CRANBROOK SCHOOL

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

2000

MATHEMATICS

3 UNIT (Additional)
4 UNIT (First Paper)

Time allowed – Two hours

DIRECTIONS TO CANDIDATES

* **Attempt all questions.**
* **ALL questions are of equal value.**
* **All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.**
* **Standard integrals are printed on the back page.**
* **Board-approved calculators may be used.**
* **You may ask for extra Writing Booklets if you need them.**

* Submit your work in five booklets :

(i) QUESTIONS 1 & 2 (8 page)

(ii) QUESTIONS 3 & 4 (8 page)

(iii) QUESTION 5 (4 page)

(iv) QUESTION 6 (4 page)

(v) QUESTION 7 (4 page)

1. (8 page booklet)

(a) If the equation $5x^3 - 6x^2 - 29x + 6 = 0$ has roots α, β, γ find the value of $\alpha^2 + \beta^2 + \gamma^2$. **[3 marks]**

(b) (i) Show that there exists one value of the constant b for which the polynomial $P(x) = x^4 + 2x^3 - x^2 - 8x - b$ is divisible by $Q(x) = x^2 - 4$. **[2 marks]**

(ii) Hence or otherwise find the roots of $P(x)$ for this value of b. **[2 marks]**

(c) (i) Find $\frac{d}{dx}(\operatorname{cosec} x \ \cot x)$ in terms of $\operatorname{cosec} x$. **[3 marks]**

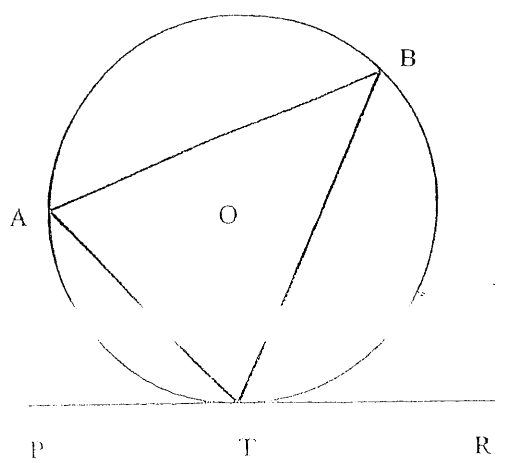
(ii) Use your result in (i) to find the exact value of $\int_{\pi/6}^{\pi/3} \operatorname{cosec} x(\cot^2 x + \operatorname{cosec}^2 x)dx$. **[2 marks]**

2.

(a) Find the general solutions of $\sin 2\theta + \cos\theta = 0$ in radian form. **[3 marks]**

(b) Find the solutions of $3\sin\theta + 4\cos\theta = -4$ for $0 \le \theta \le 4\pi$, giving your answers in radians, correct (where necessary) to 3 decimal places. **[4 marks]**

(c) PR is a tangent to the circle centre O, at the point T. Prove that $\angle ATP = \angle ABT$.
(Redraw the diagram below as part of your answer).
[5 marks]



3. (new 8 page booklet please)

(a) Find the term independent of x in the expansion of $\left(\frac{3x^2}{2} - \frac{1}{3x}\right)^9$ **[4 marks]**

(b) Twelve candidates for election to a committee of four include two well-known geniuses, Mr G.J. Baker and Mr S.K. Blazey. If all candidates have an equal chance of selection, what is the probability that the committee

(i) includes Mr Baker but excludes Mr Blazey?
(ii) includes at least one of these two geniuses? **[4 marks]**

(c) A weather bureau finds that it predicts maximum temperatures with about 60% accuracy. What is the probability that, in a particular week, it is accurate

(i) on every day but Saturday and Sunday?
(ii) on exactly five days? **[4 marks]**

4.

