Student Number: ____________

Teacher: CJL HRK SKB

CRANBROOK SCHOOL

MATHEMATICS EXTENSION 1

2005

HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION

General Instructions

- Reading time: five minutes
- Working time: two hours
- Calculators may be used.

- The examination consists of *7 questions* worth 12 marks each.
  <u>*Begin a new booklet for each question*</u>
  All questions should be attempted.
  All necessary working should be shown in every question.
  A table of standard integrals is provided at the back of this paper.

Question 1 (12 marks) CJL **Marks**

(a) Let A be the point $(-3,8)$ and let B be the point $(5,-6)$. Find the coordinates of the point P that divides the interval AB internally in the ratio $1:3$. **2**

(b) What is the remainder when the polynomial $P(x)=x^3+3x^2-1$ is divided by $x-2$? **2**

(c) Use the table of standard integrals to find the exact value of **2**

$$\int_0^1 \frac{1}{\sqrt{x^2+9}}\,dx.$$

(d) Solve $\dfrac{2}{x+5}\le 1$. **3**

(e) Use the substitution $u=x-1$ to evaluate $\displaystyle\int_2^4 \frac{x}{(x-1)^2}\,dx$. **3**

Question 2 (12 marks) CJL **Marks**

(a) Sketch the graph of $y = 2\sin^{-1} 3x$ showing clearly the domain and range of the function as well as any intercepts. **2**

(b) Let $f(x) = 4x^2 - 1$. **2**
Use the definition $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ to find the derivative of $f(x)$ at $x = a$.

(c) Find $\frac{d}{dx}(3x^2 \cos^{-1} x)$ **2**

(d) Find $\int 4\cos^2 3x\,dx$. **2**

(e) Solve the equation $\sin 2\theta = \sqrt{2}\cos\theta$ for $0 \le \theta \le 2\pi$. **4**

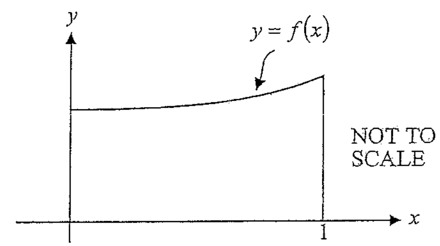
Question 3 (12 marks) HRK **Marks**

(a) Find the acute angle between the lines $3x - 4y + 3 = 0$ and $y = 2x - 5$ to the nearest minute. **2**

(b) The function $f(x) = \log_e x + 5x$ has a zero near $x = 0\cdot 2$. Using $x = 0\cdot 2$ as a first approximation, use one application of Newton's method to find a second approximation to the zero. Write your answer correct to 3 decimal places. **3**

(c) (i) Find the natural domain of the function $f(x) = \frac{1}{\sqrt{4 - x^2}}$. **1**

(ii) The sketch below shows part of the graph of $y = f(x)$. The area under the curve for $0 \le x \le 1$ is shaded. **2**
Find the area of the shaded region.



(d) A particle moves in simple harmonic motion about a fixed point O. The amplitude of the motion is 2 m and the period is $\frac{2\pi}{3}$ seconds. Initially the particle moves from O with a positive velocity.

(i) Explain why the displacement x, in metres, of the particle at time t seconds, is given by **1**

$$x = 2\sin 3t$$

(ii) Find the speed of the particle when it is $\sqrt{3}$ m from O. **2**

(iii) What is the maximum speed reached by the particle? **1**

Question 4 (12 marks) HRK **Marks**

(a) Use mathematical induction to prove that **3**

$$1+6+15+\ldots+n(2n-1)=\frac{1}{6}n(4n-1)(n+1)$$

for all positive integers n.

(b) The population N, of Keystown first reached 25 000 on 1 January 2000.
The population of Keystown is set to increase according to the equation

$$\frac{dN}{dt}=k(N-8000)$$

where t represents time in years after the population first reached 25 000.
On 1 January 2005, the population of Keystown was 29 250.

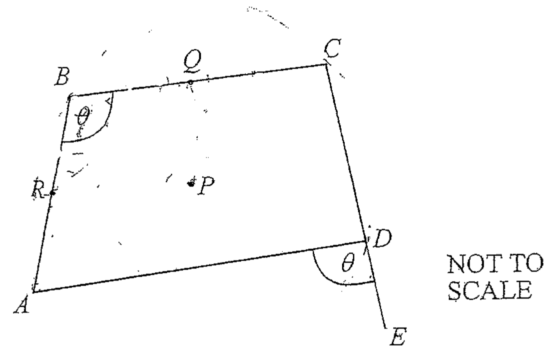
(i) Verify that $N=8000+Ae^{kt}$ is a solution to the above equation where A is a constant. **1**

(ii) Find the values of A and k. **2**

Question 4 continues on the next page.

Question 4 (continued) **Marks**

(c)



In the diagram, R is the midpoint of AB, Q is the midpoint of BC and $AB = BC$. Point D lies on CE. Let $\angle ADE = \angle ABC = \theta$.

