CRANBROOK
SCHOOL

Term 3, 2009

Year 12 Mathematics Extension 1

Trial HSC Examination

Tuesday August 4th, 2009

Time Allowed: 2 hours, plus 5 minutes reading time

Total Marks: 84

There are 7 questions, all of equal value.
Submit your work in seven 4 Page booklets.
All necessary working should be shown in every question.
Full marks may not be awarded if work is careless or badly arranged.
Board of Studies approved calculators may be used.
A list of standard integrals is attached to the back of this paper.

Question 1 (12 marks) Use a separate writing booklet **Marks**

(a) Evaluate: $\lim\limits_{x \to 0} \dfrac{\sin 3x}{2x}$ **1**

(b) Find: $\dfrac{d}{dx}\left[\ln\sqrt{\dfrac{1+x}{1-x}}\right]$ **2**

(c) Evaluate: $\displaystyle\int_{-3}^{3} \frac{dx}{x^2+9}$ **2**

(d) State the domain and range of the function : $f(x) = 2\cos^{-1} 3x$ **2**

(e) The variable point ($2\cos\theta$, $3\sin\theta$) lies on a curve. Find the cartesian equation of this curve. **2**

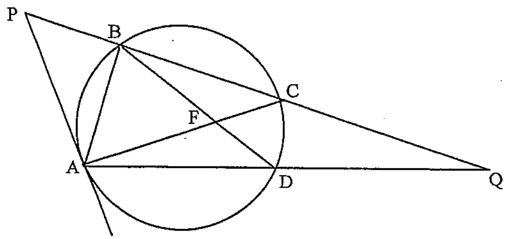
(f) Use the substitution $\sqrt{x} = u$ to evaluate: $\displaystyle\int_{1}^{4} \frac{dx}{x+\sqrt{x}}$ **3**

Question 2 (12 marks) Use a separate writing booklet

(a) Solve: $3^{x+1} = 5$. Give your answer correct to two decimal places. **2**

(b) Solve: $x^3 + 2x^2 - 5x - 6 = 0$ **2**

(c) NOT TO SCALE



In the above figure, AP is a tangent to the circle at A. PBCQ and ADQ are straight lines. Prove that $\angle PAB = \frac{1}{2}(\angle CFD + \angle CQD)$ **3**

(d) Evaluate: $2\int_0^{\frac{\pi}{4}} \cos^2 4x \, dx$ **3**

(e) Find the general solution to: $\cos 5\theta - \cos 2\theta = 0$ **2**

Question 3 (12 marks) Use a separate writing booklet **Marks**

(a) If the domain of $y = x^2 - 4x$ is restricted to a monotonic increasing curve:

(i) sketch $y = f(x)$ **1**

(ii) find the inverse function $y = f^{-1}(x)$ **2**

(iii) state the domain and range of the inverse function **1**

(b) (i) Show that $f(x) = 3\sin 2x - x$ has a root between 1·33 and 1·34. **1**

(ii) Starting with $x = 1{\cdot}33$, use one application of Newton's method to find a better approximation for this root correct to 4 decimal places. **3**

(c) Consider the graph of $y = \dfrac{x^2}{4 - x^2}$

(i) Write down the asymptotes of the function. **1**

(ii) Find any stationary points and determine their nature. **2**

(iii) Sketch the graph. **1**

Question 4 (12 marks) Use a separate writing booklet **Marks**

(a) (i) Prove by mathematical induction

$$1^3+2^3+3^3+\cdots+n^3=\frac{1}{4}n^2(n+1)^2 \quad for\ n\geq 1$$ 4

(ii) Hence evaluate: $2^3+4^3+6^3+\cdots+20^3$ 1

(b) The polynomial $P(x)=2x^3-5x^2-3x+1$ has zeros α, β and γ. Find the values of

(i) $3\alpha+3\beta+3\gamma-4\alpha\beta\gamma$ 2

(ii) $\alpha^{-1}+\beta^{-1}+\gamma^{-1}$ 1

(iii) $\alpha^2+\beta^2+\gamma^2$ 1

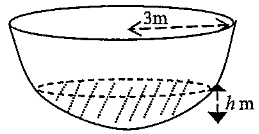
(c) $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2=4ay$. The chord PQ subtends a right angle at the origin. If $pq=-4$ prove that the locus of the midpoint of PQ is a parabola with vertex $(0, 4a)$. 3