(a) Solve $\frac{3x+2}{x-1} > 2$ **[3 marks]**

(b) Prove by Mathematical Induction that

$$2 \times 1! + 5 \times 2! + 10 \times 3! + \ \ldots\ + (n^2+1) \times n! = n \times (n+1)!$$ **[5 marks]**

(c) (i) Show that ${}^nC_r : {}^nC_{r-1} = (n-r+1) : r$

(ii) Hence evaluate $\frac{{}^nC_1}{{}^nC_0} + \frac{2 \times {}^nC_2}{{}^nC_1} + \frac{3 \times {}^nC_3}{{}^nC_2} + \ \ldots\ + \frac{n \times {}^nC_n}{{}^nC_{n-1}}$ **[4 marks]**

5. (new 4 page booklet please)

(a) Find the derivative of $\cos^{-1}(2x+1)$, stating the values of x for which it is defined. **[2 marks]**

(b) Differentiate $\sin^{-1}(e^{2x})$ and hence find $\int_{-\ln 2}^{0} \frac{e^{2x}}{\sqrt{1-e^{4x}}}\,dx$ correct to two decimal places. **[4 marks]**

(c) The rate of emission E, in tonnes per year, of chloro-fluorocarbons (CFC's) in Australia from 18th July 2000 will be given by $E = 80 + \left(\frac{[illegible]}{1+t}\right)$, where t is the time in years.

(i) What is the rate of emission E on 18th July 2000? **[1 mark]**
(ii) What is the rate of emission E on 18th July 2005? **[1 mark]**
(iii) Draw a sketch of E as a function of t. **[2 marks]**
(iv) Calculate the total amount of CFCs emitted in Australia during the years 2000 to 2005. **[2 marks]**

6. (new 4 page booklet please)

(a) Evaluate $\int_0^{\pi} 2\sin x \cos^2 x \; dx$. **[2 marks]**

(b) Integrate the following using the substitutions given

(i) $\int \frac{x^4}{\left(x^5+1\right)^3} dx \qquad (u = x^5+1)$ (ii) $\int_{\frac{1}{2}}^{1} \frac{\sqrt{1-x^2}}{x^2} \; dx \qquad (x = \cos\theta)$ **[6 marks]**

(c) Two roads intersect, making an angle of 30° between them. After an argument at the intersection, George storms off at 6 km/h along one of the roads, and Jerry walks off calmly at 2 km/h along the other. Show that the rate at which the distance between them is increasing is constant. Find this rate of increase correct to three significant figures. **[4 marks]**

7. (new 4 page booklet please)

(a) The rate of change of the volume of water (V kL) in a dam at any given time t (in hours) is given by $\frac{dV}{dt} = k(V-5000)$, where k is a constant.

(i) Show that $V = 5000 + Ae^{kt}$ is a solution of this differential equation. **[2 marks]**

(ii) If the initial volume is 87 000 kL, and after 10 hours the volume is 129 000 kL, find the exact values of A and k. **[3 marks]**

(iii) Determine how long it will take the volume to reach 4.2 million kL. [Give your answer in days and hours, correct to the nearest hour.] **[2 marks]**

(b) The inner and outer radii of a cylindrical tube of constant length change in such a way that the volume of the material forming the tube remains constant. Find the rate of increase of the outer radius at the instant when the radii are 3 cm and 5 cm, and the rate of increase of the inner radius is $3\frac{1}{3}$ cm/s. **[5 marks]**

STANDARD INTEGRALS

$\int x^n \; dx = \frac{1}{n+1} x^{n+1} \qquad (n \neq -1;\ x \neq 0 \text{ if } n < 0)$

$\int \frac{1}{x} \; dx = \log_e x \qquad (x > 0)$

$\int \cos ax \; dx = \frac{1}{a} \sin ax \qquad (a \neq 0)$

$\int \sin ax \; dx = -\frac{1}{a} \cos ax \qquad (a \neq 0)$

$\int e^{ax} \; dx = \frac{1}{a} e^{ax} \qquad (a \neq 0)$

$\int \sec^2 ax \; dx = \frac{1}{a} \tan ax \qquad (a \neq 0)$

$\int \sec ax \tan ax \; dx = \frac{1}{a} \sec ax \qquad (a \neq 0)$

$\int \frac{1}{a^2+x^2} \; dx = \frac{1}{a} \tan^{-1} \frac{x}{a} \qquad (a \neq 0)$

$\int \frac{1}{\sqrt{a^2-x^2}} \; dx = \sin^{-1} \frac{x}{a} \qquad (a > 0,\ -a < x < a)$

$\int \frac{1}{\sqrt{x^2-a^2}} \; dx = \log_e \left\{x + \sqrt{x^2-a^2}\right\} \qquad (|x| > |a|)$

$\int \frac{1}{\sqrt{x^2+a^2}} \; dx = \log_e \left\{x + \sqrt{x^2+a^2}\right\}$

3UNIT TRIAL CRANBROOK 2000.

