CRANBROOK
SCHOOL

Term 3, 2008

# Year 12 Mathematics Extension 1

# Trial Examination

July, 2008

**Time Allowed:** 2 hours, plus 5 minutes reading time

**Total Marks:** 84

There are 7 questions, all of equal value
Submit your work in seven 4 Page Booklets.
All necessary working should be shown in every question. Full marks may not be awarded if work is careless or badly arranged. Board of Studies approved calculators may be used. A list of standard integrals is attached to the back of this paper.

**Question 1** (12 marks) **Marks**

(a) Evaluate $\int_0^3 \frac{dx}{\sqrt{9-x^2}}$ . **2**

(b) Solve the inequality, $\frac{2-3x}{7x+2} \leq -2$ **3**

(c) The polynomial $P(x) = 2x^3 + kx^2 - 1$ is divided by $x+2$ and the remainder is 7. Find the value of $k$. **2**

(d) Find $\frac{d}{dx}\left(e^x \tan^{-1} x\right)$ . **2**

(e) Evaluate $\int_0^{2\pi} \cos^2 2x \, dx$. **3**

**Question 2** (12 marks) **Marks**

(a) Given that $A$ is the point (4,1) and $B$ is the point (-1,5), find the point $P$ which divides the interval $AB$ externally in the ratio 2:7. **2**

(b) (i)
The function $f(x) = x - e^{-2x}$ has one root which lies between $x = 0$ and $x = 1$. Use the 'halving the interval' method to find an approximation to the root. **2**
(ii)
How many times must we halve the interval to ensure that it is correct to 1 decimal place. **1**

(c) Find $\int_0^2 2x\sqrt{1-\frac{x}{2}}\, dx$ using the substitution $u = 1 - \frac{x}{2}$. **3**

(d) By graphing or otherwise, simplify

(i) $\sin^{-1} x + \sin^{-1}(-x)$ **1**

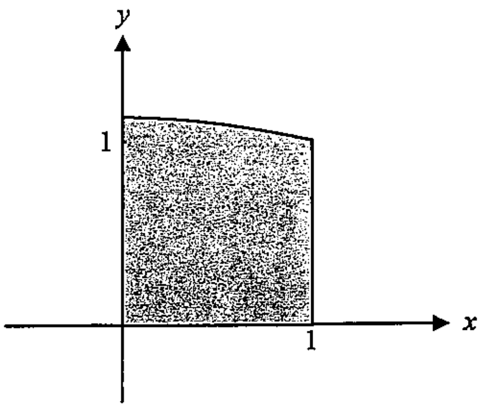
(ii) $\tan^{-1} x + \tan^{-1}(-x)$ **1**

(iii) $\sin^{-1} x - \cos^{-1}(-x)$ **2**

**Question 3** (12 marks) **Marks**

(a) Find all the solutions to the equation $\cos\theta = \cos 2\theta$ for $0 \le \theta \le 2\pi$. **4**

(b)




The region bounded by the function $y = \dfrac{2}{\sqrt{x^2+3}}$, the $x$-axis, the $y$-axis and the line $x = 1$, is shaded in the diagram above. Find the volume of the solid formed when this region is rotated about the $x$-axis. **3**

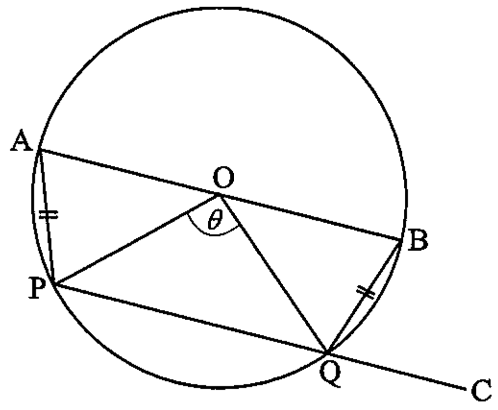
(c) Let $f(x) = \cos x + \tan\left(\dfrac{x}{2}\right)$, $-\dfrac{\pi}{2} \le x \le 0$.

(i) State whether or not the inverse function $f^{-1}(x)$ exists. Give reasons for your answer. **2**

(ii) The function $f(x)$ has a zero near $x = -1$. Use Newton's method once to find a closer approximation to the zero. Express your answer correct to 3 significant figures. **3**

**Question 4** (12 marks) **Marks**

(a)




The points $A$, $B$, $P$ and $Q$ lie on the circle which has its centre at $O$. $AB$ is a diameter and $PC$ passes through the point $Q$. $AP$ is equal to $BQ$ and $\angle POQ = \theta$.

(i) Express $\angle AOP$ in terms of $\theta$. **2**

(ii) Prove that $AB$ is parallel to $PC$. **2**

(b) Prove by induction that **3**

$$4n + 3n^2 + 2n^3$$

is divisible by 3 for $n = 1,2,3...$

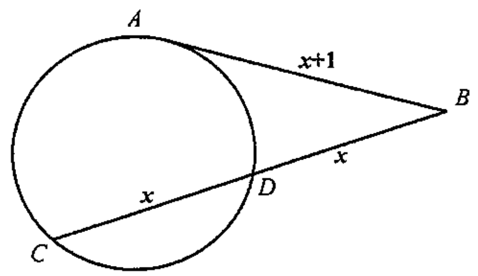
(c)
Billy Bob and Betty Boop were bush wacking in a forest near a fire tower standing 200m in height. Billy was 200m due east of the tower, while Betty was in a direction south west of the tower. The angle of elevation from Betty to the tower is $60°$.

(i) Draw a diagram showing this information **1**

(ii) Find their distance apart to the nearest metre. **4**

**Question 5** (12 marks) **Marks**

(a) Let AB be a tangent to the circle as shown. Find $x$ in exact form where the distances $CD$ and $DB$ are both equal to $x$, while the distance $AB$ is $x+1$.




2

(b) A tray of meat is taken out of the freezer at $-9°C$ and allowed to thaw in the air at $25°C$. The rate at which the meat warms follows Newton's law of cooling and so $\frac{dT}{dt} = -k(T-25)$, with time $t$ measured in minutes.