Question 5 (12 marks) Use a separate writing booklet **Marks**

(a) The interval AB is divided internally in the ratio 5:6 at the point R. If A and B have co-ordinates (2, 3) and (5, 4) respectively find the co-ordinates of R. 2

(b) (i) Show that 1 is a root of $h^3-9h^2+8=0$ and find the other roots. 2

NOT TO SCALE



(ii) A hemi-spherical bowl has a radius of 3m. Oil is poured into the container at a constant rate of $\pi/3$ m^3/min. When the depth is h metres, the volume of oil is

$$V=\frac{\pi}{3}\left(9h^2-h^3\right)m^3.$$

(α) How deep is the oil after 8 minutes? 2

(β) At what rate is h increasing at this time? 2

(c) The acceleration of a $\pi-$meson moving in a straight line is given by: $\ddot{x}=\frac{-4}{(x+2)^2}$, where x is the displacement in metres from a fixed point O. Initially the $\pi-$meson is 1 metre to the left of O and travelling with a velocity of 6 ms^{-1} in $\rightarrow$. Find the velocity of the $\pi-$meson when it is 6m to the right of O. 4

Question 6 (12 marks) Use a separate writing booklet **Marks**

(a) The increase and decrease of pollution readings, x, in the skyline of Mexico City may be taken as simple harmonic according to the equation $\ddot{x} = -n^2(x-b)$, where $x = b$ is the centre of motion. A high pollution reading of 45 parts per million occurs at 6am on a particular day and a low pollution reading of 5 parts per million occurs at 11.30am on the same day.

(i) Prove that $x = b + a\cos nt$ satisfies $\ddot{x} = -n^2(x-b)$. **2**

(ii) Find the earliest time interval on this day after 6am that a Mexican pigeon trainer Ms Swinivia Flutos can release her pigeons into the atmosphere for training if the pigeons cannot tolerate pollution readings of more than 15 parts per million. **5**

(b) (i) Using the result for tan (A+B), prove that

$$\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$$ **2**

(ii) Given A, B and C are angles of a triangle and

$\frac{\tan A}{5} = \frac{\tan B}{6} = \frac{\tan C}{7} = k$, show that $k = \sqrt{\frac{3}{35}}$ **2**

(iii) Hence calculate the smallest of the angles to the nearest minute. **1**

Question 7 (12 marks) Use a separate writing booklet **Marks**

(a) A cup of soup with a temperature 95^0C is placed in a room which has a temperature of 20^0C. In 10 minutes the cup of soup cools to 70^0C. Assuming the rate of heat loss is proportional to the excess of its temperature above room temperature, that is

$$\frac{dT}{dt} = -k(T-20),$$

(i) show that $T = 20 + Ae^{-kt}$ is a solution of

$\frac{dT}{dt} = -k(T-20)$. **1**

(ii) find the temperature of the soup after a further 5 min. to the nearest degree. **2**

(iii) how long will it take the soup to cool to 35^0C? Give your answer correct to the nearest minute. **1**

(iv) find the rate of cooling when the soup is 35^0C. Give your answer correct to 1 decimal place. **1**

(b) If $\sin x - 7\cos x = -5$, $0 \le x \le 2\pi$, find x correct to 2 decimal places. **3**

(c) Sketch $y = \tan^{-1}(\sin 3x)$, $0 \le x \le \pi$, by firstly finding the existence of any stationary points and determining their nature. **4**

(a) $\lim_{x\to 0} \frac{\sin 3x}{2x} = \left(\lim_{x\to 0}\frac{\sin 3x}{3x}\right)\cdot\frac{3}{2}$