1(a) $5x^3-6x^2-29x+6=0$, has roots α, β, γ.

$\therefore \alpha+\beta+\gamma = \frac{--6}{5} = \frac{6}{5}$

$\alpha\beta+\alpha\gamma+\beta\gamma = \frac{-29}{5}$

$\alpha\beta\gamma = \frac{-6}{5}$

Now $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)$

$= \left(\frac{6}{5}\right)^2-2\left(\frac{-29}{5}\right)$

$= \frac{36}{25}+\frac{58}{5}$

$= \frac{326}{25}$ ✓

(b) (i) If $P(x)$ is divisible by $Q(x)$ then $P(2)=P(-2)=0$

as $Q(x)=x^2-4$

$=(x-2)(x+2)$.

Now $P(2) = 16+16-4-16-b$

$= 12-b$ ✓

$\therefore$ If $P(2)=0$ then $b=12$

Also $P(-2) = 16-16-4+16-b$

$= 12-b$

$\therefore$ if $P(-2)=0$ then $b=12$ again.

ie there exists only 1 value of the constant b if $P(x)$ is divisible by $Q(x)$. ✓

(ii) $P(x) = x^4+2x^3-x^2-8x-12$

[illegible]

$=(x-2)(x+2)(x^2+2x+3)$ ✓

$\therefore$ Roots are $x=2, -2$.

($x^2+2x+3=0$ has no real roots) ✓

c (i) Let $y = \operatorname{cosec} x \cot x$

$\therefore \frac{dy}{dx} = \operatorname{cosec} x \cdot -\operatorname{cosec}^2 x + \cot x \cdot -\operatorname{cosec} x \cot x$ ✓

$= -\operatorname{cosec}^3 x - \operatorname{cosec} x(\cot^2 x)$

$= -\operatorname{cosec}^3 x - \operatorname{cosec} x(\operatorname{cosec}^2 x - 1)$ ✓

$= -2\operatorname{cosec}^3 x + \operatorname{cosec} x$

$\therefore \frac{d}{dx}(\operatorname{cosec} x \cot x) = -2\operatorname{cosec}^3 x + \operatorname{cosec} x$ ✓

or let $y = \operatorname{cosec} x \cot x$

$= \frac{1}{\sin x \tan x}$

$= \frac{\cos x}{\sin^2 x}$

$\therefore \frac{dy}{dx} = \frac{\sin^2 x \cdot -\sin x - \cos x \cdot 2\sin x \cos x}{\sin^4 x}$ ✓

$= \frac{-\sin^3 x - 2\sin x(1-\sin^2 x)}{\sin^4 x}$

$= \frac{-\sin^3 x - 2\sin x + 2\sin^3 x}{\sin^4 x}$ ✓

$= \frac{\sin^3 x - 2\sin x}{\sin^4 x}$

$= \frac{1}{\sin x} - \frac{2}{\sin^3 x}$

$= \operatorname{cosec} x - 2\operatorname{cosec}^3 x$. ✓

ie $\frac{d}{dx}(\operatorname{cosec} x \cot x) = \operatorname{cosec} x - 2\operatorname{cosec}^3 x$

(ii) $I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \operatorname{cosec} x(\cot^2 x + \operatorname{cosec}^2 x)\,dx$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \operatorname{cosec} x(\operatorname{cosec}^2 x - 1 + \operatorname{cosec}^2 x)\,d$

[illegible]

$= -\left[\operatorname{cosec} x \cot x\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$ (using result of part(i))

$= -\left[\frac{1}{\sin x \tan x}\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$

