FORT STREET

1849-1999 150 YEARS

PAST, PRESENT, FUTURE

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

2000

MATHEMATICS

3 UNIT (ADDITIONAL) – 4 UNIT (COMMON)

Time allowed: Two Hours
(Plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES

- Attempt **all questions**.
- All questions are of equal value.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- A table of standard integrals is included.
- Board-approved calculators may be used.
- Each question is to be started on a NEW PAGE and solutions are to be written on ONE side only.
- You may ask for extra Writing Paper if required.
- You must STAPLE each question in a separate bundle clearly labelled with your NAME and CLASS TEACHER

Name

Class/Teacher

Q	1	2	3	4	5	6	7	Total /84
Mark /12								

Question 1 (12 marks) *Start a new page*

(a) Solve for x: $2x^2+5x-3\geq 0$

(b) Solve the following inequality and graph the solution on the number line:

$$\frac{4x-1}{x-2}\leq 3$$

(c) Evaluate $\int_0^{\frac{\pi}{4}}\cos x\sin^2 x\,dx$

(d) Use the substitution $u=1+3x^3$ to find:

$$\int x^2\sqrt{1+3x^3}\,dx$$

(e) Find the size of the acute angle between the lines $y=2x-1$ and $2x+3y=4$

Question 2 (12 marks) *Start a new page*

(a) The point P (6,9) divides the interval AB in the ratio −3:2. Find the point B given that A is (1, 4)

(b) Prove the following identity:

$$\frac{2\cos A}{\operatorname{cosec}A-2\sin A}=\tan 2A$$

(c) Prove using mathematical induction that

$$\sum_{r=1}^{n}\frac{1}{(3r-2)(3r+1)}=\frac{n}{3n+1}$$

(d) If x^2-x-2 is a factor of $x^4+3x^3+ax^2-2x-b$ find a and b.

Question 3 (12 marks) *Start a new page*

(a) The volume of an expanding spherical balloon is increasing at a constant rate of 10 cm^3/s.
Show that $\frac{dr}{dt} = \frac{5}{2\pi r^2}$ and find the rate of increase in its surface area when the balloon's radius is 8cm. 3

(b) If α, β and γ are roots of the equation $x^3 - 6x^2 + 3x + 10$ find the value of $\alpha^2+\beta^2+\gamma^2$. 3

(c) The displacement (in cm) of a particle from 0 on a line after t seconds is given by :

$$x = \sqrt{10}\sin(2t+\alpha) \text{ where } \alpha = \tan^{-1}\frac{1}{3}.$$

(i) Find its initial displacement.
(ii) What is the time lapse between two successive values of x = 0 ?
(iii) What are the maximum and minimum displacement positions?
(iv) Find its acceleration when it is at $x = 2$ for the first time.

6

Question 4 (12 marks) *Start a new page*

(a) (i) Show that the normal at point P $(2ap, ap^2)$ on the parabola $x^2 = 4ay$ has gradient $\frac{-1}{p}$ and determine its equation.

(ii) A line is drawn from the focus S perpendicular to the normal meeting it at Q; show that the equation of SQ is $px - y = -a$.
(iii) Prove that the co-ordinates of Q are $(ap, a(p^2+1))$.
(iv) Hence show that Q is the midpoint of PG where G is the point of intersection of the normal and the axis of the parabola.

9

(b) Show that the equation $2x^3 + x - 8 = 0$ has a root between $x = 1$ and $x = 2$. Use one step of Newton's method to find a closer approximation to the root. 3

Question 5 (12 marks) *Start a new page*

(a) Find the primitive of $\frac{1}{9+4x^2}$ 2

(b) (i) Verify that $\frac{d}{dx}\left(x\sin^{-1}x + \sqrt{1-x^2}\right) = \sin^{-1}x$
Using a similar expression, find the primitive of $\cos^{-1}x$.

(ii) The curves $y = \sin^{-1}x$ and $y = \cos^{-1}x$ intersect at P, the curve $y = \cos^{-1}x$ also intersects the x axis at Q

(α) show that P has co-ordinates $\left(\frac{1}{\sqrt{2}}, \frac{\pi}{4}\right)$

(β) find the area enclosed by the x axis and the arcs OP and PQ, where O is the origin.