(i) Explain why $ABCD$ is a cyclic quadrilateral. **1**

(ii) Given that P is the centre of the circle that passes through points A, B, C and D, show that $BQPR$ is a cyclic quadrilateral. **2**

(iii) Show that $\angle APR = \dfrac{180^\circ - \theta}{2}$. **3**

Question 5 (12 marks) HRK **Marks**

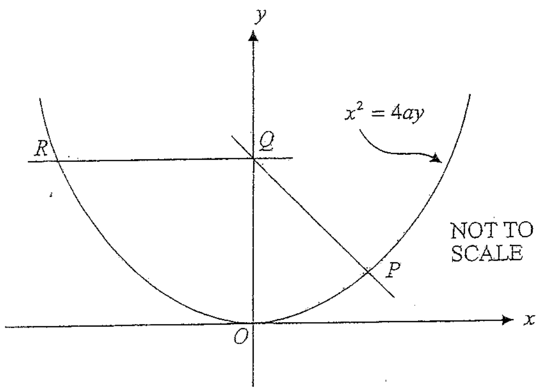
(a) If $\sin x = \frac{3}{5}$ and x is acute find the exact value(s) of $\tan\frac{x}{2}$. 3

(b) Consider the function $f(x) = x(x-2)^2,\ x \le a$ where a is a constant

(i) Find the values of a given that the inverse function $f^{-1}(x)$ of $f(x)$ exists. 2

(ii) State the domain of $f^{-1}(x)$. 1

(c)



The diagram above shows the graph of the parabola $x^2 = 4ay$. The normal to the parabola at the variable point $P(2at, at^2)$, $t > 0$, cuts the y-axis at Q. Point R lies on the parabola.

(i) Show that the equation of the normal to the parabola at P is $x + ty = at^3 + 2at$. 2

(ii) Find the coordinates of R given that QR is parallel to the x-axis and $\angle PQR > 90^\circ$. 2

(iii) Let M be the midpoint of RQ. Find the Cartesian equation of the locus of M. 2

Question 6 (12 marks) SKB **Marks**

(a) A particle moves in a straight line with an acceleration given by

$$\frac{d^2x}{dt^2} = 9(x-2)$$

where x is the displacement in metres from an origin O after t seconds. Initially, the particle is 4 metres to the right of O so that $x = 4$ and has velocity $v = -6$.

(i) Show that $v^2 = 9(x-2)^2$. 2

(ii) Find an expression for v and hence find x as a function of t. 2

(iii) Explain whether the velocity of the particle is ever zero. 2

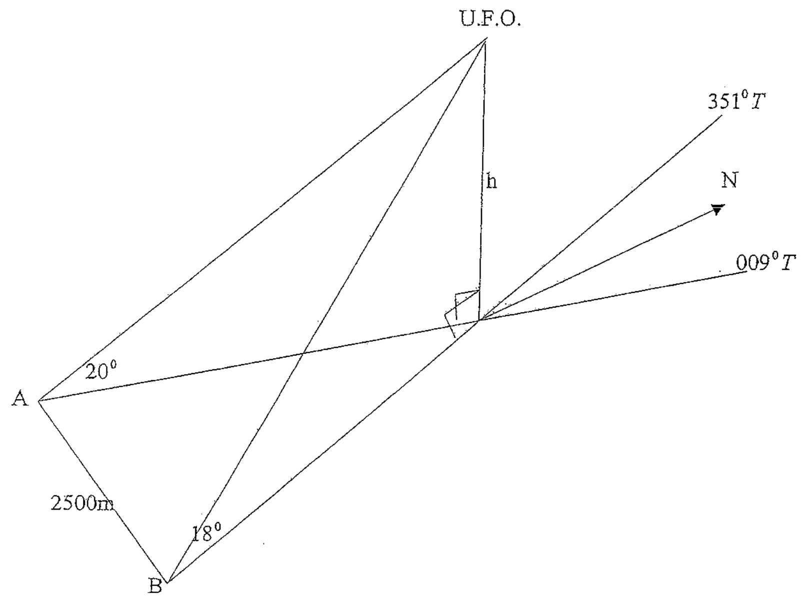
Question 6 continues on the next page.

Question 6 (continued) **Marks**

(b) Two surveyors A and B standing 2500 metres apart on level ground take the bearing and elevation of a U.F.O. at the same instant. Surveyor A notes the bearing as $009^0\,T$ and the angle of elevation as 20^0, whilst Surveyor A finds the corresponding bearing and elevation to be $351^0\,T$ and 18^0 respectively. If the U.F.O. is h metres above the ground :

(i) Prove that $h = \dfrac{2500}{\sqrt{\cot^2 20^0 + \cot^2 18 - 2\cot 20^0 \cot 18^0 \cos 18^0}}$. 4

(ii) Hence evaluate the height, h to the nearest metre. 2



Question 7 (12 marks) SKB **Marks**

(a) (i) Write $8\sin\theta - 15\cos\theta$ in the form $R\sin(\theta - \alpha)$ where α is acute. 1

(ii) Hence or otherwise find the general solutions of $8\sin\theta - 15\cos\theta = \dfrac{-17\sqrt{3}}{2}$ to the nearest minute. 3

(b) The roots of the equation $x^3 - 12x^2 + 12x + m = 0$ are in arithmetic progression. Find the numerical value of m and the roots of the equation. 4

(c) Evaluate $\int_0^3 \dfrac{dx}{9 + x^2}$ by using the substitution $x = 3\tan\theta$. 4

2005 CRANBROOK
MATHEMATICS EXTENSION 1
HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION SOLUTIONS

Question 1 (12 marks)

(a) $P = \left(\dfrac{lx_1 + kx_2}{k+l}, \dfrac{ly_1 + ky_2}{k+l} \right)$

where $k = 1,\ l = 3,\ x_1 = -3,\ y_1 = 8,\ x_2 = 5$ and $y_2 = -6$.