$= 1 \cdot \frac{3}{2}$

$= \frac{3}{2}$ ✓

(b) $\frac{d}{dx}\left[\ln\sqrt{\frac{1+x}{1-x}}\right]$

$= \frac{d}{dx}\left[\frac{1}{2}\left(\ln(1+x) - \ln(1-x)\right)\right]$

$= \frac{1}{2}\left[\frac{1}{1+x} + \frac{1}{1-x}\right]$ ✓

$= \frac{1}{2}\left[\frac{1-x+1+x}{1-x^2}\right]$

$= \frac{1}{1-x^2}$ ✓

(c) $I = \int_{-3}^{3} \frac{dx}{x^2+9}$

$= \frac{1}{3}\left[\tan^{-1}\frac{x}{3}\right]_{-3}^{3}$ ✓

$= \frac{1}{3}\left[\tan^{-1}1 - \tan^{-1}(-1)\right]$

$= \frac{1}{3}\left[\frac{\pi}{4} + \frac{\pi}{4}\right]$

$= \frac{\pi}{6}$ ✓

(d) $f(x) = 2\cos^{-1}3x$

Domain is: $-1 \le 3x \le 1$

$\therefore -\frac{1}{3} \le x \le \frac{1}{3}$ ✓

Range is: $0 \le y \le 2\pi$ ✓

(e) $x = 2\cos\theta$ $\quad \therefore \cos\theta = \frac{x}{2}$

$y = 3\sin\theta$ $\quad \sin\theta = \frac{y}{3}$

But $\sin^2\theta + \cos^2\theta = 1$ ✓

$\therefore \left(\frac{y}{3}\right)^2 + \left(\frac{x}{2}\right)^2 = 1$

$\therefore \frac{x^2}{4} + \frac{y^2}{9} = 1$ is the Cartesian equation of this curve. ✓

(f) Let $\sqrt{x} = u$ $\quad \therefore \frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$

when $x=1$ $u=1$ $\quad \therefore du = \frac{1}{2\sqrt{x}}\,dx$

$x=4$ $u=2$

$\therefore 2u\,du = dx$ ✓

$\therefore I = \int_1^2 \frac{2u\,du}{u^2+u}$

$= 2\int_1^2 \frac{du}{u+1}$ ✓

$= 2\left[\ln(u+1)\right]_1^2$

$= 2\left[\ln 3 - \ln 2\right]$

$= 2\ln\left(\frac{3}{2}\right)$. ✓

2.(a) $3^{x+1} = 5$

$\therefore \ln(3^{x+1}) = \ln 5$

$\therefore (x+1)\ln 3 = \ln 5$ ✓

$\therefore x+1 = \frac{\ln 5}{\ln 3}$

$\therefore x = \frac{\ln 5}{\ln 3} - 1$

$\therefore x = 0.46$ (2 d.p.) ✓

(b) $x^3 + 2x^2 - 5x - 6 = 0$

Let $P(x) = x^3 + 2x^2 - 5x - 6$

Possible zeros are : $\pm 6, \pm 3, \pm 2, \pm 1$.

Let $x=-1$ $\therefore P(-1) = -1+2+5-6 = 0$ ✓

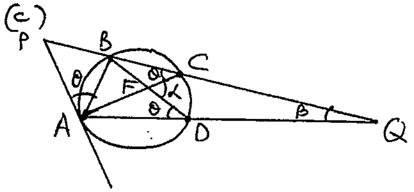
$\therefore x+1$ is a factor

$\therefore P(x) = (x+1)(x^2+x-6)$

$= (x+1)(x+3)(x-2)$

$\therefore$ If $P(x)=0$ $\therefore x = -1, -3$ or 2. ✓

(c)



Let $\angle PAB = \theta$, $\angle CFD = \alpha$, $\angle CQD = \beta$

Now $\angle PAB = \angle PCA = \theta$ ✓

($\angle$ between tangent and chord at pt of contact = $\angle$ in alt. segment)

similarly $\angle PAB = \angle BDA = \theta$

$\therefore \angle FCQ = 180° - \theta = \angle FDQ$

$\therefore$ In quad. FCQD :

$\alpha + 180° - \theta + \beta + 180° - \theta = 360°$ ✓

($\angle$ sum of quad. $= 360°$)