$= -\left[\frac{1}{\frac{\sqrt{3}}{2} \cdot \sqrt{3}} - \frac{1}{\frac{1}{2} \cdot \frac{1}{\sqrt{3}}}\right]$

$$= -\left[\frac{2}{3} - 2\sqrt{3}\right]$$
$$= 2\sqrt{3} - \frac{2}{3}$$

2 (a) $\sin 2\theta + \cos\theta = 0$

$\therefore 2\sin\theta\cos\theta + \cos\theta = 0$

$\therefore \cos\theta(2\sin\theta + 1) = 0$

$\therefore \cos\theta = 0$ or $\sin\theta = -\frac{1}{2}$

$\therefore \cos\theta = \cos\frac{\pi}{2}$ or $\sin\theta = \sin\left(-\frac{\pi}{6}\right)$

$\therefore \theta = 2n\pi \pm \frac{\pi}{2}$ or $\theta = n\pi + (-1)^n\left(-\frac{\pi}{6}\right)$

for $n \in J$.

(b) $3\sin\theta + 4\cos\theta = -4$, $0 \le \theta \le 4\pi$

Let $t = \tan\frac{\theta}{2}$ $(\theta \neq \pi)$

$\therefore \sin\theta = \frac{2t}{1+t^2}$, $\cos\theta = \frac{1-t^2}{1+t^2}$

$\therefore 3\left(\frac{2t}{1+t^2}\right) + 4\left(\frac{1-t^2}{1+t^2}\right) = -4$

$\therefore 6t + 4 - 4t^2 = -4 - 4t^2$

$\therefore 6t = -8$

$\therefore t = -\frac{4}{3}$

$\therefore \tan\frac{\theta}{2} = -\frac{4}{3}$

$\therefore$ basic angle $\frac{\theta}{2} = \tan^{-1}\frac{4}{3}$ (require 2nd, 4th quads)

$\therefore \frac{\theta}{2} = \pi - \tan^{-1}\frac{4}{3}$ or $2\pi - \tan^{-1}\frac{4}{3}$

$\therefore \angle\theta = 2\pi - 2\tan^{-1}\frac{4}{3}$ or $4\pi - 2\tan^{-1}\frac{4}{3}$

$\therefore \angle\theta = 4.429$ or 10.712 (3 d.p.)

Now LHS $= -4 =$ RHS

$\angle\theta = \pi, 3\pi, 4.429$ or 10.712 (3dp)

or $3\sin\theta + 4\cos\theta = -4$

$\therefore 5\left(\frac{3}{5}\sin\theta + \frac{4}{5}\cos\theta\right) = -4$

$\therefore \sin(\theta + \alpha) = -\frac{4}{5}$

where $\cos\alpha = \frac{3}{5}$, $\sin\alpha = \frac{4}{5}$

$\therefore \tan\alpha = \frac{4}{3}$ ie $\alpha = \tan^{-1}\frac{4}{3}$

$\therefore \sin\left(\theta + \tan^{-1}\frac{4}{3}\right) = -\frac{4}{5}$

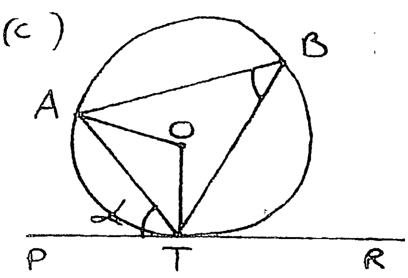
$\therefore$ basic angle $\left(\theta + \tan^{-1}\frac{4}{3}\right) = \sin^{-1}\frac{4}{5}$ (require 3rd, 4th, for $0 \le$

$\therefore \angle\theta = \pi + \sin^{-1}\frac{4}{5} - \cancel{\tan^{-1}\frac{4}{3}}\, 2\pi - \sin^{-1}\frac{4}{5} - \tan^{-1}\frac{4}{3}$,

$3\pi + \sin^{-1}\frac{4}{5} - \tan^{-1}\frac{4}{3}$, $4\pi - \sin^{-1}\frac{4}{5} -$

$= \pi$, 4.429 (3dp), 3π, 10.71

(c)



TO PROVE:

$\angle ATP = \angle AB$

PROOF: Let $\angle ATP = \alpha$

Join O to A and O to T.