(i) Show that $T = 25 - Ae^{-kt}$ is a solution of this equation, and find the value of A. **2**

(ii) The meat reaches $8°C$ in 45 minutes. Find the temperature it reaches after another 45 minutes. **2**

(d) If the volume of a spherical ball is increasing at the rate of $29\text{cm}^3\text{s}^{-1}$, find the rate of increase of the surface area when its radius is 60cm. **2**

(e)The velocity $v$ $\text{ms}^{-1}$ of a particle moving in simple harmonic motion along the $x$-axis is given by

$$v^2 = 8 + 2x - x^2$$

(i) Between which two points is the particle oscillating? **1**

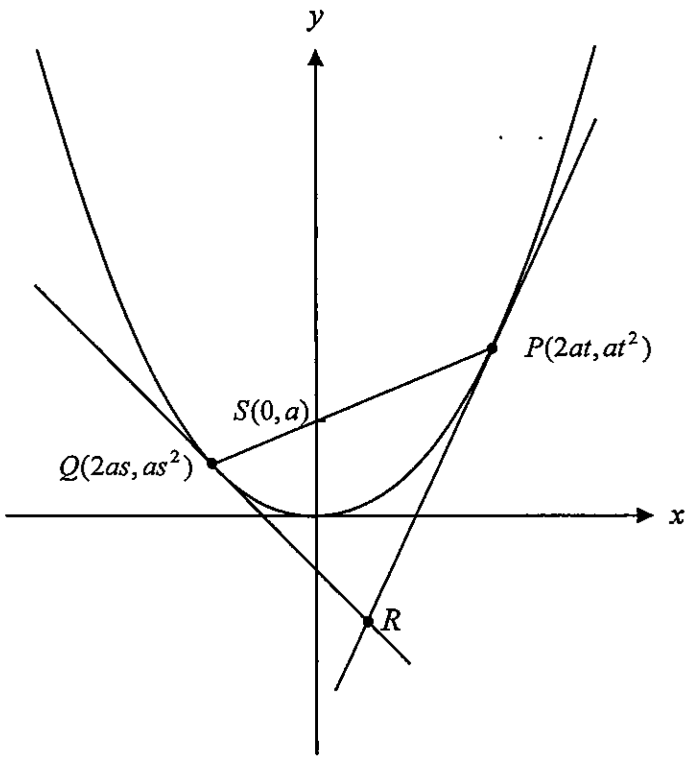
(ii) What is the amplitude of the motion? **1**

(iii) Find the acceleration of the particle in terms of $x$ **1**

(iv) Find the period of the oscillation. **1**

**Question 6** (12 marks) **Marks**

(a)




The points $P(2at, at^2)$ and $Q(2as, as^2)$ are variable points on the parabola $x^2 = 4ay$ such that the chord $PQ$ passes through the point $S(0, a)$.
The tangents to the parabola at points $P$ and $Q$ intersect at the point $R$.

(i) Show that the equation of the tangent to the parabola at $P$ is $y = tx - at^2$. **2**

(ii) Show that $st = -1$. **2**

(iii) Show that the locus of the point $R$ is coincident with the directrix of the parabola. **3**

*Question 6 continues on the next page.*

(b) A particle moves in a straight line so that at time $t, (t \geq 0)$ its acceleration $a$, is given by

$$a = 4x$$

where $x$ is the displacement of the particle from the origin. The particle starts its journey one metre to the right of the origin $(\text{at } x = 1)$ with a velocity of $v = -2$.

(i) Show that $v = -2x$. **2**

(ii) Express $x$ as a function of $t$. **2**

(iii) Explain whether or not the particle ever moves to the left of the origin. **1**

**Question 7** (12 marks) **Marks**

(a) A particle moves in a straight line and its position at time $t$ seconds is given by

$$x = \sqrt{3}\sin\frac{t}{2} + \cos\frac{t}{2}$$

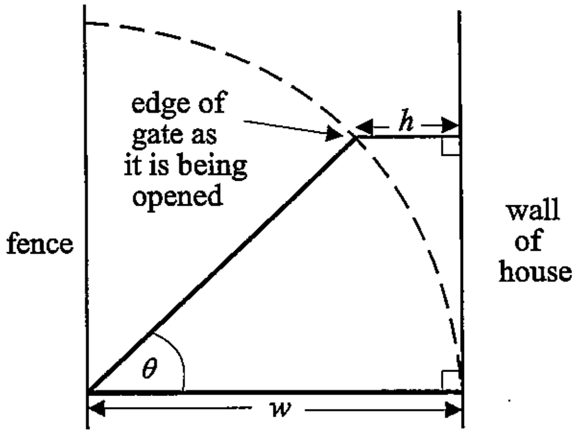
(i) Find an expression for $x$ in the form $R\sin\left(\frac{t}{2} + \alpha\right)$, where $R$ is a real number and $0 \leq \alpha \leq 2\pi$. **2**

(ii) Hence prove that the particle is moving in simple harmonic motion about $x = 0$. **2**

(iii) For $0 < t < 4\pi$, when is the speed of the particle equal to $0.5\,\text{ms}^{-1}$? **1**

*Question 7 continues on the next page.*

(b)




A gate $w$ metres long, is closed when it is at right angles to a fence and the wall of a house. The fence and the wall are parallel. The gate opens out towards the fence. The horizontal opening, created as the gate is opened is the distance from the edge of the gate to the house so that it meets the house wall at right angles. This distance $h$, in metres, is shown in the diagram above. Let $\theta(t)$ radians be the angle of opening of the gate at time $t$ seconds.

The gate is initially shut, but is opened at a rate of $\dfrac{1}{\sqrt{1-t^2}}$ radians per second.