$\therefore \alpha + \beta = 2\theta$

$\therefore \theta = \frac{1}{2}(\alpha + \beta)$

ie $\angle PAB = \frac{1}{2}(\angle CFD + \angle CQD)$ ✓

(d) $I = 2\int_0^{\frac{\pi}{4}} \cos^2 4x\,dx$

Now $\cos 2x = 2\cos^2 x - 1$

$\therefore \cos 8x = 2\cos^2 4x - 1$

$\therefore 2\cos^2 4x = 1 + \cos 8x$ ✓

$\therefore I = \int_0^{\frac{\pi}{4}} 1 + \cos 8x\,dx$

$= \left[x + \frac{\sin 8x}{8}\right]_0^{\frac{\pi}{4}}$ ✓

$= \left[\left(\frac{\pi}{4} + \frac{\sin 2\pi}{8}\right) - (0+0)\right]$

$= \frac{\pi}{4}$. ✓

(e) $\cos 5\theta - \cos 2\theta = 0$

$\therefore \cos 5\theta = \cos 2\theta$ ✓

$\therefore 5\theta = 2\pi n \pm 2\theta$, where n is any integer

$\therefore 3\theta = 2\pi n$ or $7\theta = 2\pi n$

$\therefore \theta = \frac{2\pi n}{3}$ or $\frac{2\pi n}{7}$. ✓

3. (a) $y = x^2 - 4x$

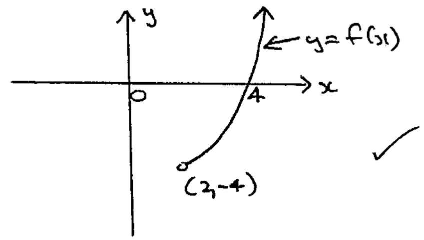
$\therefore \frac{dy}{dx} = 2x - 4$

For monotonic increasing $\frac{dy}{dx} > 0$

$\therefore x > 2$.

(i) when $x = 2$ $y = -4$

$\therefore$ graph of $y = f(x)$ is :



✓

(ii) For inverse function interchange x for y $\quad \therefore x = y^2 - 4y$

$\therefore x = (y-2)^2 - 4$ ✓

$\therefore y - 2 = \pm\sqrt{x+4}$

$\therefore y = 2 \pm \sqrt{x+4}$

$\therefore f^{-1}(x) = 2 + \sqrt{x+4}$ ✓

(iii) For inverse function :

Domain is: $x > -4$

Range is: $y > 2$. ✓

(b) (i) $f(x) = 3\sin 2x - x$

Now $f(1.33) = 0.0595\ldots > 0$

$f(1.34) = -0.0038\ldots < 0$

$\therefore$ As $f(1.33)$ and $f(1.34)$ have opposite signs and $f(x)$ is cts $\forall x$

$\Rightarrow f(x)$ has at least 1 root in the interval $1.33 < x < 1.34$ ✓

(ii) $f(x) = 3\sin 2x - x$

$\therefore f'(x) = 6\cos 2x - 1$

By Newton's Method $z_2 = z_1 - \frac{P(z_1)}{P'(z_1)}$

$\therefore$ if $z_1 = 1.33$

$\therefore z_2 = 1.33 - \frac{P(1.33)}{P'(1.33)}$

$= 1.33 - \frac{0.0595\ldots}{-6.3175\ldots}$

$= 1.3394298\ldots$

$= 1.3394$ (4 d.p.)

c) $y = \frac{x^2}{4-x^2}$

(i) For vertical asymptotes $4-x^2=0$

$\therefore x = \pm 2$

$\therefore$ vertical asymptotes at $x = \pm 2$

For horiz. asymptote $\lim_{x\to\pm\infty} y$

$= \lim_{x\to\pm\infty} \frac{x^2(1)}{x^2\left(\frac{4}{x^2}-1\right)}$

$= \frac{1}{0-1}$ $\left(\text{As } x\to\pm\infty,\ \frac{4}{x^2}\to 0\right)$

$= -1$

$\therefore$ horizontal asymptote at $y=-1$

(ii) $y = \frac{x^2}{4-x^2}$

$\therefore \frac{dy}{dx} = \frac{(4-x^2).2x - x^2(-2x)}{(4-x^2)^2}$

$= \frac{8x-2x^3+2x^3}{(4-x^2)^2}$

$= \frac{8x}{(4-x^2)^2}$

For a stat. pt $\frac{dy}{dx}=0$ $\therefore x=0$

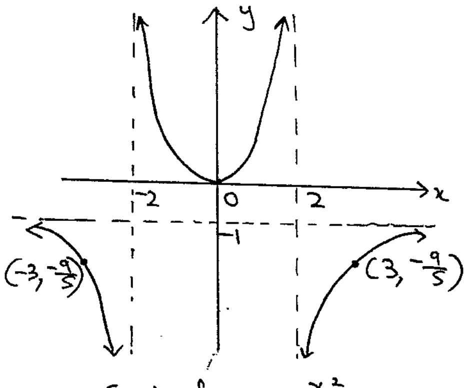
when $x=0$

x	0^-	0	0^+
y'	−	0	+

$\Rightarrow$ min. turn pt at $(0,0)$

(iii) Let $y=f(x)$

$\therefore f(-x)=f(x)$ $\therefore$ function is even.