So $P = \left(\dfrac{3 \times -3 + 1 \times 5}{4}, \dfrac{3 \times 8 + 1 \times -6}{4} \right)$

$= \left(-1,\ 4\frac{1}{2} \right)$

2 marks	Correct answer
1 mark	Either the x or the y coordinate of P correct

(b) $P(x) = x^3 + 3x^2 - 1$

Using the Remainder Theorem,

$P(2) = 8 + 12 - 1$

$= 19$

The remainder is 19.

2 marks	Correct answer
1 mark	Correct method with an arithmetic mistake

(c) From the table of standard integrals, we have

$$\int_0^1 \frac{1}{\sqrt{x^2+9}}\,dx = \left[\ln\left(x + \sqrt{x^2+9}\right) \right]_0^1$$

$$= \ln\left(1 + \sqrt{10}\right) - \ln(0+3)$$

$$= \ln \frac{1+\sqrt{10}}{3}$$

2 marks	Correct answer
1 mark	Correct first line above

Question 1 (continued)

(d) Method 1 Multiplying both sides by $(x+5)^2$

$\dfrac{2}{x+5} \le 1, \qquad x \ne -5$

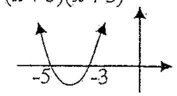
$2(x+5) \le (x+5)^2$

$2x + 10 \le x^2 + 10x + 25$

$0 \le x^2 + 8x + 15$

$0 \le (x+5)(x+3)$

From the graph of $y = (x+5)(x+3)$ we see that $y \ge 0$ for $x \le -5$ or $x \ge 3$ but $x \ne -5$ from above so $x < -5$ or $x \ge 3$.



3 marks	Correct answer
2 marks	Correct method but with one mistake
1 mark	Finds either of the critical points $x = -5$ or $x = -3$ or states that $x \ne -5$

Method 2 Critical points method

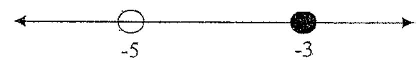
$\dfrac{2}{x+5} \le 1, \qquad x \ne -5$

Consider $\dfrac{2}{x+5} = 1$

$2 = x + 5$

$x = -3$

Look at the number line.



For $x = -6$, $\dfrac{2}{-6+5} \le 1$

For $x = -4$, $\dfrac{2}{-4+5} \ge 1$

For $x = -1$, $\dfrac{2}{-1+5} \le 1$

So $\dfrac{2}{x+5} \le 1$ for $x \le -5$ or $x \ge 3$, but $x \ne -5$

So $x < -5$ or $x \ge 3$

3 marks	Correct answer
2 marks	Correct method but with one mistake
1 mark	Finds either of the critical points $x = -5$ or $x = -3$ or states that

	$x \neq -5$

Question 1 (continued)

Method 3 Consider the cases where $x+5>0$ and $x+5<0$

$$\frac{2}{x+5} \leq 1, \qquad x \neq -5$$

Now, $2 \leq x+5$ if $x+5>0$

$x \geq -3$ if $x>-5$

So $x \geq -3$

Also, $2 \geq x+5$ if $x+5<0$

$x \leq -3$ if $x<-5$

So $x<-5$

So $x<-5$ or $x \geq -3$

3 marks	Correct answer
2 marks	Correct method but with one mistake
1 mark	Finds either of the critical points $x=-5$ or $x=-3$ or states that $x \neq -5$

(e)

$$\int_2^4 \frac{x}{(x-1)^2} = \int_1^3 (u+1)u^{-2}\frac{du}{dx}dx$$

$$= \int_1^3 \left(u^{-1}+u^{-2}\right)du$$

$$= \left[\log_e u - u^{-1}\right]_1^3$$

$$= \left(\log_e 3 - \frac{1}{3}\right) - \left(\log_e 1 - 1\right)$$

$$= \log_e 3 + \frac{2}{3} \qquad (\text{since } \log_e 1 = 0)$$

where $u = x-1$,

$\frac{du}{dx} = 1$,

and $x = u+1$

Also, $x=4$ so $u=3$

and $x=2$ so $u=1$

3 marks	Correct answer
2 marks	Finds correct integrand and terminals OR Follows correct method but forgets to change terminals
1 mark	Finds correct terminals OR finds correct integrand

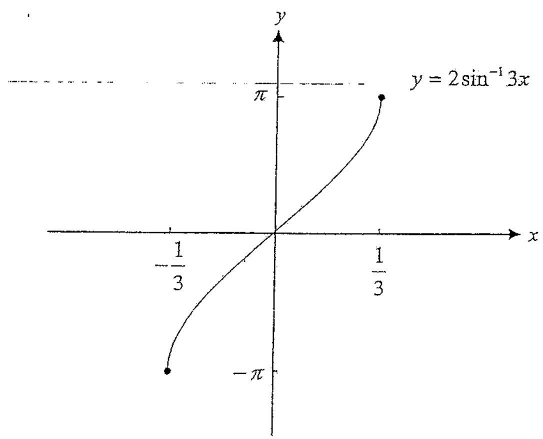
Question 2 (12 marks)

(a) The function $y = 2\sin^{-1} 3x$ is defined for

$$-1 \le 3x \le 1$$
$$-\frac{1}{3} \le x \le \frac{1}{3}$$

The coefficient 2 has the effect of stretching vertically the basic graph of $y = \sin^{-1} 3x$ by a factor of 2.

Since the graph of $y = \sin^{-1} 3x$ has a range of $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, the graph of $y = 2\sin^{-1} 3x$ has a range of $-\pi \le y \le \pi$.