$\angle OTP = 90°$ ($\angle$ between tangent and ra at pt. contact $= 90°$)

$\therefore \angle OTA = 90° - \alpha$

Now as $OA = OT$ (equal radii)

$\therefore \triangle AOT$ is isosceles.

$\therefore \angle OAT = 90° - \alpha$ (base $\angle$s of isos. $\triangle$ ar

$\angle \triangle = 18$

$= 2\alpha$

$\therefore \angle ABT = \alpha$ ($\angle$ at centre $= 2\angle$ circum. standing on a common arc.)

$\therefore \angle ATP = \angle ABT$.

3 (a) $\left(\frac{3x^2}{2} - \frac{1}{3x}\right)^9 = \left(\frac{3x^2}{2}\right)^9\left(1 - \frac{2}{9x^3}\right)^9$

x^{18} $\quad x^{-18} = (x^{-3})^6$

$$\text{Term} = \frac{3^9 x^{18}}{2^9} \cdot {}^9C_6 \left(-\frac{2}{9x^3}\right)^6$$

$$= \frac{3^9}{2^9} \cdot 84 \cdot \frac{2^6}{3^{12}} = \underline{\underline{\frac{7}{18}}}$$ 4

(b) (i) $\text{Prob} = \frac{1 \times {}^{10}C_3}{{}^{12}C_4} = \underline{\underline{\frac{8}{33}}}$ 2

(ii) $\text{Prob} = 1 - \text{Prob (both excluded)}$

$$= 1 - \frac{{}^{10}C_4}{{}^{12}C_4} = \underline{\underline{\frac{19}{33}}}$$ 2

(c) (i) $\text{Prob} = \left(\frac{3}{5}\right)^5\left(\frac{2}{5}\right)^2$

$$= \underline{\underline{\frac{3^5 \times 2^2}{5^7}}} \doteq \underline{\underline{0.012}}$$ ✓ $\left(\frac{972}{78125}\right)$

(ii) $\text{Prob} = {}^7C_5 \left(\frac{3}{5}\right)^5\left(\frac{2}{5}\right)^2$

$$= 21\left(\frac{3}{5}\right)^5\left(\frac{2}{5}\right)^2$$

$$= \underline{\underline{\frac{21 \times 3^5 \times 2^2}{5^7}}} \doteq \underline{\underline{0.261}}$$ ✓

4. (a) $\dfrac{3x+2}{x-1} > 2$

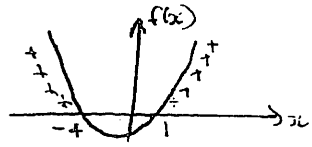
$\therefore (3x+2)(x-1) > 2(x-1)^2$

$3x^2 - x - 2 \geqslant 2x^2 - 4x + 2$

$\therefore x^2 + 3x - 4 \geqslant 0$

$(x+4)(x-1) \geqslant 0$

$\therefore x \geqslant 1$ or $x < -4$ 3



(b) Let S be the set of +ve integers n for which

$2 \times 1! + 5 \times 2! + \dots + (n^2+1) \times n! = n \times (n+1)!$

If n=1

L.H.S. $= 2 \times 1! = 2$

R.H.S. $= 1 \times 2! = 2$ $\therefore 1 \in S$

Suppose $k \in S$

$\therefore 2 \times 1! + 5 \times 2! + \dots + (k^2+1) \times k! = k \times (k+1)!$

If $n = k+1$

L.H.S. $= 2 \times 1! + \dots + (k^2+1) \times k! + ((k+1)^2+1) \times (k+1)!$

$= k \times (k+1)! + (k^2+2k+2) \times (k+1)!$

$= (k+1)!\,(k^2+3k+2)$

$= (k+1)!\,(k+1)(k+2)$

$= (k+1) \times (k+2)! =$ R.H.S.