Graph of $y = \frac{x^2}{4-x^2}$

4 (a) TO PROVE: $1^3+2^3+3^3+\ldots+n^3 = \frac{1}{4}n^2(n+1)^2$ for $n \geq 1$.

PROOF: Step 1: When $n=1$ LHS $=1^3=1$

RHS $=\frac{1}{4}.1^2(2)^2=1$ $=$ LHS

$\therefore$ it is true for $n=1$.

Step 2: Assume it is true for $n=k$ ($k \in n$, $n \in J^+$) and prove it is true for $n=k+1$.

Now $S_k + T_{k+1} = S_{k+1}$

$\therefore$ LHS $= S_k + T_{k+1}$

$= \frac{1}{4}k^2(k+1)^2 + (k+1)^3$

$= (k+1)^2\left[\frac{1}{4}k^2 + k+1\right]$

$= (k+1)^2\left[\frac{k^2+4k+4}{4}\right]$

$= \frac{1}{4}(k+1)^2(k+2)^2$

RHS $= S_{k+1}$

$= \frac{1}{4}(k+1)^2(k+1+1)^2$

$= \frac{1}{4}(k+1)^2(k+2)^2$

$=$ LHS.

$\therefore$ if it is true for $n=k$ so it is true for $n=k+1$.

Step 3: It is true for $n=1$ and so it is true for $n=1+1=2$. It is true for $n=2$ and so, it is true for $n=2+1=3$ and so on for all positive integral values of n.

(ii) Now $2^3+4^3+6^3+\ldots+20^3$

$= (2^3.1^3)+(2^3.2^3)+(2^3.3^3)+\ldots$
$\ldots+(2^3.10^3)$

$= 2^3\left[1^3+2^3+3^3+\ldots+10^3\right]$

$= 8\left[\frac{1}{4}\times 10^2 \times 11^2\right]$

$= 24200$

(b) $P(x) = 2x^3-5x^2-3x+1$ has zeros α, β and γ.

$\therefore \alpha+\beta+\gamma = \frac{-b}{a} = \frac{5}{2}$

$\alpha\beta+\alpha\gamma+\beta\gamma = \frac{c}{a} = -\frac{3}{2}$

$\alpha\beta\gamma = \frac{-d}{a} = -\frac{1}{2}$

(i) $3\alpha+3\beta+3\gamma-4\alpha\beta\gamma = 3\left(\frac{5}{2}\right)-4\left(\frac{-1}{2}\right)$

$= 9\frac{1}{2}$

(ii) $\alpha^{-1}+\beta^{-1}+\gamma^{-1} = \frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$

$= \frac{\beta\gamma+\alpha\gamma+\alpha\beta}{\alpha\beta\gamma}$

$= \frac{-3/2}{-1/2}$

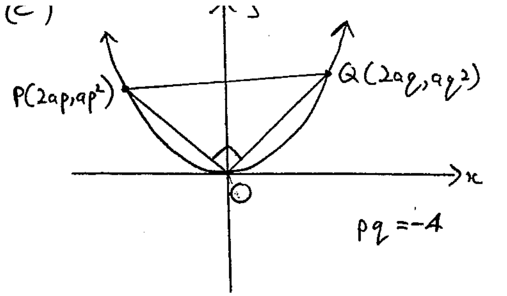
$= 3$

(iii) $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)$

$= \left(\frac{5}{2}\right)^2 - 2\left(\frac{-3}{2}\right)$

$= 9\frac{1}{4}$

(c)