10

Question 6 (12 marks) *Start a new page*

A certain particle moves along the x axis according to the law $t = 2x^2 - 5x + 3$, where x is measured in centimetres and t in seconds. Initially the particle is 1.5 cm to the right of the origin O and moving away from O

(i) Prove that the velocity, v cm/s is given by $v = \frac{1}{4x-5}$

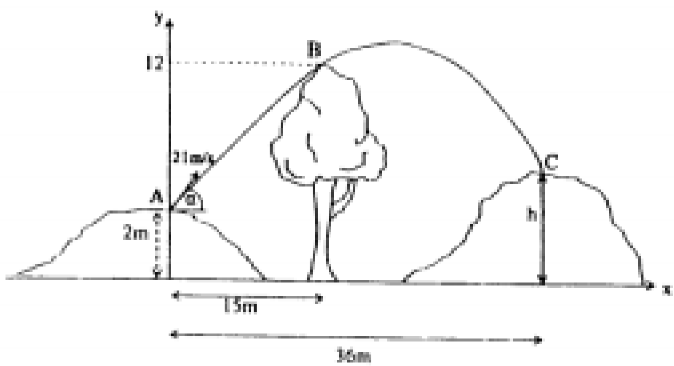
(ii) Find an expression for the acceleration, a cm/s^2, in terms of x.

(iii) Find the velocity of the particle when t = 6 seconds.

5

Question 6 continued

(b)



A golf ball is projected from point A on the top of elevated ground 2 metres high with a speed of 21m/s and at an angle $\alpha < 50°$ to the horizontal, aiming to reach a point C on top of a hill.
The horizontal distance separating A and C is 36metres.
In the course of its trajectory the ball just clears a point B which is the top of a tree 12 metres high and 15 metres away from A.

(Assuming there is no air resistance and $g = 9.8 m/s^2$)

i) Using axes, as shown, show that the cartesian equation of the path in terms of α is

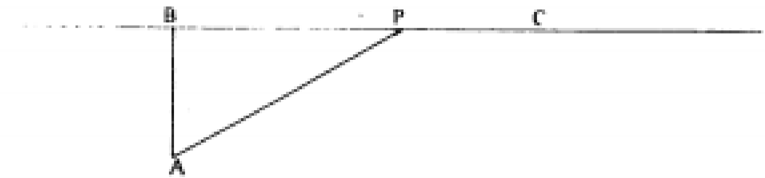
$$y = \frac{-x^2}{90}(1 + \tan^2\alpha) + x\tan\alpha + 2$$

ii) Find the value of the angle of projection α.

iii) Find the height in metres of the point C.

iv) Find the maximum height reached by the ball.

QUESTION 7 (12 MARKS)



The diagram shows a straight road BC running due East. A four-wheel drive ambulance is in open country at A, 3 km due South of B. It must reach C, 9 km East of B, as quickly as possible.

The driver knows that she can travel at 80 km per hour in open country and at 100 km per hour along the road. She intends to proceed in a straight line to some point P on the road and then to continue along the road to C. She wishes to choose P so that total time for the journey APC is a minimum.

(a) If the distance BP is x km, derive an expression for $t(x)$, the total journey time from A to C via P, in terms of x. 6

(b) Show that the minimum time for the total journey APC is $6\frac{3}{4}$ minutes. 6

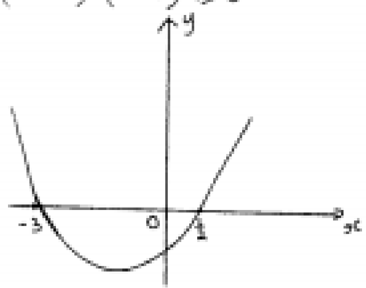
END OF EXAMINATION

3 unit Trial 2010

Solutions

1(a) $2x^2+5x-3\ge 0$

$(2x-1)(x+3)\ge 0$



Soln: $x\le -3$ or $x\ge \frac{1}{2}$ (2)

(b) $\frac{4x-1}{x-2}\le 3$ $\quad x\ne 2$

let $\frac{4x-1}{x-2}=3$

$4x-1=3x-6$

$x=-5$

Critical points: -5, 0, 2

Sub $x=0$: $\frac{0-1}{0-2}\le 3$ T

Soln: $-5\le x<2$ (3)

(c) $\int_0^{\frac{\pi}{4}}\cos x\sin^2 x\,dx$

let $u=\sin x$

$\frac{du}{dx}=\cos x$

When $x=\frac{\pi}{4}$, $u=\frac{1}{\sqrt2}$

$x=0$, $u=0$

$\int_0^{\frac{\pi}{4}}\cos x\sin^2 x\,dx=\int_0^{\frac{1}{\sqrt2}}u^2\frac{du}{dx}dx$

