = \frac{1}{2}e^0 + c_1$

$\therefore c = 0$ (3)

$\therefore v^2 = e^{-x}$

$v = \sqrt{e^{-x}}$ (-ve does not apply) since when $x=0$, $v > 0$

(ii) $\frac{dx}{dt} = \sqrt{e^{-x}} = e^{-\frac{x}{2}}$

$\frac{dt}{dx} = e^{\frac{x}{2}}$

$\therefore t = \int e^{\frac{x}{2}}\, dx$

$= 2e^{\frac{x}{2}} + c_2$

$0 = 2e^0 + c_2$

$c_2 = -2$

$\therefore t = 2e^{\frac{x}{2}} - 2$

$e^{\frac{x}{2}} = \frac{t+2}{2}$

$\frac{x}{2} = \ln\frac{t+2}{2}$ (3)

$x = 2\ln\frac{t+2}{2}$

(d) $\cos\left(\tan^{-1}\sqrt{3}\right) = \cos\frac{\pi}{3}$

$= \frac{1}{2}$ (1)

(b) $\sec^2 x - 3\tan x - 3 = 0$ $\quad -\pi \leq x \leq \pi$.

$1 + \tan^2 x - 3\tan x - 3 = 0$

$\tan^2 x - 3\tan x - 2 = 0$

$\tan x = \frac{3 \pm \sqrt{9 - 4(-2)1}}{2}$

$= \frac{3 \pm \sqrt{17}}{2}$

$= 3.525\ldots, -0.512\ldots$

Q1, 3 $\quad$ Q2, 4

$x = 1.297, -\pi + 1.297$ OR $x = \pi - 0.512, -0.512$

$= 1.297, -1.844$ $\quad$ $= 2.630, -0.512$

$\therefore x = -1.844, -0.512, 1.297, 2.630$ (all to 3dp)

(3)

4. (a) $\frac{dV}{dt} = 50$

$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$ $\qquad$ $V = \frac{4}{3}\pi r^3$

$50 = 4\pi r^2 \times \frac{dr}{dt}$ $\qquad$ $\therefore \frac{dV}{dr} = 4\pi r^2$

$\frac{50}{4\pi \times 20^2} = \frac{dr}{dt}$

$\therefore \frac{dr}{dt} = \frac{1}{32\pi}$

Now, $\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$ $\qquad$ $A = 4\pi r^2$, $\frac{dA}{dr} = 8\pi r$

$= 8\pi r \times \frac{1}{32\pi}$

$= 8\pi \times 20 \times \frac{1}{32\pi}$

$= 5$ $\qquad$ (4)

$\therefore$ surface area increasing at rate of $5\,mm^2/s$

(b) RTS $\tan^{-1}\frac{3}{5} + \tan^{-1}\frac{1}{4} = \frac{\pi}{4}$

Let $x = \tan^{-1}\frac{3}{5}$, $y = \tan^{-1}\frac{1}{4}$

$\tan x = \frac{3}{5}$ $\qquad$ $\tan y = \frac{1}{4}$

Taking tan of both sides

$LHS = \tan(x+y)$

$= \frac{\tan x + \tan y}{1 - \tan x \tan y}$

$= \frac{\frac{3}{5} + \frac{1}{4}}{1 - \frac{3}{5}\times\frac{1}{4}}$ $\qquad$ (3)

$= 1$

$RHS = \tan\frac{\pi}{4}$

$= 1$

$\therefore \tan^{-1}\frac{3}{5} + \tan^{-1}\frac{1}{4} = \frac{\pi}{4}$

(c) (i) $\ddot{x} = 0$ $\qquad$ $\ddot{y} = -10$

$\dot{x} = V\cos\theta$ $\qquad$ $\dot{y} = -10t + V\sin\theta$

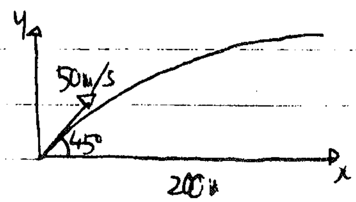
$= 50\cos 45°$ $\qquad$ $= 50\sin 45° - 10t$

$= 25\sqrt{2}$ $\qquad$ $= 25\sqrt{2} - 10t$

$x = Vt\cos\theta$ $\qquad$ $y = 25t\sqrt{2} - 5t^2$

$= 25t\sqrt{2}$ $\qquad$ (3)