2 marks	Showing correct graph with endpoints clearly marked
1 mark	Correct shape of graph without clearly marked endpoints

Question 2 (continued)

(b)
$$f(x) = 4x^2 - 1$$
$$f(a) = 4a^2 - 1$$
$$f(a+h) = 4(a+h)^2 - 1$$
$$= 4a^2 + 8ah + 4h^2 - 1$$
$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$
$$= \lim_{h\to 0} \frac{4a^2 + 8ah + 4h^2 - 1 - 4a^2 + 1}{h}$$
$$= \lim_{h\to 0} \frac{8ah + 4h^2}{h}$$
$$= \lim_{h\to 0} \frac{h(8a + 4h)}{h}$$
$$= \lim_{h\to 0} (8a + 4h)$$
$$= 8a$$

2 marks	Correct derivation
1 mark	Correct substitution into definition

(c) $\frac{d}{dx}\left(3x^2 \cos^{-1} x\right) = 6x\cos^{-1} x + 3x^2 \times \frac{-1}{\sqrt{1-x^2}}$ (Product rule)

$$= 6x\cos^{-1} x - \frac{3x^2}{\sqrt{1-x^2}}$$

2 marks	Correct answer
1 mark	Reasonable attempt to use product rule OR Showing $\frac{d}{dx}\left(\cos^{-1} x\right) = \frac{-1}{\sqrt{1-x^2}}$

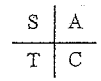
(d) $\int 4\cos^2 3x \, dx = 4\int\left(\frac{1}{2} + \frac{1}{2}\cos 6x\right)dx$ since $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$

$$= 4\left(\frac{x}{2} + \frac{1}{12}\sin 6x\right) + c$$
$$= 2x + \frac{1}{3}\sin 6x + c$$

2 marks	Correct answer
1 mark	Correct substitution of identity for $\cos^2 3x$

Question 2 (continued)

(e)
$$\sin 2\theta = \sqrt{2}\cos\theta$$
$$2\sin\theta\cos\theta = \sqrt{2}\cos\theta$$
$$\sqrt{2}\sin\theta\cos\theta - \cos\theta = 0$$
$$\cos\theta\left(\sqrt{2}\sin\theta - 1\right) = 0$$



$\cos\theta = 0$ or $\sin\theta = \dfrac{1}{\sqrt{2}}$

$\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$ $\qquad$ $\theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}$

4 marks	Four correct answers
3 marks	Correct factorisation and finding $\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$ OR Correct factorisation and finding $\theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}$
2 marks	Correct factorisation
1 mark	Correct replacement of $\sin 2\theta$

Question 3 (12 marks)

(a) For $3x - 4y + 3 = 0$ $\quad m_1 = \dfrac{3}{4}$ $\qquad$ and for $y = 2x - 5$ $\quad m_2 = 2$.

Now for acute angle θ between lines: $\qquad \tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$

$\therefore \tan\theta = \left|\dfrac{\frac{3}{4} - 2}{1 + \frac{3}{4}\times 2}\right|$ $\qquad \therefore \tan\theta = \dfrac{1}{2}$ $\qquad \therefore < \theta = 26^{0}34'$ (to nearest minute).

2 marks	Correct answer
1 mark	Determining correct gradients and stating correct formula.

Question 3 (continued)

(b) Now, $f(x) = \log_e x + 5x$

$$f'(x) = \frac{1}{x} + 5$$

and $\quad x_0 = 0 \cdot 2$

Using Newton's method

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$
$$= 0 \cdot 2 - \frac{\log_e 0 \cdot 2 + 5 \times 0 \cdot 2}{\frac{1}{0 \cdot 2} + 5}$$
$$= 0 \cdot 261 \text{ (correct to 3 decimal places)}$$

3 marks	Correct answer
2 marks	Finds $f(0 \cdot 2)$ and $f'(0 \cdot 2)$ correctly and substitutes into formula
1 mark	Correctly evaluates $f'(0 \cdot 2)$

(c) (i) Method 1

Because of the square root sign, $4 - x^2 \geq 0$ so $-2 \leq x \leq 2$.

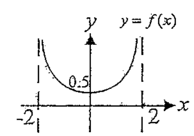
Also however, because $\sqrt{4 - x^2} \neq 0$, $x \neq -2, 2$.

So the natural domain of f is $-2 < x < 2$.

1 mark	Correctly reasoned answer

Method 2

Sketch the graph.



The natural domain of f is $-2 < x < 2$.

1 mark	Correct answer and correct graph

Question 3 (continued)

(ii) $\text{Area} = \int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx$

$$= \left[\sin^{-1}\left(\frac{x}{2}\right)\right]_0^1$$

$$= \sin^{-1}\frac{1}{2} - \sin^{-1} 0$$

$$= \frac{\pi}{6}$$

Area required is $\frac{\pi}{6}$ square units.

2 marks	Correct answer
1 mark	Correct first line above

(d) (i) The particle starts from the centre of motion and moves with positive velocity so the general form of the displacement time function is $x = a\sin nt$.

Now $a = 2$ and period $= \frac{2\pi}{n}$

so $n = 3$

So the required equation is $x = 2\sin 3t$ as required.