$\therefore k \in S \Rightarrow (k+1) \in S$

$\therefore$ S is the set of all +ve integers 5

(c) (i) ${}^nC_r : {}^nC_{r-1} = \dfrac{\cancel{n!}}{r!\,(n-r)!} \cdot \dfrac{(r-1)!\,(n-r+1)!}{\cancel{n!}} = \dfrac{n-r+1}{r}$ 2

(ii) $\dfrac{{}^nC_1}{{}^nC_0} + \dfrac{2 \cdot {}^nC_2}{{}^nC_1} + \dfrac{3 \cdot {}^nC_3}{{}^nC_2} + \dots + \dfrac{n \cdot {}^nC_n}{{}^nC_{n-1}} = \dfrac{n}{1} + 2 \cdot \dfrac{n-1}{2} + 3 \cdot \dfrac{n-2}{3} + \dots + n$

$= n + (n-1) + (n-2) + \dots + 1$

$= n(n+1)$

(a) $\cos^{-1}(2x+1)$ is defined for $-1 \le 2x+1 \le 1$

ie. $-2 \le 2x \le 0$

$-1 \le x \le 0$

Hence $\cos^{-1}(2x+1)$ is defined for $-1 \le x \le 0$.

Let $y = \cos^{-1}(2x+1)$

$y = \cos^{-1}(u)$ where $u = 2x+1$

Now, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$	$u = 2x+1$ $\frac{du}{dx} = 2$
$= -\frac{1}{\sqrt{1-u^2}} \times 2$	$y = \cos^{-1} u$
(1)	$\frac{dy}{du} = -\frac{1}{\sqrt{1-u^2}}$

$$= -\frac{2}{\sqrt{1-(2x+1)^2}} \quad \text{provided } -1 < x < 0 \quad (1)$$

$$= -\frac{2}{\sqrt{1-(4x^2+4x+1)}}$$

optional

$$= -\frac{2}{\sqrt{1-4x^2-4x-1}}$$

$$= -\frac{2}{\sqrt{-4x(x+1)}}$$

Q5. (b) Let $y = \sin^{-1} u$ where $u = e^{2x}$

$u = e^{2x}$ $\qquad y = \sin^{-1} u$

$\frac{du}{dx} = 2e^{2x}$ $\qquad \frac{dy}{du} = \frac{1}{\sqrt{1-u^2}}$

Now, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$= \frac{1}{\sqrt{1-u^2}} \times 2e^{2x}$$

$$= \frac{2e^{2x}}{\sqrt{1-e^{4x}}}$$

$$\therefore \int_{-\ln 2}^{0} \frac{2e^{2x}}{\sqrt{1-e^{4x}}}\,dx = \left[\sin^{-1}(e^{2x})\right]_{-\ln 2}^{0} \quad (2)$$

$$\therefore \int_{-\ln 2}^{0} \frac{e^{2x}}{\sqrt{1-e^{4x}}}\,dx = \frac{1}{2}\left[\sin^{-1}(e^{2x})\right]_{-\ln 2}^{0}$$

$$= \frac{1}{2}\left(\sin^{-1} 1 - \sin^{-1}(e^{-2\ln 2})\right)$$

$-2\ln 2 = -\ln 4$

$e^{-\ln 4} = \frac{1}{e^{\ln 4}} = \frac{1}{4}$.

$$= \frac{1}{2}\left(\frac{\pi}{2} - \sin^{-1}\left(\frac{1}{4}\right)\right)$$

$$= \frac{1}{2}(1.570796327 - 0.252680255)$$

$= 0.66$ (2d.p.) (2)

Q5. (c) $E = 80 + \left(\dfrac{30}{1+t}\right)^2$

(i) On 18^{th} July 2000, $t = 0$

$$E = 80 + \left(\frac{30}{1+0}\right)^2$$

$$E = 80 + (30)^2$$

$$E = 980 \text{ tonnes/year}$$ (1)