(ii)



$200 = 25t\sqrt{2}$

$\therefore t = \frac{8}{\sqrt{2}}$

$= 4\sqrt{2}\ s$ $\qquad$ (2)

(iii) height, $y = 25 \times 4\sqrt{2} \times \sqrt{2} - 5 \times (4\sqrt{2})^2$

$= 40\ m$. $\qquad$ (1)

Alternative solution to (a)

(finding $\frac{dV}{dt}$ without first finding $\frac{dr}{dt}$)

since r is not a constant

$V = \frac{4}{3}\pi r^3$ $\qquad$ $S = 4\pi r^2$

$\therefore \frac{dV}{dt} = 4\pi r^2 \times \frac{dr}{dt}$ $\qquad$ $\frac{dS}{dt} = 8\pi r\frac{dr}{dt}$

$\frac{\frac{dV}{dt}}{\frac{dS}{dt}} = \frac{r}{2}$

ie. $\frac{dV}{dt} = \frac{r}{2} \times \frac{dS}{dt}$

$50 = \frac{20}{2} \times \frac{dS}{dt}$ $\qquad$ when $r = 20$

$\therefore \frac{dS}{dt} = 5$

ie. surface area is increasing at rate of $5\,mm^2/s$.

or $\frac{dS}{dt} = \frac{dS}{dr} \times \frac{dr}{dV} \times \frac{dV}{dt}$

$= 8\pi r \times \frac{1}{4\pi r^2} \times 50$

Cranbrook 3unit Trial 1997. (1)

Q5 (a) i let $f(x) = 3\sin 2x - x$. Consider $f(1.3) \doteqdot 0.247$ & $f(1.4) = -0.395$ ∴ root lies between 1.3 & 1.4

) Applying Newton's Method for a better root (x_2) & using $x_1 = 1.3$ since $f(1.3) \doteqdot 0.247$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \text{ where } f'(x) = 6\cos 2x - 1 \text{ gives } x_2 = 1.3 - \frac{f(1.3)}{f'(1.3)} = 1.340138539.$$

$$\therefore x_3 = 1.34 - \frac{f(1.34)}{f'(1.34)} = 1.339669957 \quad \therefore \text{ a better root is } 1.34 \quad (2)$$

Q5(b)

If $x=1$ is a double root for $f(x) = ax^3 + bx^2 + cx + d$ then $f(1) = 0$ (1) & $a+b+c+d = 0$

$f'(1) = 0$ (1) $\Rightarrow 3a + 2b + c = 0$ & since min value exists at -1 $f'(-1) = 0$ (1) $\Rightarrow 3a - 2b + c = 0$

& $f(-1) = -4$ (1) $\Rightarrow -a + b - c + d = -4$. ∴ $3a + c = 0$ & $4b = 0$ together with $2b + 2d = -4$ or $b + d = \ldots$

∴ $b = 0$, $a = -1$, $c = 3$ & $d = -2$ (1)

Q5(c)

Let $P(n)$ be the proposition that $n^3 + 2n = 3t_n$ (where $t_n \in \mathbb{N}$) & let S be the truth set for $P(n)$.

Consider $P(1)$ LHS $= 3 =$ RHS if $t_1 = 1$ ∴ $1 \in S$.

Assume $K \in S$ ie $K^3 + 2K = 3t_K$. Consider $P(K+1)$. LHS $= (K+1)^3 + 2(K+1)$

(1) or LHS $= (K^3 + 3K^2 + 3K + 1) + 2K + 2$

$= K^3 + 2K + 3K^2 + 3K + 3$

(1) ∴ $P(K+1)$ is true & if $K \in S$ then $K+1 \in S$, $1 \in S$ $= 3t_K + 3(K^2 + K + 1)$

∴ $P(n)$ is true for $\forall n \in \mathbb{N}$. $= 3t_{K+1}$ where $t_{K+1} = t_K + (K^2 + K + 1)$

Q6(a)

For $\left(x^2 + \frac{2}{x}\right)^{10}$ $T_n = {}^{10}C_{n-1} (x^2)^{11-n} \left(\frac{2}{x}\right)^{n-1}$ or ${}^{10}C_{n-1}\, x^{23-3n} \cdot 2^{n-1}$ (1)