1 mark	Correctly derived equation

Question 3 (continued)

(ii) $x = 2\sin 3t$

When $x = \sqrt{3}$,

S	A
T	C

$$\sin 3t = \frac{\sqrt{3}}{2}$$

$$3t = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \ldots$$

$$t = \frac{\pi}{9}, \frac{2\pi}{9}, \frac{7\pi}{9}, \ldots$$

Now $x = 2\sin 3t$

$$\frac{dx}{dt} = 6\cos 3t$$

When $t = \frac{\pi}{9}$,

S	A
T	C

$$\frac{dx}{dt} = 6\cos\frac{\pi}{3}$$

$$= 3$$

(Check when $t = \frac{2\pi}{9}$,

$$\frac{dx}{dt} = 6\cos\frac{2\pi}{3}$$

$$= 6 \times -\frac{1}{2}$$

$$= -3$$

$$\text{speed} = |-3|$$

$= 3$)

So speed is $3\,\text{m s}^{-1}$.

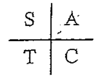
2 marks	Correct answer
1 mark	Finding correct values of t when particle is $\sqrt{3}$ m from O OR Substituting incorrect value of t into correct expression for $\frac{dx}{dt}$ OR Giving an answer of $-3\,\text{m s}^{-1}$

Question 3 (continued)

(iii) Method 1

For *SHM* the maximum speed occurs at the centre of motion i.e. at *O*.

$$x = 2\sin 3t$$
$$0 = 2\sin 3t$$
$$3t = 0, \pi, 2\pi ...$$
$$t = 0, \frac{\pi}{3}, \frac{2\pi}{3}, ...$$



$$\frac{dx}{dt} = 6\cos 3t$$

When $t = 0$,

$$\frac{dx}{dt} = 6\cos 0$$
$$= 6$$

(Check when $t = \frac{\pi}{3}$,

$$\frac{dx}{dt} = 6\cos \pi$$
$$= -6$$

Speed $= |-6|$

$= 6$)

Maximum speed reached by particle is 6 m s^{-1}.

1 mark	Correct answer

Method 2

Maximum speed occurs when $\frac{d^2x}{dt^2} = 0$

$$\frac{d^2x}{dt^2} = -18\sin 3t$$
$$0 = -18\sin 3t$$
$$3t = 0, \pi, 2\pi, ...$$
$$t = 0, \frac{\pi}{3}, \frac{2\pi}{3}, ...$$
$$\frac{dx}{dt} = 6\cos 3t$$

When $t = 0$,

$$\frac{dx}{dt} = 6\cos 0$$
$$= 6$$

(Check when $t = \frac{\pi}{3}$,

$$\frac{dx}{dt} = 6\cos \pi$$
$$= -6$$

Speed $= |-6|$

$= 6$)

Maximum speed reached by particle is 6 m s^{-1}.

1 mark	Correct answer.

Question 4 (12 marks)

(a) Prove that $1 + 6 + 15 + ... + n(2n-1) = \frac{1}{6}n(4n-1)(n+1)$ using mathematical induction.

When $n = 1$, $RHS = \frac{1}{6} \times 3 \times 2$

$= 1$

$= LHS$

So the statement is true for $n = 1$.

Suppose that it is true for $n = k$.

That is, suppose that $1 + 6 + 15 + ... + k(2k-1) = \frac{1}{6}k(4k-1)(k+1)$ - (*A*)

Then, when $n = k + 1$, we have

$$1 + 6 + 15 + ... + (k+1)\{2(k+1)-1\} = \frac{1}{6}(k+1)\{4(k+1)-1\}\{(k+1)+1\}$$
$$= \frac{1}{6}(k+1)(4k+3)(k+2)$$

$$LHS = 1 + 6 + 15 + ... + (k+1)\{2(k+1)-1\}$$
$$= 1 + 6 + 15 + ... + k(2k-1) + (k+1)\{2(k+1)-1\}$$
$$= \frac{1}{6}k(4k-1)(k+1) + (k+1)(2k+1) \quad \text{from } (A)$$
$$= (k+1)\left\{\frac{1}{6}k(4k-1) + (2k+1)\right\}$$
$$= (k+1)\left\{\frac{1}{6}k(4k-1) + \frac{1}{6}(12k+6)\right\}$$
$$= \frac{1}{6}(k+1)\{4k^2 - k + 12k + 6\}$$
$$= \frac{1}{6}(k+1)(4k+3)(k+2)$$
$$= RHS$$

It is true for $n = 1$ and when it is true for $n = k$ it has been proven that it is true for $n = k + 1$. Hence by mathematical induction it must be true for all positive integers n.

3 marks	Shows correct proof
2 marks	Shows proof for $n = 1$ AND shows correct assumption statement for $n = k$ and substitutes $n = k + 1$ correctly into the expression
1 mark	One of either of the points mentioned immediately above

Question 4 (continued)

(b) (i) Consider $\frac{dN}{dt} = k(N-8000)$

$$LS = \frac{dN}{dt}$$
$$= \frac{d}{dt}(8\,000 + Ae^{kt})$$
$$= Ake^{kt}$$
$$RS = k(N-8\,000)$$
$$= k(8\,000 + Ae^{kt} - 8\,000)$$
$$= Ake^{kt}$$
$$= LS$$

Have verified.