(i) Show that $h = w - w\cos\theta$. **1**

(ii) Show that $\theta = \sin^{-1} t$. **1**

(iii) Show that $\dfrac{dh}{dt} = \dfrac{wt}{\sqrt{1-t^2}}$. **1**

(iv) Using the substitution $u = 1 - t^2$, find an expression for $h$ in terms of $t$. **2**

(v) Hence, using calculus, or otherwise, explain why $\displaystyle\int_0^1 \frac{t}{\sqrt{1-t^2}}\,dt = 1$. **2**

## STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1;\ x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \ a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \ a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \ a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, a > 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

**NOTE :** $\ln x = \log_e x, \quad x > 0$

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**MATHEMATICS EXTENSION 1**
**HIGHER SCHOOL CERTIFICATE**
**TRIAL EXAMINATION**
**SOLUTIONS**

**Question 1** (12 marks)

(a) $\displaystyle\int_0^3 \frac{dx}{\sqrt{9-x^2}} = \left[\sin^{-1}\frac{x}{3}\right]_0^3$ **(1 mark)**

$= \sin^{-1}1 - \sin^{-1}0$

$= \dfrac{\pi}{2}$

(b) $\dfrac{2-3x}{7x+2} \le -2$

$\dfrac{(2-3x)}{7x+2} + \dfrac{2(7x+2)}{7x+2} \le 0$

Now $\times\ (7x+2)^2$

$(2-3x+2(7x+2))(7x+2) \le 0$

$(11x+6)(7x+2) \le 0$

$\rightarrow -\dfrac{6}{11} \le x < -\dfrac{2}{7}$

NOTE: no need to expand! * Many wasted time (3 MARKS)

(c)

$P(x) = 2x^3 + kx^2 - 1$

So, $P(-2) = -16 + 4k - 1$

So, $4k - 17 = 7$

$k = 6$

More care needed on simple equation solving!

(d) $\dfrac{d}{dx}\left(e^x \tan^{-1} x\right) = e^x \tan^{-1} x + \dfrac{e^x}{1+x^2}$

V Well done

**(1 mark)** for first term
**(1 mark)** for second term

(e) $\displaystyle\int_0^{2\pi} \cos^2 2x\,dx = \int_0^{2\pi} \frac{1}{2}(1+\cos 4x)\,dx$ since $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$ **(1 mark)**

$= \dfrac{1}{2}\left[x + \dfrac{1}{4}\sin 4x\right]_0^{2\pi}$ **(1 mark)**

$= \dfrac{1}{2}\left\{\left(2\pi + \dfrac{1}{4}\sin 8\pi\right) - \left(0 + \dfrac{1}{4}\sin 0\right)\right\}$

$= \pi$

**(1 mark)**

Some did not know this LEARN!

**Question 2** (12 marks)

(a)

$$P=\left(\frac{mx_2+nx_1}{m+n},\frac{my_2+ny_1}{m+n}\right)$$

where $m=-2$ and $n=7$

$$\text{So, } P=\left(\frac{-2\times -1+7\times 4}{-2+7},\frac{-2\times 5+7\times 1}{-2+7}\right)$$

$$=\left(6,\frac{-3}{5}\right)$$ ✓✓

good but a few did not know rule! – NO EXCUSE!!

(b) $f(x)=x-e^{-2x}$ $\quad x_1=\frac{1+0}{2}=\frac{1}{2}$ ✓✓

| x (fraction) | 0 | 1 | $\frac{1}{2}$ | $\frac{1}{4}$ | $\frac{3}{8}$ | $\frac{7}{16}$ |
|---|---|---|---|---|---|---|
| x (decimal) | 0 | 1 | 0.5 | 0.25 | 0.375 | 0.4375 |
| $f(x)$ | -1 | 0.86 | 0.132 | -0.357 | -0.097 | 0.021 |

Since we know that the root lies between $x=\frac{7}{16}$ and $x=\frac{3}{8}$ then we know that the root is equal to 0.4 (correct to 1 decimal place). ✓

∴ 4 TIMES IS REQUIRED NUMBER OF APPLICATIONS

(c)

$$\int_0^2 2x\sqrt{1-\frac{x}{2}}\,dx$$

let $u=1-\frac{x}{2}$ so $\frac{du}{dx}=-\frac{1}{2}$ | If $x=2$, $u=0$

So $x=-2(u-1)$ | and if $x=0$, $u=1$

$$=\int_1^0 -4(u-1)u^{\frac{1}{2}}\times -2\frac{du}{dx}\,dx$$

$$=8\int_1^0\left(u^{\frac{3}{2}}-u^{\frac{1}{2}}\right)du$$

**(1 mark)** for function
**(1 mark)** for terminals

$$=8\left[\frac{2u^{\frac{5}{2}}}{5}-\frac{2}{3}u^{\frac{3}{2}}\right]_1^0$$

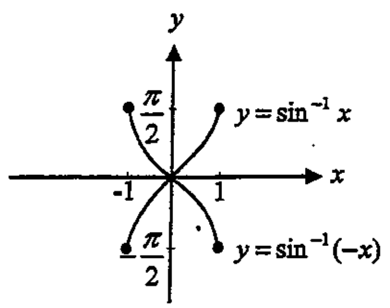
$$=8\left\{(0-0)-\left(\frac{2}{5}-\frac{2}{3}\right)\right\}$$

$$=\frac{32}{15}$$

✓ Well done though some needed to be more careful with coefficients

**(1 mark)**

(d) (i) Method 1 – graphically




not well done – as many did not sketch.