$pq=-4$

$M_{PQ} = \left(\frac{2ap+2aq}{2}, \frac{ap^2+aq^2}{2}\right)$

$= \left(a(p+q), \frac{a(p^2+q^2)}{2}\right)$

Now $x = a(p+q)$ $\therefore p+q = \frac{x}{a}$ — ①

$y = a\left(\frac{p^2+q^2}{2}\right)$ $\therefore p^2+q^2 = \frac{2y}{a}$ — ②

Now $p^2+q^2 = (p+q)^2 - 2pq$

$\therefore \frac{2y}{a} = \left(\frac{x}{a}\right)^2 - 2(-4)$

$\therefore \frac{2y}{a} = \frac{x^2}{a^2} + 8$

$\therefore 2ay = x^2 + 8a^2$

$\therefore x^2 = 2ay - 8a^2$

$\therefore x^2 = 2a(y-4a)$

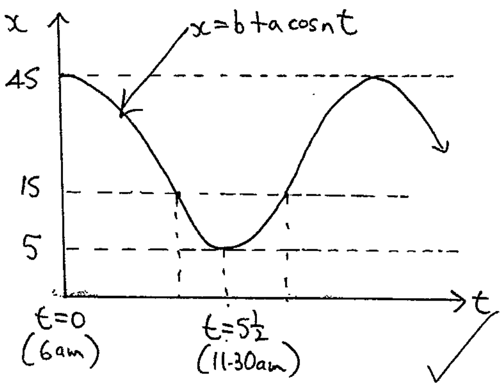
$\Rightarrow$ locus of the midpoint of PQ is a parabola with vertex $(0,4a)$.

5 (a) $R = \left(\frac{kx_2+lx_1}{k+l}, \frac{ky_2+ly_1}{k+l}\right)$

$\therefore R = \left(\frac{5(5)+6(2)}{5+6}, \frac{5(4)+6(3)}{5+6}\right)$

$= \left(\frac{37}{11}, \frac{38}{11}\right)$

(ii)



Now $45 = b+a$ —(1)

$5 = b-a$ —(2)

(1)+(2): $50 = 2b$ $\therefore b = 25$ $\therefore a = 20$

Now period, $T = 2\times 5\frac{1}{2} = 11 = \frac{2\pi}{n}$

$\therefore n = \frac{2\pi}{11}$

$\therefore x = 25 + 20\cos\frac{2\pi}{11}t$

If $x = 15$, $15 = 25 + 20\cos\frac{2\pi}{11}t$

$\therefore -\frac{1}{2} = \cos\frac{2\pi}{11}t$

$\therefore \frac{2\pi}{11}t = \cos^{-1}\left(\frac{1}{2}\right)$ (require 2nd, 3rd quads)

$\therefore \frac{2\pi}{11}t = \pi - \frac{\pi}{3}$ or $\pi + \frac{\pi}{3}$

$\therefore t = \frac{2}{3}\times\frac{11}{2}$ or $\frac{4}{3}\times\frac{11}{2}$

$\therefore t = \frac{11}{3}$ or $\frac{22}{3}$

$\therefore$ For Ms Flutos to release pigeons for training earliest time interval is: 6am + 3 hr 40mins to 6am + 7 hr 20mins

= 9.40 am to 1.20 pm.

b (i) Now $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$

$\therefore$ let $B = B+C$

$\therefore \tan(A+B+C) = \frac{\tan A + \tan(B+C)}{1 - \tan A\tan(B+C)}$

$$= \frac{\tan A + \frac{\tan B + \tan C}{1 - \tan B\tan C}}{1 - \tan A\left(\frac{\tan B + \tan C}{1 - \tan B\tan C}\right)}$$

$$= \frac{\tan A(1 - \tan B\tan C) + \tan B + \tan C}{1 - \tan B\tan C - \tan A(\tan B + \tan C)}$$

$$= \frac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1 - \tan A\tan B - \tan B\tan C - \tan C\tan A} \quad (*)$$

(ii) If A, B and C are $\angle$s of a Δ

$\therefore A+B+C = \pi$ $\therefore \tan(A+B+C) = 0$

and as $\frac{\tan A}{5} = \frac{\tan B}{6} = \frac{\tan C}{7} = k$

$\therefore \tan A = 5k, \tan B = 6k, \tan C = 7k$

$\Rightarrow 0 = \frac{5k+6k+7k-(5k)(6k)(7k)}{1-(5k)(6k)-(6k)(7k)-(7k)(5k)}$ sub into (*)

$\therefore 18k - 210k^3 = 0$.

$\therefore 6k(3 - 35k^2) = 0$

$\therefore k = 0$ or $\pm\sqrt{\frac{3}{35}}$

But as $\tan A, \tan B, \tan C > 0$ $\therefore k > 0$

$\therefore k = \sqrt{\frac{3}{35}}$ only.

(iii) Now smallest angle is A where $\tan A = 5k$

$\therefore \tan A = 5\sqrt{\frac{3}{35}}$

$\therefore \angle A = 55°40'$ (to nearest min)

) (i) Let $P(h) = h^3 - 9h^2 + 8$

if $h=1$ $P(1) = 1 - 9 + 8 = 0$

$\therefore h = 1$ is a root.