1 mark	Shows that $N = 8\,000 + Ae^{kt}$ satisfies the differential equation OR Finds the general solution to the differential equation

(ii) When $t = 0,\quad N = 25\,000$

So, $25\,000 = 8\,000 + Ae^{0}$

$A = 17\,000$

When $t = 5,\quad N = 29\,250$

So, $29\,250 = 8\,000 + 17\,000e^{5k}$

$21\,250 = 17\,000e^{5k}$

$1\cdot 25 = e^{5k}$

$5k = \log_e 1\cdot 25$

$k = 0\cdot 2\log_e 1\cdot 25$

So $A = 17\,000$ and $k = 0\cdot 2\log_e 1\cdot 25$

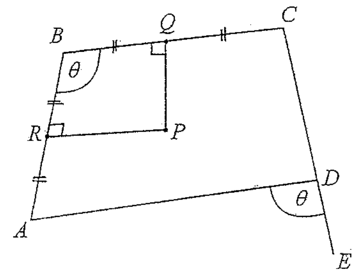
2 marks	Correct answers for A and k
1 mark	Correct answer for A OR Correct answer for k

Question 4 (continued)

(c) (i) $\angle ADE = \angle ABC = \theta$ and hence $ABCD$ is a cyclic quadrilateral because the exterior angle at a vertex of a cyclic quadrilateral equals the interior opposite angle.

1 mark	Correct answer

(ii)



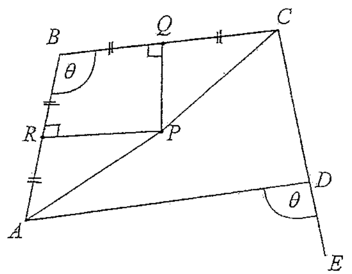
NOT TO SCALE

Since P is the centre of the circle that passes through the points A, B, C and D, then QP is the perpendicular bisector of BC and PR is the perpendicular bisector of AB.
(The line from the centre of a circle to the midpoint of a chord is perpendicular to the chord.)
So $\angle BQP = \angle BRP = 90°$
So $BQPR$ is a cyclic quadrilateral. (The opposite sides of a cyclic quadrilateral are supplementary).

2 marks	Correct reasoning
1 mark	Stating with reasons that QP and PR are the perpendicular bisectors of BC and AB respectively.

Question 4 (continued)

(iii)



Now,	$\angle QPR = 180^\circ - \theta$	(Opposite sides of a cyclic quadrilateral are supplementary.
and	$\angle APC = 2\theta$	(*A*, *B* and *C* lie on a circle and the angle at the centre of a circle is twice the angle at the circumference subtended by the same arc.
Also,	$PQ = PR$	(Equal chords are equidistant from the centre of the circle.)
	$CQ = AR$	(Since $AB = BC$ and R and Q are respective midpoints.)
	$AP = CP$	(Radii of a circle are equal.)
So $\Delta APR \equiv \Delta CPQ$		(Three pairs of sides are equal in length.)

So $\angle CPQ = \angle APR$

Now, $\angle QPR + \angle CPQ + \angle CPA + \angle APR = 360^\circ$

So, $180^\circ - \theta + \angle APR + 2\theta + \angle APR = 360^\circ$

$$\angle APR = \frac{180^\circ - \theta}{2}$$

as required.

3marks	Correct reasoning including showing $\Delta APR \equiv \Delta CPQ$
2 marks	Showing congruence correctly but incorrectly reasoning the rest of the case
1 mark	Showing correct reasoning without showing $\Delta APR \equiv \Delta CPQ$

Question 5 (12 marks)

(a) If $t = \tan\frac{x}{2}, x \neq \pi$ then $\sin x = \frac{2t}{1+t^2}$.

$\therefore \frac{3}{5} = \frac{2t}{1+t^2}$

$\therefore 3t^2 - 10t + 3 = 0$

$\therefore (3t-1)(t-3) = 0$

$\therefore t = \frac{1}{3}$ *or* 3

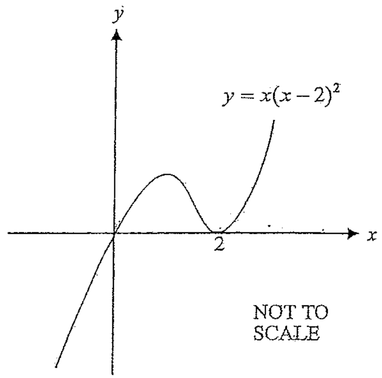
$\therefore \tan\frac{x}{2} = \frac{1}{3}$ *or* 3

But as x is acute $\tan\frac{x}{2} \neq 3$ $\therefore \tan\frac{x}{2} = \frac{1}{3}$ only.

3 marks	Correct derivation of answer
2 marks	Gives both answers for $\tan\frac{x}{2}$
1 mark	Uses a correct t result as part of the solution

Question 5 (continued)

(b) (i) The inverse function $f^{-1}(x)$ exists if the graph of $y = f(x)$ is $1:1$.
From the graph we see that $f(x)$ is not a 1:1 function.
That is, a horizontal line can be drawn that will cut the graph at more than one point.



One of the turning points is $(2,0)$.
Find the other turning point.

$$y = x(x-2)^2$$
$$= x^3 - 4x^2 + 4x$$
$$\frac{dy}{dx} = 3x^2 - 8x + 4$$
$$= (3x-2)(x-2)$$

So, $(3x-2)(x-2) = 0$

$$x = \frac{2}{3} \text{ or } x = 2$$

So $f^{-1}(x)$ exists if $a \le \frac{2}{3}$.

2 marks	Correct answer
1 mark	Giving answer $a \le \frac{2}{3}$ OR Giving an incorrect inequality as an answer after an arithmetic mistake in the differentiation or solution to the quadratic

(ii) The domain of $f(x)$ is $x \le \frac{2}{3}$.

The range of $f(x)$ is $y \le \frac{32}{27}$.