Using addition of ordinates, we see that

$\sin^{-1}x + \sin^{-1}(-x) = 0$ **(1 mark)**

Method 2 – algebraically

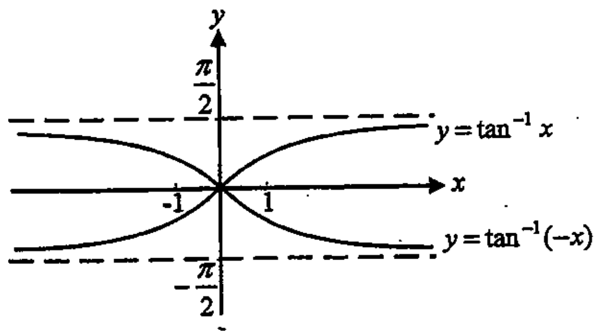
$\sin^{-1}x + \sin^{-1}(-x)$

$= \sin^{-1}x - \sin^{-1}x$ (because $\sin^{-1}x$ is an odd function and hence

$= 0$ $\sin^{-1}(-x) = -\sin^{-1}(x)$)

**(1 mark)**

(ii) Method 1 – graphically




Using addition of ordinates, we see that

$\tan^{-1}x + \tan^{-1}(-x) = 0$ **(1 mark)**

Method 2 – algebraically

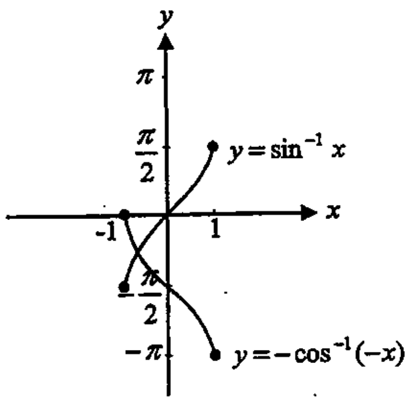
$\tan^{-1}x + \tan^{-1}(-x)$

$= \tan^{-1}x - \tan^{-1}x$ (because $\tan^{-1}x$ is an odd function and hence

$= 0$ $\tan^{-1}(-x) = -\tan^{-1}x$ )

**(1 mark)**

(iii) Method 1 – graphically




**(1 mark)**

Using addition of ordinates, we see that

$\sin^{-1}x - \cos^{-1}(-x) = \frac{-\pi}{2}$

**(1 mark)**

Method 2 – algebraically

$\sin^{-1}x - \cos^{-1}(-x)$

$= \sin^{-1}x - \pi + \cos^{-1}x$ **(1 mark)**

$= \frac{\pi}{2} - \pi$

$= -\frac{\pi}{2}$

**(1 mark)**

(because $\cos^{-1}(-x) = \pi - \cos^{-1}x$ and $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$)

**Question 3** (12 marks)

(a)

$$\cos\theta = \cos 2\theta \qquad 0 \le \theta \le 2\pi$$

So $\cos\theta = 2\cos^2\theta - 1$ (Double angle formula) **(1 mark)**

So $2\cos^2\theta - \cos\theta - 1 = 0$

$(2\cos\theta + 1)(\cos\theta - 1) = 0$ **(1 mark)**

$\cos\theta = -\dfrac{1}{2}$ or $\cos\theta = 1$

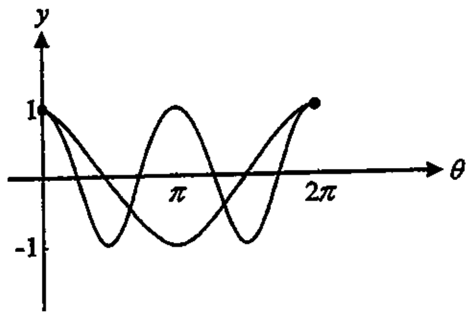
$\theta = \dfrac{2\pi}{3}, \dfrac{4\pi}{3}$ or $\theta = 0, 2\pi$

**(1 mark)** **(1 mark)**

Surprising number had trouble with quadratic

So, all the solutions are $0, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}$ and $2\pi$.

Note, that a quick sketch of the graphs of $y = \cos\theta$ and $y = \cos 2\theta$ for $0 \le \theta \le 2\pi$ will confirm that there are 4 solutions.




(b) Volume required $= \pi \displaystyle\int_0^1 \frac{4}{x^2+3}\,dx$ **(1 mark)**

$$= \frac{4\pi}{\sqrt{3}} \int_0^1 \frac{\sqrt{3}}{x^2+3}\,dx$$

$$= \frac{4\pi}{\sqrt{3}} \left[ \tan^{-1} \frac{x}{\sqrt{3}} \right]_0^1$$ **(1 mark)**

$$= \frac{4\pi}{\sqrt{3}} \left( \tan^{-1} \frac{1}{\sqrt{3}} - \tan^{-1} 0 \right)$$

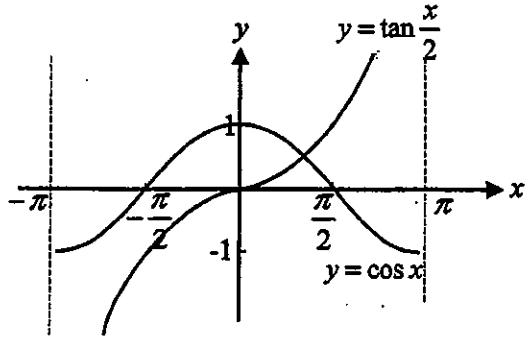
$$= \frac{4\pi}{\sqrt{3}} \times \frac{\pi}{6}$$

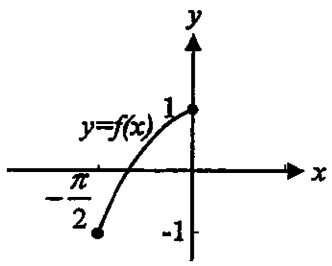
$$= \frac{2\sqrt{3}\pi^2}{9} \text{ cubic units.}$$ **(1 mark)**

Very Well done

(c) (i) Using addition of ordinates and the graphs of $y = \cos x$ and $y = \tan\left(\frac{x}{2}\right)$ sketch the graph of $f(x)$.

OR
Differentiate and show always increasing
ie $y' = -\sin x + \frac{1}{2}\sec^2\left(\frac{x}{2}\right)$ which is positive in the given domain since $\sin x < 0$ for $-\frac{\pi}{2} < x < 0$




**(1 mark)**

Since $f(x)$ is a $1:1$ function, that is, no horizontal line crosses the function more than once, the inverse function $f^{-1}(x)$ exists.

**(1 mark)**

(ii) Now $f(x) = \cos x + \tan\left(\frac{x}{2}\right)$

So, $f'(x) = -\sin x + \frac{1}{2}\sec^2\left(\frac{x}{2}\right)$ **(1 mark)**

Using Newton's method we have,

$$x_1 = -1 - \frac{f(-1)}{f'(-1)}$$

where $x_1$ is a closer approximation than $x = -1$.