(ii) On 18^{th} July 2005, $t = 5$

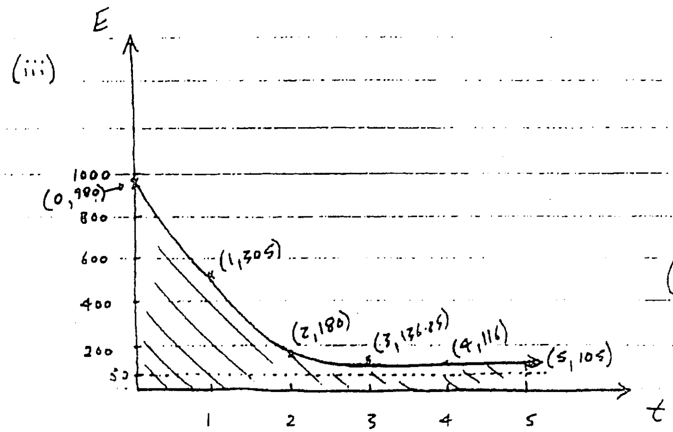
$$E = 80 + \left(\frac{30}{1+5}\right)^2$$

$$E = 80 + \left(\frac{30}{6}\right)^2$$

$$E = 80 + 5^2$$

$$E = 105 \text{ tonnes/year}$$ (1)

(iii)



(2)

E	980	305	180	136.25	116	105

(iv) Total amount of CFCs emitted $= \displaystyle\int_0^5 \left(80 + \left(\frac{30}{1+t}\right)^2\right).$

$$= \int_0^5 80.\,dt + \int_0^5 \frac{900}{(1+t)^2} \cdot d$$

$$\left[80t\right]_0^5 + \int_0^5 900(1+t)^{-2}.$$

$$\left(\int (1+t)^{-2}.\,dt = \frac{(1+t)^{-1}}{-1} + C = -\frac{1}{1+t} + C\right)$$

$$\left[80t\right]_0^5 + \left[-\frac{900}{(1+t)}\right]_0^5$$

$$(400 - 0) + \left(-\frac{900}{6} - \left(-\frac{9}{\ }\right.\right.$$

$$400 + \left(900 - \frac{900}{6}\right)$$

1150 tonnes (2)

YEAR 12 3 UNIT 2000 TRIAL QUESTION 6

(a) $\int_0^{\pi} 2\sin x \cos^2 x \, dx$.

Let $\cos x = u$ when $x = \pi$ $u = -1$

$-\sin x = \frac{du}{dx}$ $x = 0$ $u = 1$

$\sin x \, dx = du$.

$$\int_1^{-1} 2u^2 \, du = -\left[\frac{2u^3}{3}\right]_1^{-1}$$

$$-\left[\left[-\frac{2}{3}\right] - \left[\frac{2}{3}\right]\right]$$

$$= 1\tfrac{1}{3} \text{ units.}$$

(b) (i) $\int \frac{x^4}{(x^5+1)^3} dx \Rightarrow$ Given ②

$u = x^5 + 1$

$\frac{du}{dx} = 5x^4$

$du = 5x^4$

$$\therefore \frac{1}{5}\int \frac{du}{u^3} = \frac{1}{5}\left(-\frac{1}{2u}\right) + c$$

$$= -\frac{1}{10u} + c$$

$$= -\frac{1}{10(x^5+1)} + c$$ ③

(ii) $\int_{1/2}^{1} \frac{\sqrt{1-x^2}}{x^2} dx$ $x = \cos\theta$

$\frac{dx}{d\theta} = -\sin\theta$

$$\int_{\pi/3}^{0} \left[\frac{\sqrt{(1-\cos^2\theta)}}{\cos^2\theta} - \sin\theta\right] d\theta.$$

$$= \int_{\pi/3}^{0} \frac{-\sin^2\theta \, d\theta}{\cos^2\theta}$$

$$= -\int \tan^2\theta \, d\theta$$

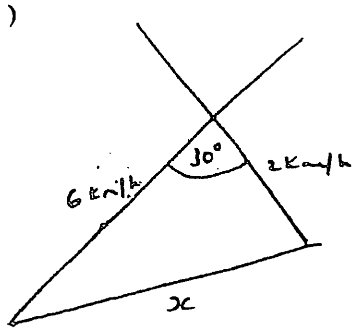
$$= -\int \sec^2\theta - 1 \, d\theta$$ ③

$$= -\left[\tan\theta - \theta\right]_{\pi/3}^{0}$$

$$= -\left[(0 - 0) - (\tan \pi/3 - \pi/3)\right]$$

$$= (-\sqrt{3} + \pi/3)$$

(c)