(Since $f\left(\frac{2}{3}\right) = \frac{2}{3}\left(\frac{2}{3} - 2\right)^2$
$= \frac{32}{27}$)

So the domain of $f^{-1}(x)$ is $x \le \frac{32}{27}$.

1 mark	Correct answer

Question 5 (continued)

(c) (i) Method 1 – Using the Cartesian equation

$$x^2 = 4ay$$
$$2x = 4a\frac{dy}{dx} \quad \text{(implicit differentiation)}$$
$$\frac{dy}{dx} = \frac{x}{2a}$$

At $P(2at, at^2)$,

$$\frac{dy}{dx} = \frac{2at}{2a}$$
$$= t$$

So the gradient of the normal at P is $-\frac{1}{t}$.

Equation of the normal at P:

$$y - at^2 = -\frac{1}{t}(x - 2at)$$
$$ty - at^3 = -x + 2at$$
$$x + ty = at^3 + 2at \quad \text{as required}$$

2 marks	Derives correct gradient of normal AND Derives correct equation of normal
1 mark	Derives correct gradient of normal

Method 2 – Using parametric equations

At $P(2at, at^2)$,

$x = 2at$ and $y = at^2$

$$\frac{dx}{dt} = 2a \qquad \frac{dy}{dt} = 2at$$

So $$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} \quad \text{(Chain rule)}$$
$$= 2at \cdot \frac{1}{2a}$$
$$= t$$

So the gradient of the normal at P is $-\frac{1}{t}$.

Equation of the normal at P:

$$y - at^2 = -\frac{1}{t}(x - 2at)$$
$$ty - at^3 = -x + 2at$$
$$x + ty = at^3 + 2at \quad \text{as required}$$

2 marks	Derives correct gradient of normal AND Derives correct equation of normal
1 mark	Derives correct gradient of normal

Question 5 (continued)

(ii) The normal at P passes through Q which lies on the y-axis.

Now, $x + ty = at^3 + 2at$,

when $x = 0$, $y = at^2 + 2a$

So Q and therefore R have a y-coordinate of $y = at^2 + 2a$.

Since R lies on the parabola, substitute $y = at^2 + 2a$ into $x^2 = 4ay$.

$$x^2 = 4a(at^2 + 2a)$$
$$= 4a^2t^2 + 8a^2$$
$$= 4a^2(t^2 + 2)$$

Since $\angle PQR > 90^\circ$

$$x = -2a\sqrt{t^2 + 2}$$

If $\angle PQR < 90^\circ$, $x = 2a\sqrt{t^2 + 2}$

So R has coordinates $\left(-2a\sqrt{t^2 + 2}, at^2 + 2a\right)$

2 marks	Finding two correct coordinates
1 mark	Finding one correct coordinate

(iii)

$$M = \left(\frac{-2a\sqrt{t^2+2}+0}{2}, \frac{at^2+2a+at^2+2a}{2}\right)$$
$$= \left(-a\sqrt{t^2+2}, at^2+2a\right)$$
$$x = -a\sqrt{t^2+2}$$
$$y = at^2 + 2a$$

So $x = -a\sqrt{\frac{y-2a}{a}+2}$

$$= -a\sqrt{\frac{y-2a+2a}{a}}$$

$x = -a\sqrt{\frac{y}{a}}$ is the Cartesian equation of the locus of M.

($x = -\sqrt{ay}$ is also an acceptable equation)

2 marks	Correct equation
1 mark	Correct coordinates of M

Question 6 (12 marks)

(a) (i)

$$\frac{d^2x}{dt^2} = 9(x-2)$$
$$\frac{d\left(\frac{1}{2}v^2\right)}{dx} = 9(x-2)$$
$$\frac{1}{2}v^2 = 9\int(x-2)\,dx$$
$$= 9\left(\frac{x^2}{2} - 2x\right) + c$$

Initially $x = 4$ and $v = -6$

So, $18 = 9(8-8) + c$

$$c = 18$$
$$\frac{1}{2}v^2 = 9\left(\frac{x^2}{2} - 2x\right) + 18$$
$$v^2 = 9x^2 - 36x + 36$$
$$= 9(x^2 - 4x + 4)$$
$$= 9(x-2)^2$$

as required

2 marks	Correct integration AND Correct evaluation of c
1 mark	Statement that $\frac{d\left(\frac{1}{2}v^2\right)}{dx} = 9(x-2)$

Question 6 (continued)

(ii) From (i), $v^2 = 9(x-2)^2$

So $v = \pm 3(x-2)$, but

when $t = 0$, $x = 4$ and $v = -6$.

So, $v = -3(x-2)$

So, $\frac{dx}{dt} = -3(x-2)$

$$\frac{dt}{dx} = \frac{-1}{3(x-2)}$$

$$t = -\frac{1}{3}\int \frac{1}{x-2}\,dx$$

$$= -\frac{1}{3}\log_e(x-2) + c$$

$t = 0$, $x = 4$

$$0 = -\frac{1}{3}\log_e 2 + c$$

$$t = \frac{1}{3}\log_e \frac{2}{x-2}$$

$$3t = \log_e \frac{2}{x-2}$$

$$e^{3t} = \frac{2}{x-2}$$

$$x - 2 = 2e^{-3t}$$

$$x = 2\left(1 + e^{-3t}\right)$$

2 marks	Correct expression for $x(t)$
1 mark	Correct expression for $v(x)$

(iii) From (ii), $v = -3(x-2)$

If $v = 0$, $0 = -3(x-2)$

So $x = 2$

From (ii) also, $x = 2\left(1 + e^{-3t}\right)$

If $x = 2$, $2 = 2\left(1 + e^{-3t}\right)$

$e^{-3t} = 0$ which has no solution.