So, $$x_1 = -1 - \frac{\cos(-1) + \tan\left(-\frac{1}{2}\right)}{-\sin(-1) + \frac{1}{2}\sec^2\left(-\frac{1}{2}\right)}$$ **(1 mark)**

$= -0 \cdot 995974$

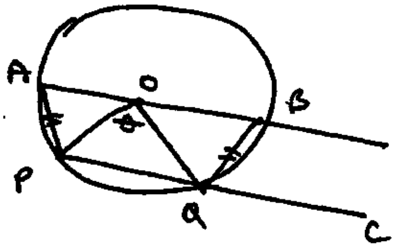
$= -0 \cdot 996$ (correct to 3 significant figures)

**(1 mark)**

Good knowledge of rules but not of Calculator!!

QUESTION 4

a)




i) $\angle AOP = \angle BOQ$ (equal chords subtend equal angles at the centre) ✓

let $\angle AOP = x$ ∴ $\angle BOQ = x$

∴ $2x + \theta = 180°$ (straight angle)

$2x = 180 - \theta$

$x = \dfrac{180° - \theta}{2}$ ✓

ii) $\Delta OPQ$ is isosceles (2 sides equal as OP and OQ radii)

∴ $\angle OPQ = \angle OQP$ (base ∠'s equal)

∴ $\angle OPQ = \dfrac{180° - \theta}{2}$ (angle sum of $\Delta OPQ$ is 180°) ✓

∴ $\angle OPQ = \angle AOP$ and they are in alternate positions ✓

∴ $AB \parallel PC$

b) R.T.P:

$4n + 3n^2 + 2n^3$ is divisible by 3

1. Prove true for $n = 1$

∴ $4 + 3 + 2 = 9$

which is divisible by 3 ✓

∴ true for $n = 1$

2. Assume true for $n = k$, $k \in J$

i.e. assume $4k + 3k^2 + 2k^3 = 3P$ ✓

∴ $2k^3 = 3P - 4k - 3k^2$

3. Prove true for $n = k+1$, $k \in J$

Prove $4(k+1) + 3(k+1)^2 + 2(k+1)^3 = 3Q$

LHS $= 4k + 4 + 3(k^2 + 2k + 1) + 2(k^3 + 3k^2 + 3k + 1)$

$= 4k + 4 + 3k^2 + 6k + 3 + 2k^3 + 6k^2 + 6k + 2$

$= 2k^3 + 9k^2 + 16k + 9$

$= 3P - 4k - 3k^2 + 9k^2 + 16k + 9$

$= 3P + 6k^2 + 12k + 9$ ✓

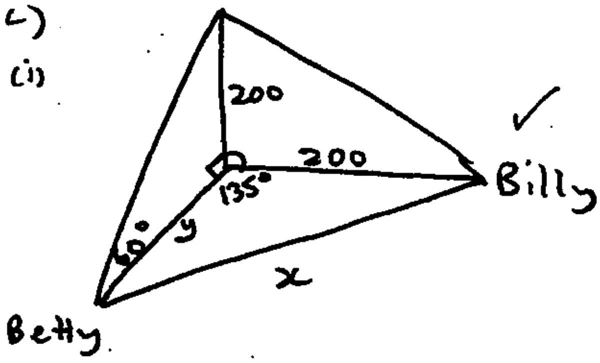
$= 3(P + 2k^2 + 4k + 3)$

which is divisible by 3

S4 ∴ If it is true for $n = k$, then it is also true for $n = k+1$. But it is true for $n = 1$, so it is true for $n = 2$ and so on ✓

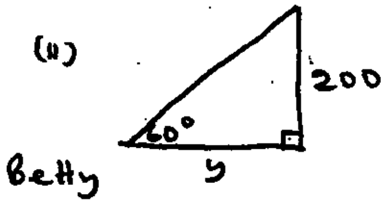
c)

(i)




✓

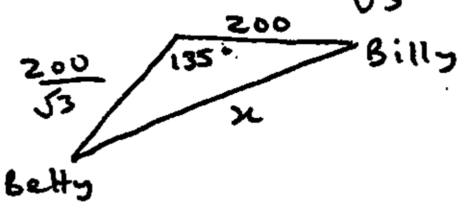
(ii)




$\tan 60° = \dfrac{200}{y}$ ✓

$y = \dfrac{200}{\tan 60°}$

$y = \dfrac{200}{\sqrt{3}}$ (115.47....)




✓

$x^2 = \left(\dfrac{200}{\sqrt{3}}\right)^2 + 200^2 - 2\left(\dfrac{200}{\sqrt{3}}\right)200\cos 135$

$= 85993.196.....$

$x = 293.245....$

∴ They are 293m apart. ✓

QUESTION 5

a) $AB^2 = CB \cdot DB$

$(x+1)^2 = 2x(x)$ ✓

$x^2 + 2x + 1 = 2x^2$

$\therefore x^2 - 2x - 1 = 0$

$x = \dfrac{2 \pm \sqrt{4 - 4(1)(-1)}}{2}$

$x = \dfrac{2 \pm \sqrt{8}}{2}$

$x = \dfrac{2 \pm 2\sqrt{2}}{2}$

$x = 1 \pm \sqrt{2}$

but $x > 0$ (it's a length)