$\therefore P(h) = (h-1)(h^2 - 8h - 8)$

$\therefore$ other roots are solved from $h^2 - 8h - 8 = 0$

$\therefore h = \frac{8 \pm \sqrt{64 - 4\cdot 1\cdot -8}}{2}$

$= \frac{8 \pm \sqrt{96}}{2}$

$= \frac{8 \pm 4\sqrt{6}}{2}$

$= 4 \pm 2\sqrt{6}$

$\therefore$ roots are $h = 1, 4 \pm 2\sqrt{6}$.

(ii) $V = \frac{\pi}{3}(9h^2 - h^3)$, $\frac{dV}{dt} = \frac{\pi}{3}$ m³/min

(α) when $t = 8$ $V = 8\left(\frac{\pi}{3}\right) = \frac{8\pi}{3}$ m³

$\therefore \frac{8\pi}{3} = \frac{\pi}{3}(9h^2 - h^3)$

$\therefore 8 = 9h^2 - h^3$

$\therefore h^3 - 9h^2 + 8 = 0$

$\therefore h = 1, 4 \pm 2\sqrt{6}$ (from (i) above).

But $0 < h \leq 3$ (radius of bowl is 3m)

$\therefore$ if $h = 4 + 2\sqrt{6}$ this is greater than 3

and if $h = 4 - 2\sqrt{6}$ " " less than 0.

$\therefore$ depth after 8 mins is 1m.

(β) $\frac{dV}{dh} = \frac{\pi}{3}(18h - 3h^2)$

$\therefore$ when $h = 1$ $\frac{dV}{dh} = \frac{\pi}{3}(15) = 5\pi$

and $\frac{dV}{dt} = \frac{\pi}{3}$.

Now $\frac{dV}{dt} = \frac{dV}{dh}\cdot\frac{dh}{dt}$

$\therefore$ when $h = 1$ $\frac{\pi}{3} = 5\pi\cdot\frac{dh}{dt}$

$\therefore \frac{dh}{dt} = \frac{1}{15}$ m/min.

ie h is increasing at $\frac{1}{15}$ m/min.

(c) $\ddot{x} = \frac{-4}{(x+2)^2} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$\therefore \frac{1}{2}v^2 = -4\int(x+2)^{-2}\,dx$

$= \frac{4}{x+2} + c$

when $x = -1$, $v = 6$ $\therefore 18 = 4 + c$

$\therefore c = 14$

$\therefore \frac{1}{2}v^2 = \frac{4}{x+2} + 14$

$\therefore v^2 = \frac{8}{x+2} + 28$

$\therefore v = \sqrt{\frac{8}{x+2} + 28}$

(taking +ve square root as initial condition gave a +ve velocity)

When $x = 6$ $v = \sqrt{1 + 28} = \sqrt{29}$

$\therefore$ velocity of π-meson when it is 6m to the right of O is $\sqrt{29}$ ms^{-1} in $\rightarrow$.

6 (a)(i) $\ddot{x} = -n^2(x - b)$ —(1)

$x = b + a\cos nt$ —(2)

sub (2) into (1):

LHS of (1) $= \ddot{x}$

$= \frac{d}{dt}(\dot{x})$

$= \frac{d}{dt}(-an\sin nt)$

$= -an^2\cos nt$

$= -n^2(x - b)$ [as $a\cos nt = x - b$ from (2)]

= RHS of (1)

$\therefore x = b + a\cos nt$ satisfies the given equation.

1) (a) (i) $\dfrac{dT}{dt} = -k(T-20)$ —(1)

$T = 20 + Ae^{-kt}$ —(2)

sub (2) into (1):

LHS of (1) $= \dfrac{dT}{dt}$

$= \dfrac{d}{dt}\left(20 + Ae^{-kt}\right)$

$= -kAe^{-kt}$

$= -k(T-20)$ (As $Ae^{-kt} = T-20$ from (2))

$=$ RHS of (1)

$\Rightarrow T = 20 + Ae^{-kt}$ satisfies given equation.