$=\left[\frac{u^3}{3}\right]_0^{\frac{1}{\sqrt2}}$

$=\frac{1}{3}\left[\frac{1}{2\sqrt2}-0\right]$

$=\frac{1}{6\sqrt2}\times\frac{\sqrt2}{\sqrt2}$

$=\frac{\sqrt2}{12}$ (3)

(d) $\int x^2\sqrt{1+3x^3}\,dx$ $\quad u=1+3x^3$, $\frac{du}{dx}=9x^2$

$=\frac{1}{9}\int u^{\frac12}\frac{du}{dx}dx$

$=\frac{1}{9}\int u^{\frac12}du$

$=\frac{1}{9}\cdot\frac{2}{3}u^{3/2}+c$

$=\frac{2}{27}u^{3/2}+c$

$=\frac{2}{27}\sqrt{(1+3x^3)^3}+c$ (3)

(e) $\tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|$ $\quad m_1=2$, $m_2=-\frac{3}{2}$

$=\left|\frac{-\frac32-2}{1+2\cdot-\frac32}\right|$

$=\frac{-\frac72}{-2}$

$\tan\theta=\frac{7}{4}$

$\theta=60°15'$ to nearest minute (2)

2(a) $P(6,9)$ $\quad -3:2$ $\quad A(1,4)$

$x=\frac{mx_2+nx_1}{m+n}$ $\quad y=\frac{my_2+ny_1}{m+n}$

$6=\frac{-3.x_2+2.1}{-1}$ $\quad 9=\frac{-3y_2+2.4}{-1}$

$-3x_2+2=-6$ $\quad -9=-3y_2+8$

$-3x_2=-8$ $\quad 3y_2=17$

$x_2=2\frac{2}{3}$ $\quad y_2=5\frac{2}{3}$

$B:\ (2\frac23,\ 5\frac23)$ (2)

(b) Prove $\frac{2\cos A}{\csc A-2\sin A}=\tan 2A$

LHS $=\frac{2\cos A}{\frac{1}{\sin A}-2\sin A}$

$=\frac{2\cos A}{\frac{1-2\sin^2A}{\sin A}}$

$=\frac{2\sin A\cos A}{1-2\sin^2A}$

$=\frac{\sin 2A}{\cos 2A}$

$=\tan 2A$

$=$ RHS

$\therefore \frac{2\cos A}{\csc A-2\sin A}=\tan 2A$ (3)

c. Prove: $\sum_{r=1}^{n}\frac{1}{(3r-2)(3r+1)}=\frac{n}{3n+1}$

Step 1. Let $n=1$, LHS $=\frac{1}{(3-2)(3+1)}=\frac14$

RHS $=\frac{1}{3.1+1}=\frac14$

$\therefore$ Formula true for $n=1$

Step 2. Assume true for $n=k$.

i.e. $S_k=\frac{1}{(3-2)(3+1)}+\frac{1}{(6-2)(6+1)}+\frac{1}{(9-2)(9+1)}+\ldots$

Show true for S_{k+1}

$S_{k+1}=S_k+\frac{1}{[3(k+1)-2][3(\ldots}$

$=\frac{k}{3k+1}+\frac{1}{(3k+1)(3k\ldots}$

$=\frac{3(3k+4)+1}{(3k+1)(3k+4)}$

$=\frac{3k^2+4k+1}{(3k+1)(3k+4)}$

$=\frac{(k+1)(3k+1)}{(3k+1)(3k+4)}$

$=\frac{k+1}{3(k+1)+1}$ ...

Hence, if it is true for $n=k$, i... for $n=k+1$

Step 3 It is true for $n=1$ and ... it is true for $n=2$ ... on. Hence it is true ... n, a positive integer

(d) $x^2-x-2=(x+1)(x-2)$

$P(x)=x^4+3x^3+ax^2\ldots$

$P(-1)=1-3+a+2$

$=a-b$

Now $a-b=0$ or $a=\ldots$

$P(2)=16+24+4a-4\ldots$

$=4a-b+36$

Now $4a-b+36=0$ or $4a\ldots$

From (1) $4a-a=-36$

$3a=-36$

(i) $V = \frac{4}{3}\pi r^3 \qquad \frac{dV}{dr} = 4\pi r^2$

$\frac{dr}{dt} = \frac{dr}{dV} \cdot \frac{dV}{dt}$

$= \frac{1}{4\pi r^2} \cdot 10$

$\therefore \frac{dr}{dt} = \frac{5}{2\pi r^2}$

$\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt} \qquad A = 4\pi r^2, \ \frac{dA}{dr} = 8\pi r$

$= 8\pi r \cdot \frac{5}{2\pi r^2}$

$= \frac{20}{r}$

$= 2\cdot 5$ when $r = 8$ (3)

Rate of increase $= 2\cdot 5 \text{ cm}^2/\text{s}$.