$\therefore x = 1 + \sqrt{2}$ only ✓

i) $T = 25 - Ae^{-kt}$

$\therefore Ae^{-kt} = T - 25$

Now $\dfrac{dT}{dt} = kAe^{-kt}$ ✓

$= k(T - 25)$

$= -k(25 - T)$ as required

$T = 25 - Ae^{-kt}$

$t=0$, $T=-9$ } $\therefore -9 = 25 - Ae^0$

$-9 = 25 - A$

$\therefore A = 34$ ✓

) $T = 25 - 34e^{-kt}$

$T=8$, $t=45$ } $\therefore 8 = 25 - 34e^{-45k}$

$17 = 34e^{-45k}$

$\dfrac{1}{2} = e^{-45k}$

$-45k = \ln\left(\frac{1}{2}\right)$

$k = \dfrac{\ln\left(\frac{1}{2}\right)}{-45}$ ✓

$= 0.0154\ldots$

$t=90$ $\therefore T = 25 - 34^{-0.0154\ldots \times 90}$

$= 25 - 8.5$

$= 16.5°C$ ✓

c) $\dfrac{dV}{dt} = 29$ $\quad \dfrac{dA}{dt} = ?$

$\dfrac{dA}{dt} = \dfrac{dV}{dt} \times \dfrac{dA}{dV}$

$= \dfrac{dV}{dt} \times \dfrac{dA}{dr} \times \dfrac{dr}{dV}$ ✓

Now $V = \frac{4}{3}\pi r^3$ and $A = 4\pi r^2$

$\dfrac{dV}{dr} = 4\pi r^2$ and $\dfrac{dA}{dr} = 8\pi r$

$\therefore \dfrac{dA}{dt} = 29 \times 8\pi r \times \dfrac{1}{4\pi r^2}$

$= \dfrac{58}{r}$

$= \dfrac{58}{60} = \dfrac{29}{30} = 0.9\dot{6}\ cm^2 s^{-1}$ ✓

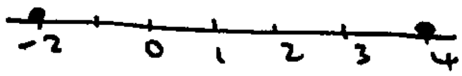
e) $v^2 = 8 + 2x - x^2$

(i) $v = 0$ $\therefore x^2 - 2x - 8 = 0$

$(x-4)(x+2) = 0$

$x = 4,\ x = -2$ ✓

(ii)




$\therefore$ length of path is 6 units

$\therefore$ amplitude is 3 ✓

(iii) $\ddot{x} = \dfrac{d}{dx}\left(\frac{1}{2}v^2\right)$

$= \dfrac{d}{dx}\left(4 + x - \frac{1}{2}x^2\right)$

$= 1 - x$ ✓

$\ddot{x} = -1(x-1)$

$= -n^2X$ where $n = 1$

(iv) period $= \dfrac{2\pi}{n}$

$= 2\pi$ ✓

Markers Notes

Question 5

a) Learn your circle geometry rules! Don't be lazy when solving quadratic equations. Use the quadratic formula if you have to. Finally THINK!! $x$ is a length so it can't be negative

b) Bread and butter question. Easy marks if you know the method.

i) $A \neq -9$ find $A$ by substitution

ii) done well

c) $\frac{dA}{dV}$ cannot be found. Need to know $\frac{dA}{dr}$ and $\frac{dV}{dr}$. Learn S.A. and vol formulae for spheres (and cones in future)

d) Too many students tried to apply ridiculous formula like $v^2 = n^2(a^2 - x^2)$ instead of just thinking!!

QUESTION 4

a) Circle geometry rules poorly known.

This question can be done by congruent Δ's but full proofs must be given

b) Learn how to do induction. It is rigorous and precise

Learn the correct conclusion

Show $n=1$ is true (easy mark)

Use the assumption step (that's why you do it)

Induction is always asked!

c) SW means an angle of 45°

The ground level Δ is not right angled.

Make your diagrams simple!

If you can't draw in 3D, then do several 2D triangles

6 (a) (i) $x^2 = 4ay$

$\therefore y = \frac{x^2}{4a}$

$\therefore \frac{dy}{dx} = \frac{2x}{4a} = \frac{x}{2a}$

At $P(2at, at^2)$ $\frac{dy}{dx} = \frac{2at}{2a} = t$

$\therefore m_{tang} = t$ ✓

$\therefore$ Eqn of tangent at P is:

$y - at^2 = t(x - 2at)$

$\therefore y - at^2 = tx - 2at^2$

$\therefore y = tx - at^2$ ✓

(ii) Eqn of PQ is:

$\frac{y - at^2}{x - 2at} = \frac{as^2 - at^2}{2as - 2at}$

$= \frac{a(s-t)(s+t)}{2a(s-t)}$

$= \frac{s+t}{2}$ ✓

Now as PQ passes through $S(0, a)$

$\therefore \frac{a - at^2}{-2at} = \frac{s+t}{2}$

$\therefore 2a - 2at^2 = -2ast - 2at^2$

$\therefore 2a = -2ast$

$\therefore -1 = st$ ✓

(iii) From (i) eqn of tangent at P is:

$y = tx - at^2$ — ①

similarly eqn of tangent at Q is:

$y = sx - as^2$ — ②

① − ②: $0 = x(t-s) - a(t^2 - s^2)$

$\therefore x = \frac{a(t-s)(t+s)}{(t-s)}$ ✓

$\therefore x = a(t+s)$ sub into ①

$\therefore y = at(t+s) - at^2$

$\therefore y = at^2 + ast - at^2$

$\therefore y = ast$ ✓

$\Rightarrow R = (a(t+s), ast)$

But from (ii) $st = -1$

$\therefore R = (a(t+s), -a)$

$\therefore$ locus of R is $y = -a$ ✓

which is coincident with the directrix of parabola $y = -a$.

(b) (i) $a = 4x = \frac{d}{dx}(\frac{1}{2}v^2)$ ✓

$\therefore \frac{1}{2}v^2 = \frac{4x^2}{2} + c$

$\frac{1}{2}v^2 = 2x^2 + c$

when $t = 0$, $x = 1$, $v = -2$

$\therefore 2 = 2 + c$ $\therefore c = 0$

$\therefore \frac{1}{2}v^2 = 2x^2$

$\therefore v^2 = 4x^2$ ✓

$\therefore v = -2x$, as initially velocity of particle was in negative directio

(ii) Now $v = \frac{dx}{dt}$

$\therefore -2x = \frac{dx}{dt}$

$\therefore \frac{-1}{2x} = \frac{dt}{dx}$

$\therefore t = -\frac{1}{2}\int \frac{1}{x}\,dx$ ✓

$t = -\frac{1}{2}\ln|x| + c$ ✓

when $t = 0$, $x = 1$ $\therefore 0 = 0 + c$

$\therefore c = 0$ $\therefore t = -\frac{1}{2}\ln|x|$ ✓

$\therefore x = e^{-2t}$ ✓

(iii) If particle is at the left of the origin $x < 0$ and $e^{-2t} < 0$ ✓

But [illegible] for all $t$ ✓

$\therefore$ [illegible] ves to left of origin.