(i) Let x be the distance between them pe

$x^2 = 6^2 + 2^2 - 2 \times 6 \times 2 \cdot \cos 30$

$x^2 = 36 + 4 - 24 \times \frac{\sqrt{3}}{2}$

$x^2 = 40 - 12\sqrt{3}$

$\therefore x = \sqrt{40 - 12\sqrt{3}}$ [NB. Only +ve solut since length can not be -ve]

Distance $\therefore x = \left(2\sqrt{10 - 3\sqrt{3}}\right)t$

Rate $\therefore \frac{dx}{dt} = 2\sqrt{10 - 3\sqrt{3}}$ $\therefore$ constant value.

$\therefore$ The rate of distance increasing is $2\sqrt{10-3\sqrt{3}}$ km/h.

$\therefore 4.383536279\ldots$

4.38 km/h (3 s.f.).

④

P2 3U TRIAL EXAMINATION 2000 SOLUTIONS (Q7)

(a) $\frac{dV}{dt} = k(V - 5000)$

If $V = 5000 + Ae^{kt}$

$\frac{dV}{dt} = kAe^{kt}$

$= k(V - 5000)$

$\therefore V = 5000 + Ae^{kt}$ is a solution (1)

(ii) when $t = 0$, $V = 87000$

$\therefore 87000 = 5000 + Ae^{0}$

$\therefore A = 82000$

when $t = 10$, $V = 129000$

$\therefore 129000 = 5000 + 82000e^{10k}$

$\frac{124000}{82000} = e^{10k}$

$\therefore 10k = \ln\frac{124}{82}$

$k = \frac{1}{10}\ln\frac{62}{41}$ (3)

(iii) $V = 5000 + 82000e^{\frac{t\ln\frac{62}{41}}{10}}$

$4200000 = 5000 + 82000e^{\frac{t\ln\frac{62}{41}}{10}}$

$\therefore e^{\frac{t\ln\frac{62}{41}}{10}} = \frac{4195000}{82000}$

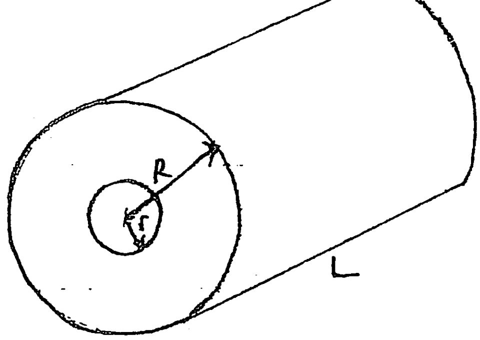
$t = \frac{10\ln\frac{4195}{82}}{\ln\frac{62}{41}}$ (3)

$= 3$ days 23 h

(to nearest hour)

(3 days 24 h accepted since theoretically, 4 200 000 L wouldn't have been reached by 3 days 23 h)

(c)



Volume of material, $V = (\pi R^2 - \pi r^2)L$

$\therefore \frac{V}{L} = \pi R^2 - \pi r^2$

constant

$\pi R^2 = \frac{V}{L} + \pi r^2$

$R^2 = \frac{V}{\pi L} + r^2$

$R = \sqrt{\frac{V}{\pi L} + r^2}$

$\frac{dR}{dt} = \frac{1}{2}\left(\frac{V}{\pi L} + r^2\right)^{-\frac{1}{2}}$

$= \frac{r}{\sqrt{\frac{V}{\pi L} + r^2}}$

$\frac{dR}{dt} = \frac{dR}{dr} \times \frac{dr}{dt}$

$= \frac{r}{\sqrt{\frac{V}{\pi L} + r^2}} \times \frac{dr}{dt}$

when $R = 5$, $r = 3$

$\frac{V}{L} = \pi \times 5^2 - \pi \times 3^2$

$= 16\pi$

$\therefore \frac{dR}{dt} = \frac{3}{\sqrt{16 + 3^2}} \times \frac{10}{3}$

$= \frac{10}{\sqrt{25}}$

$= 2$

outer radius is increasing at rate of 2 cm/s