For term in x^2 (1) $23 - 3n = 2 \Rightarrow n = 7$. ∴ we need to evaluate T_7. (1)

so $T_7 = {}^{10}C_6 \cdot 2^6 \cdot x^2$ ∴ Coefficient of x^2 is 210×64 or 13440. (1)

Q6 (b) (i) $(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r$ let $x = -1$ then $(1-1)^n = \sum_{r=0}^{n} (-1)^n \binom{n}{r}$ ie $\sum_{r=0}^{n} (-1)^n \binom{n}{r} = 0$ (1)

(ii) let $x = 2$ (1) $\sum_{r=0}^{n} \binom{n}{r} 2^r = 3^n$. Consider $\frac{d(1+x)^n}{dx}$

$$\Rightarrow n(1+x)^{n-1} = \sum_{r=0}^{n} \binom{n}{r} r (x)^{r-1} \quad \text{let } x = 1$$

$$\text{then } \sum_{r=0}^{n} \binom{n}{r} r = n \cdot 2^{n-1} \quad (2)$$

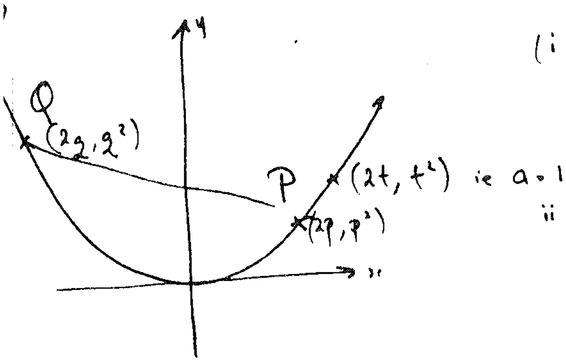
Q6(c)

$$\int_0^1 \frac{x^3}{1 + x^8}\,dx \quad \Rightarrow \quad \text{let } x^4 = u, \quad 4x^3\,dx = du \text{ \& when } x = 1, u = 1;\ x = 0, u = 0 \quad (1)$$

$$= \frac{1}{4}\int_0^1 \frac{du}{1 + u^2} \quad (1)$$

$$= \frac{1}{4}\left[\tan^{-1}(u)\right]_0^1 \ (1) \text{ or } \frac{1}{4}\left\{(\tan^{-1}(1)) - (\tan^{-1}(0))\right\} \text{ or } \frac{1}{4} \times \frac{\pi}{4}$$

[C]ranbrook 3unit Trial 1997.



ie a = 1

(i) For Equation of Tangent at P: $y = px - p^2$ (2)

ii For Chord $\dfrac{y-p^2}{x-2p} = \dfrac{p^2 - q^2}{2(p-q)}$

$\Rightarrow \dfrac{y-p^2}{x-2p} = \dfrac{p+q}{2}$ or $y = \left(\dfrac{p+q}{2}\right)x - pq$.

$\therefore$ Eq of Chord is

$2y - (p+q)x + 2pq = 0$ (2)

(iii) Eq of tangent at P $y = px - p^2$
$\therefore$ ... ... Q $y = qx - q^2$ $\Rightarrow (p-q)x = p^2 - q^2$ or $x = p+q$.

if $x = p+q$ then $y = p(p+q) - p^2$ ie $y = pq$ $\therefore M \equiv (p+q, pq)$ (2)

(iv) For M to lie on $x^2 + 4y = 0$ then $(p+q)^2 + 4pq = 0 \Rightarrow q = (-3 \pm 2\sqrt{2})p$, $p \neq$ [0]

Now M has coords $((-2 \pm 2\sqrt{2})p, (-3 \pm 2\sqrt{2})p^2)$ (4)

(v) Chord PQ is $2y = (p+q)x - 2pq$. & intersects $x^2 = -4y$
when $x^2 = -2((p+q)x - 2pq)$ or $x^2 = -2(p+q)x + 4pq$
ie $x^2 + 2(p+q)x - 4pq = 0$ & for PQ to be a tangent to $x^2 = -4ay$ we must have one & one only root to quadratic (ie $\Delta = 0$). (2)

$\therefore 4(p+q)^2 + 16pq = 0$ or $(p+q)^2 + 4pq = 0$

which was one equation in part (iv) & gave M coords $((-2 \pm 2\sqrt{2})p, (-3 \pm 2\sqrt{2})$[$p^2$])

$\therefore$ Chord PQ is a tangent to $x^2 = -4y$.