(c) (i) $P(\text{all boys}) = \frac{{}^{16}C_4 \times {}^{9}C_0}{{}^{25}C_4}$

$= \frac{1820 \times 1}{12650}$

$= \frac{182}{1265}$

**(1 mark)**

(ii) More boys will occur if there are 3 boys and 1 girl in the team or if there are 4 boys.

$P(\text{more girls than boys}) = P(\text{3 boys and a girl}) + P(\text{4 boys})$

$= \frac{{}^{16}C_3 \times {}^{9}C_1}{{}^{25}C_4} + \frac{182}{1265}$ (from part(i))

$= \frac{686}{1265}$

**(1 mark)**

(iii) Since the youngest boy is included we need choose only 3 people and we don't choose him or the oldest girl. So, we are choosing 3 people from a total of 23.

$P(\text{youngest boy included and eldest girl excluded}) = \frac{{}^{23}C_3}{{}^{25}C_4}$

$= \frac{7}{50}$

**(1 mark)**

(iv) From part (i) we know that the probability of the teacher choosing a team which contains only boys is $\frac{182}{1265}$.

Required probability $= {}^{9}C_3 \left(\frac{182}{1265}\right)^3 \left(\frac{1083}{1265}\right)^6$ **(1 mark)**

$= 0 \cdot 0985$ correct to 4 decimal places **(1 mark)**

**Question 5** (12 marks)

(a) (i) $P$ is the point $(2at, at^2)$

So, $x = 2at,\quad y = at^2$

$\frac{dx}{dt} = 2a \qquad \frac{dy}{dt} = 2at$

So, $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$

$= 2at \cdot \frac{1}{2a}$ **(1 mark)**

$= t$

So the equation of the tangent to the parabola at $P$ is

$y - at^2 = t(x - 2at)$

$y = tx - at^2$ as required **(1 mark)**

(b) (i)

$$a = 4x$$

So, $\dfrac{d\left(\frac{1}{2}v^2\right)}{dx} = 4x$ **(1 mark)**

$$\frac{1}{2}v^2 = 2x^2 + c \text{ where } c \text{ is a constant}$$

When $x = 1, v = -2$

$$2 = 2 + c, \text{ so } c = 0$$

$$\frac{1}{2}v^2 = 2x^2$$

$$v^2 = 4x^2$$

$v = \pm 2x$ since $x = 1$ when $v = -2$, reject $v = 2x$

So, $v = -2x$

**(1 mark)**

(ii)

$$\frac{dx}{dt} = -2x$$

$$\frac{dt}{dx} = \frac{-1}{2x}$$

$t = -\frac{1}{2}\ln x + d$ where $d$ is a constant **(1 mark)**

When $x = 1, \quad t = 0$

$$0 = -\frac{1}{2} \times 0 + d, \text{ so } d = 0$$

So $t = -\frac{1}{2}\ln x$

$$-2t = \ln x$$

$$x = e^{-2t}$$

**(1 mark)**

(iii) The particle is to the left of the origin for $x < 0$. Since $x = e^{-2t}$, there are no real values of $t$ for which $e^{-2t} < 0$ hence the particle never moves to the left of the origin.

**(1 mark)**

# Question 6 Marker's Comments

(a) (i) Generally well done.

(ii) Some students assumed that $\angle QRP = 90^\circ$ to obtain $st = -1$ from the gradients of the tangents at Q and P to the curve when this was not given.

(iii) Some students found the coordinates of R but need not use the result from part (ii) to show that the locus of R was coincident with the directrix.

(b) (i) Many students did not begin with $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$ and did not make much progress with this question. Some candidates did not correctly justify the negative sign for $x$.

(ii) Some students did not show that the integration constant was zero to obtain full marks.

(iii) Some varying approaches were used here with some successes.

**Question 7 (12 marks)**

(a) (i) Let $\sqrt{3}\sin\frac{t}{2}+\cos\frac{t}{2}=R\sin\left(\frac{t}{2}+\alpha\right)$, $R>0$

$$=R\sin\frac{t}{2}\cos\alpha+R\cos\frac{t}{2}\sin\alpha$$

So $R\cos\alpha=\sqrt{3}$ and $R\sin\alpha=1$

So $R^2\cos^2\alpha=3$ and $R^2\sin^2\alpha=1$

$R^2\left(\cos^2\alpha+\sin^2\alpha\right)=4$

$R=2$ $(R>0)$

**(1 mark)**

So $\cos\alpha=\frac{\sqrt{3}}{2}$ and $\sin\alpha=\frac{1}{2}$

So $\alpha=\frac{\pi}{6}$

So $\sqrt{3}\sin\frac{t}{2}+\cos\frac{t}{2}=2\sin\left(\frac{t}{2}+\frac{\pi}{6}\right)$

**(1 mark)**

(ii) Now $x=2\sin\left(\frac{t}{2}+\frac{\pi}{6}\right)$

$\dot{x}=\cos\left(\frac{t}{2}+\frac{\pi}{6}\right)$

$\ddot{x}=-\frac{1}{2}\sin\left(\frac{t}{2}+\frac{\pi}{6}\right)$

$=-\frac{1}{2}\times\frac{x}{2}$

$=-\frac{1}{4}x$ . **(1 mark)**

So $\ddot{x}=-\left(\frac{1}{2}\right)^2x$ and is clearly in the form $\ddot{x}=-n^2x$ which defines simple harmonic motion of a particle about $x=0$.