(ii) $T = 20 + Ae^{-kt}$

when $t=0$, $T=95$

$\therefore 95 = 20 + Ae^{0}$ $\therefore A = 75$

$\therefore T = 20 + 75e^{-kt}$

when $t=10$, $T=70$

$\therefore 70 = 20 + 75e^{-10k}$

$\therefore \dfrac{50}{75} = e^{-10k}$

$\therefore k = -\frac{1}{10}\ln\frac{2}{3}$

$\therefore T = 20 + 75e^{\left(\frac{1}{10}\ln\frac{2}{3}\right)t}$

when $t=15$ $T = 20 + 75e^{\left(\frac{1}{10}\ln\frac{2}{3}\right)15}$

$\therefore T = 20 + 75\left(\frac{2}{3}\right)^{3/2}$

$\therefore T = 60.824...$

$\Rightarrow$ temp. of soup is then $61°C$ (to nearest degree)

(iii) when $T=35$, $t=?$

$\therefore 35 = 20 + 75e^{\left(\frac{1}{10}\ln\frac{2}{3}\right)t}$

$\therefore \frac{1}{5} = e^{\left(\frac{1}{10}\ln\frac{2}{3}\right)t}$

$\therefore t = \dfrac{\ln\frac{1}{5}}{\frac{1}{10}\ln\frac{2}{3}}$

$\therefore t = 39.693....$

$\therefore$ it will take 40 mins (to nearest min)

(iv) $\dfrac{dT}{dt} = \left(\frac{1}{10}\ln\frac{2}{3}\right)(T-20)$ (from (1))

when $T=35$ $\dfrac{dT}{dt} = -0.60819...$

$\therefore$ rate of cooling is $-0.6°C$/min (to 1 d.p.)

(b) $\sin x - 7\cos x = -5$, $0 \le x \le 2\pi$

Now $R = \sqrt{1^2 + (-7)^2} = 5\sqrt{2}$

$\therefore 5\sqrt{2}\left(\frac{1}{5\sqrt{2}}\sin x - \frac{7}{5\sqrt{2}}\cos x\right) = -5$

$\therefore 5\sqrt{2}\sin(x-\alpha) = -5$

where $\cos\alpha = \frac{1}{5\sqrt{2}}$, $\sin\alpha = \frac{7}{5\sqrt{2}}$

$\therefore \tan\alpha = 7$ $\therefore \alpha = \tan^{-1}7$

$\therefore \sin(x - \tan^{-1}7) = \frac{-1}{\sqrt{2}}$

$\therefore$ Basic angle $x - \tan^{-1}7 = \frac{\pi}{4}$ (require 3rd 4th quads)

$\therefore x = \pi + \frac{\pi}{4} + \tan^{-1}7$ or $2\pi - \frac{\pi}{4} + \tan^{-1}7 - 2\pi$

$\therefore x = 5.36$ (2dp)

or $x = 0.64$

(c) $y = \tan^{-1}(\sin 3x)$, $0 \le x \le \pi$.

$\dfrac{dy}{dx} = \dfrac{1}{1+(\sin 3x)^2} \cdot 3\cos 3x$

For a stat. pt $\dfrac{dy}{dx} = 0$

$\therefore 3\cos 3x = 0$

$\therefore 3x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$ for $0 \le 3x \le 3\pi$

$\therefore x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$ for $0 \le x \le \pi$

when $x = \frac{\pi}{6}$

x	$\frac{\pi}{6}^-$	$\frac{\pi}{6}$	$\frac{\pi}{6}^+$
y'	$+$	0	$-$

$\Rightarrow$ max turn pt at $\left(\frac{\pi}{6}, \frac{\pi}{4}\right)$

when $x = \frac{\pi}{2}$

x	$\frac{\pi}{2}^-$	$\frac{\pi}{2}$	$\frac{\pi}{2}^+$
y'	$-$	0	$+$

$\Rightarrow$ min. turn pt at $\left(\frac{\pi}{2}, -\frac{\pi}{4}\right)$

when $x = \frac{5\pi}{6}$

x	$\frac{5\pi}{6}^-$	$\frac{5\pi}{6}$	$\frac{5\pi}{6}^+$
y'	$+$	0	$-$

$\Rightarrow$ max turn pt at $\left(\frac{5\pi}{6}, \frac{\pi}{4}\right)$

when $y=0$

$\sin 3x = 0$

$\therefore 3x = 0, \pi, 2\pi, 3\pi$

$\therefore x = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi$

Period $= \frac{2\pi}{3}$ $\therefore$ sub-interval width $= \frac{1}{4} \times \frac{2\pi}{3} = \frac{\pi}{6}$.