So v is never zero.

Alternatively, the graph of $x = 2\left(1 + e^{-3t}\right)$ has an asymptote of $x = 2$ and so $x \neq 2$ so $v \neq 0$.

2 marks	Correct explanation involving $v(x)$ and $x(t)$
1 mark	Correct explanation only as far as involving $v(x)$

Question 6 (continued)

(b) (i) Let O be at the foot of the UFO to the ground.

Let OB = y and OA = x.

$\therefore \tan 18^0 = \frac{h}{y}$ $\therefore y = h\cot 18^0$ ....1.

$\therefore \tan 20^0 = \frac{h}{x}$ $\therefore x = h\cot 20^0$ ....2.

$\therefore 2500^2 = x^2 + y^2 - 2xy\cos 18^0$ ....3.

Sub. 1 and 2 into 3:

$$\therefore 2500^2 = h^2\cot^2 20^0 + h^2\cot^2 18^0 - 2h\cot 20^0 \times h\cot 18^0 \cos 18^0$$

$$\therefore 2500^2 = h^2(\cot^2 20^0 + \cot^2 18^0 - 2\cot 20^0 \cot 18^0 \cos 18^0)$$

$$\therefore h = \frac{2500}{\sqrt{\cot^2 20^0 + \cot^2 18^0 - 2\cot 20^0 \cot 18^0 \cos 18^0}}$$

4 marks	Correct derivation.
3 marks	Derivation to second last equation.
2 marks	Listing of the three equations needed in derivation.
1 mark	Listing of at least two of the equations needed in the derivation.

(ii) $h = 2583.015192.....$

$\therefore$ height, $h = 2583$ m (to nearest metre)

2 marks	Finds correct answer
1 mark	Does not round off correctly to nearest metre.

Question 7 (12 marks)

(a) (i) $8\sin\theta - 15\cos\theta = R[\frac{8}{R}\sin\theta - \frac{15}{R}\cos\theta]$

where $R = \sqrt{8^2 + (-15)^2} = 17$.

$\therefore 8\sin\theta - 15\cos\theta = 17\sin(\theta - \alpha)$

where $\cos\alpha = \frac{8}{17}, \sin\alpha = \frac{15}{17}$

such that $\tan\alpha = \frac{15}{8}$ and $\alpha = \tan^{-1}\frac{15}{8}$ for acute α.

$\therefore\ 8\sin\theta - 15\cos\theta = 17\sin(\theta - \tan^{-1}\frac{15}{8})$.

1 mark	Correct derivation

(ii) If $8\sin\theta - 15\cos\theta = \frac{-17\sqrt{3}}{2}$

then $17\sin(\theta - \tan^{-1}\frac{15}{8}) = \frac{-17\sqrt{3}}{2}$

$\therefore\quad \sin(\theta - \tan^{-1}\frac{15}{8}) = \frac{-\sqrt{3}}{2}$

$\sin(\theta - \tan^{-1}\frac{15}{8}) = \sin(-60^0)$

$\theta - \tan^{-1}\frac{15}{8} = 180^0 n + (-1)^n(-60^0)$

$\theta = 180^0 n + (-1)^n(-60^0) + 61^056'$ to the nearest minute

where n is any integer.

3 marks	Correct derivation
2 marks	Correctly finding θ but not qualifying what n is.
1 mark	Correct to 3rd last line of working.

(b) Let the roots of $x^3 - 12x^2 + 12x + m = 0$ be a-d, a, a+d.

Now sum of roots = a-d + a + a+d

= 3a

= 12

$\therefore a = 4$

$\Rightarrow x = 4$ is a root and $P(4) = 0$ where $P(x) = x^3 - 12x^2 + 12x + m$.

$\therefore 0 = 64 - 192 + 48 + m$

$0 = -80 + m$

$\therefore m = 80$

Now $P(x) = x^3 - 12x^2 + 12x + 80$

and the product of the roots = (a-d) a (a+d)

$= a(a^2 - d^2)$

$= 4(16 - d^2)$

$= -80$

$\therefore 16 - d^2 = -20$

$\therefore d^2 = 36$

$\therefore d = \pm 6$

$\Rightarrow$ roots are $-2, 4$ and 10.

(c) $I = \int_0^3 \frac{dx}{9 + x^2}$ letting $x = 3\tan\theta$ $\quad\therefore \frac{dx}{d\theta} = 3\sec^2\theta$

when $x = 0$ $\theta = 0$

$x = 3$ $\theta = \frac{\pi}{4}$

$\therefore\quad I = \int_0^{\frac{\pi}{4}} \frac{3\sec^2\theta\ d\theta}{9 + 9\tan^2\theta}$

$= \int_0^{\frac{\pi}{4}} \frac{3\sec^2\theta\ d\theta}{9(1 + \tan^2\theta)}$

$= \int_0^{\frac{\pi}{4}} \frac{3\sec^2\theta\ d\theta}{9\sec^2\theta}$

$= \int_0^{\frac{\pi}{4}} \frac{1}{3}\ d\theta$

$= \left[\frac{\theta}{3}\right]_0^{\frac{\pi}{4}}$

$= \frac{\pi}{12}$

4 marks	Correct derivation
3 marks	Correctly derived to third last line of working.
2 marks	Correctly derived to fifth last line of working.
1 mark	Correctly finding $\frac{dx}{d\theta}$ and changing the values of the limits with respect to θ.