**(1 mark)**

(iii) From part (ii), $\dot{x}=\cos\left(\frac{t}{2}+\frac{\pi}{6}\right)$

Now, since $\dot{x}$ represents velocity and we are looking to find speed, we need to consider both positive and negative values. That is we need to solve $\cos\left(\frac{t}{2}+\frac{\pi}{6}\right)=\pm\frac{1}{2}$,

Since $0<t<4\pi$, $\quad 0<\frac{t}{2}<2\pi$

So $\quad \frac{t}{2}+\frac{\pi}{6}=\frac{\pi}{3},\frac{2\pi}{3},\frac{4\pi}{3},\frac{5\pi}{3}$

For simplicity, let $\frac{t}{2}+\frac{\pi}{6}=\frac{1}{2}\left(t+\frac{\pi}{3}\right)$

So, $\quad t+\frac{\pi}{3}=\frac{2\pi}{3},\frac{4\pi}{3},\frac{8\pi}{3},\frac{10\pi}{3}$

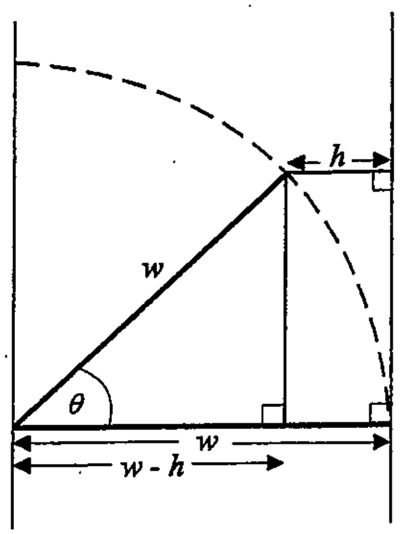
$$t=\frac{\pi}{3},\frac{3\pi}{3},\frac{7\pi}{3},\frac{9\pi}{3}$$

So the required times are

$t=\frac{\pi}{3}$ secs, $t=\pi$ secs, $t=\frac{7\pi}{3}$ secs and $t=3\pi$ secs.

**(1 mark)**

(b) (i)




From the diagram above we have

$$\cos\theta=\frac{w-h}{w}$$

$$w\cos\theta=w-h$$

$$h=w-w\cos\theta \text{ as required}$$

**(1 mark)**

(ii)

Since $\frac{d\theta}{dt} = \frac{1}{\sqrt{1-t^2}}$ (given)

$$\theta = \int \frac{1}{\sqrt{1-t^2}}\,dt$$

$$\theta = \sin^{-1} t + c$$

$t = 0, \theta = 0$ so $c = 0$

$\theta = \sin^{-1} t$ as required

**(1 mark)**

(iii) Now, $\frac{d\theta}{dt} = \frac{1}{\sqrt{1-t^2}}$

Also $\frac{dh}{d\theta} = w\sin\theta$ (using part (i))

So $\frac{dh}{dt} = \frac{dh}{d\theta}\cdot\frac{d\theta}{dt}$ (chain rule)

$$= w\sin\theta\cdot\frac{1}{\sqrt{1-t^2}}$$

$= \frac{wt}{\sqrt{1-t^2}}$ since $\theta = \sin^{-1} t$ from part (ii) and so $t = \sin\theta$

**(1 mark)**

(iv) $\frac{dh}{dt} = \frac{wt}{\sqrt{1-t^2}}$

$h = w\int \frac{t}{\sqrt{1-t^2}}\,dt$ let $u = 1-t^2$

$= w\int -\frac{1}{2}\frac{du}{dt}\cdot\frac{1}{\sqrt{u}}\,dt$ $\frac{du}{dt} = -2t$

$= \frac{-w}{2}\int u^{\frac{-1}{2}}\,du$ **(1 mark)**

$$= \frac{-w}{2}u^{\frac{1}{2}}\times 2 + c$$

$$h = -w\sqrt{1-t^2} + c$$

When $t = 0,\ h = 0$

$$0 = -w\sqrt{1} + c$$

$$c = w$$

$$h = -w\sqrt{1-t^2} + w$$

**(1 mark)**

(v) Now, $h = w - w\sqrt{1-t^2}$

The gate is fully open when $h = w$.

So $w = w - w\sqrt{1-t^2}$

$0 = -w\sqrt{1-t^2}$

$t = 1$ since $w \neq 0$ and $t \geq 0$

**(1 mark)**

Also, from part (iii) we have

$$h = w\int \frac{t}{\sqrt{1-t^2}}\,dt$$

The distance covered between $t = 0$ and $t = 1$ is $w$.

So $w = w\int_0^1 \frac{t}{\sqrt{1-t^2}}\,dt$

So $\int_0^1 \frac{t}{\sqrt{1-t^2}}\,dt = 1$ as required.

**(1 mark)**

**Question 7** (12 marks)

(a)

(i) $x = \sqrt{g}\,t\cos\theta$

So $t = \dfrac{x}{\sqrt{g}\cos\theta}$

In $y = \sqrt{g}t\sin\theta - \frac{1}{2}gt^2$

becomes $y = \dfrac{\sqrt{g}x\sin\theta}{\sqrt{g}\cos\theta} - \dfrac{1}{2}g\dfrac{x^2}{g\cos^2\theta}$

So, $y = x\tan\theta - \dfrac{x^2\sec^2\theta}{2}$

**(1 mark)**

(ii) From (i) we have

$$y = x\tan\theta - \frac{x^2\sec^2\theta}{2}$$

$$= x\tan\theta - \frac{x^2}{2}\left(1+\tan^2\theta\right)$$

At point $P$, $x = d\cos 30^\circ$ and $y = d\sin 30^\circ$

$= \dfrac{\sqrt{3}d}{2}$ $\qquad = \dfrac{d}{2}$

**(1 mark)**

Ext1 Markers Notes JSH

**Q7** a) i) Most people found $R$ correctly but many could not find $\alpha$!

ii) Done well overall.

iii) Most people got $\pi$ but many did not get $3\pi$ and some produced answers outside the range $0 \to 4\pi$.

b) i) Mostly done well

ii) Many people got this wrong.

iii) Reasonably well done.

iv) Mostly started well, but getting the constant right was the problem.

v) Few got this fully.