(b) $P(x) = x^3 - 6x^2 + 3x + 10$

$\alpha + \beta + \gamma = 6$

$\alpha\beta + \beta\gamma + \alpha\gamma = 3$

$\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\alpha\gamma)$

$= 36 - 2 \times 3$

$= 30$ (3)

3 (c)(i) $\ddot{x} = \sqrt{10}\sin(2t+\alpha)$

$t = 0, \ x = \sqrt{10}\sin\alpha$

$= \sqrt{10} \times \frac{1}{\sqrt{10}}$

Initial displacement: 1 (1)

(ii) Let $\sqrt{10}\sin(2t+\alpha) = 0$

$\sin(2t+\alpha) = 0$

$2t + \alpha = 0, \pi, 2\pi, \ldots$

$2t = -\alpha, \pi - \alpha, 2\pi - \alpha$

$t = \frac{\pi}{2} - \frac{\alpha}{2}, \quad \pi - \frac{\alpha}{2}$

$\therefore$ Time lapse $= \frac{\pi}{2}$ s. 1.57

(iii) $-1 \le \sin(2t+\alpha) \le 1$

$-\sqrt{10} \le \sqrt{10}\sin(2t+\alpha) \le \sqrt{10}$

Min. displacement $-\sqrt{10}$ cm

Max. $\sqrt{10}$ cm

OR

amplitude $\sqrt{10}$

Min: $\sqrt{10}$ ... Max: $\sqrt{10}$

(iv) $\frac{dx}{dt} = 2\sqrt{10}\cos(2t+\alpha)$

$\frac{d^2x}{dt^2} = -4\sqrt{10}\sin(2t+\alpha)$

Let $x = 2, \ \sqrt{10}\sin(2t+\alpha) = 2$

$\sin(2t+\alpha) = \frac{2}{\sqrt{10}}$

$a = -4\sqrt{10}\sin(2t+\alpha)$

$= -4\sqrt{10} \cdot \frac{2}{\sqrt{10}}$

$= -8 \text{ cm/s}^2$ (6)

OR $\ddot{x} = -n^2 x = -4 \times 2 = -8$

4 (a)(i) $y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{x}{2a}$

$= \frac{2ap}{2a}$ at $P(2ap, ap^2)$

$= p$

$\therefore$ gradient tangent $= p$

gradient normal $= -\frac{1}{p}$

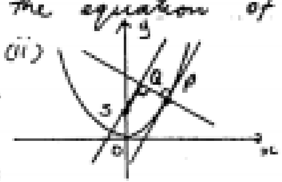
$y - y_1 = m(x - x_1)$

$y - ap^2 = -\frac{1}{p}(x - 2ap)$

$py - ap^3 = -x + 2ap$

$\therefore py + x = 2ap + ap^3$ is the equation of the normal.

(ii)



gradient of SQ $= p$

$y - y_1 = m(x - x_1)$

$y - a = p(x - 0)$

$\therefore$ Equation of SQ: $px - y = -a$

(iii) $py + x = 2ap + ap^3$ — ①

$px - y = -a$ — ②

From ② $y = px + a$ ... sub into ①

$p(px + a) + x = 2ap + ap^3$

$p^2x + pa + x = 2ap + ap^3$

4 (b) $f(x) = x^3 + x - 6$

$f(1) = 1 + 1 - 6 < 0$

$f(2) = 8 + 2 - 6 > 0$

$\therefore$ root bet. $x = 1, x = 2$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$f'(x) = 3x^2 + 1$

$x_1 = \frac{3}{2}$

$= 1.4$ (2)

iv $G: x = 0, \ py + x = 2ap + ap^3$

$py = 2ap + ap^3$

$y = 2a + ap^2$

$G: (0, a(2+p^2))$

Midpoint of PG:

$x = \frac{2ap + 0}{2}, \quad y = \frac{ap^2 + a(2+p^2)}{2}$

$\therefore x = ap \qquad = \frac{a(p^2 + 2 + p^2)}{2}$

$y = a(p^2 + 1)$

$\therefore$ the coords of Q

$\therefore$ Q is the midpoint of PG (9)

5 (a) $\int \frac{1}{9+4x^2}\,dx = \frac{1}{4}\int \frac{1}{\left(\frac{9}{4} + x^2\right)}\,dx$

$= \frac{1}{4} \cdot \frac{2}{3}\tan^{-1}\frac{2x}{3} + C$

(2) $= \frac{1}{6}\tan^{-1}\frac{2x}{3} + C$

(b)(i) Let $y = x\sin^{-1}x + \sqrt{1-x^2}$

$= x\sin^{-1}x + (1-x^2)^{\frac{1}{2}}$

$\frac{dy}{dx} = \sin^{-1}x + x\frac{1}{\sqrt{1-x^2}} + \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \cdot -2x$

$= \sin^{-1}x$

The derivative of $\cos^{-1}x$ is $\frac{-1}{\sqrt{1-x^2}}$. Using the result above, $\frac{d}{dx}(x\cos^{-1}x - \sqrt{1-x^2})$ must be $\cos^{-1}x$.

Hence the primitive of $\cos^{-1}x$ is $x\cos^{-1}x - \sqrt{1-x^2}$

(ii)(α) If $y = \sin^{-1}x, \ \sin y = x$

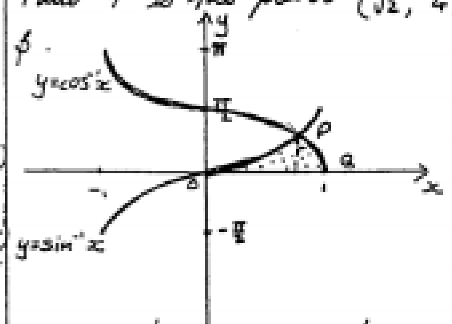
If $y = \cos^{-1}x, \ \cos y = x$

At P: $\sin y = \cos y$

$x = \sin y = \cos y = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

Thus P is the point $\left(\frac{1}{\sqrt{2}}, \frac{\pi}{4}\right)$

β



Area $= \int_0^{\frac{1}{\sqrt{2}}} \sin^{-1}x\,dx + \int_{\frac{1}{\sqrt{2}}}^{1} \cos^{-1}x\,dx$

$= \left[x\sin^{-1}x + \sqrt{1-x^2}\right]_0^{\frac{1}{\sqrt{2}}} + \left[x\cos^{-1}x - \sqrt{1-x^2}\right]$

$= \left[\frac{1}{\sqrt{2}}\cdot\frac{\pi}{4} + \sqrt{1-\frac{1}{2}} - \sqrt{1}\right] + \left[1 \cdot 0 - \sqrt{0} - \frac{1}{\sqrt{2}}\frac{\pi}{4} + \ldots\right]$

$= 2 \cdot \frac{1}{\sqrt{2}} - 1$

$= \sqrt{2} - 1$ units²

6 (a)(i) $t = 2x^2 - 5x + 3$

$\frac{dt}{dx} = 4x - 5$

$\therefore \frac{dx}{dt} = \frac{1}{4x-5}$

i.e., $v = \frac{1}{4x-5}$

(ii) $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$\frac{1}{2}v^2 = \frac{1}{2}\left(\frac{1}{4x-5}\right)^2$

$= \frac{1}{2}(4x-5)^{-2}$

$a = -2 \cdot \frac{1}{2} \cdot 4(4x-5)^{-3}$

$= \frac{-4}{(4x-5)^3}$

(iii) $t = 6$: $2x^2 - 5x + 3 = 6$

$2x^2 - 5x - 3 = 0$

6(a)(iii) cont.

$\frac{1}{4x-5} \neq 0$

particle does not change direction. As particle is initially at 1.5 cm to the right and moves away from the origin $x = 3$ only.

At $x = 3$, $v = \frac{1}{4.3-5}$

$\therefore$ velocity $= \frac{1}{7}$ cm s^{-1} when $t = 6$ s.

(5)

(i) $\ddot{x} = 0$ $\quad$ $\ddot{y} = -g$

$\dot{x} = V\cos\alpha$ $\quad$ $\dot{y} = -gt + V\sin\alpha$

$x = V\cos\alpha\, t$ $\quad$ $y = -\frac{1}{2}gt^2 + V\sin\alpha\, t$

$t = \frac{x}{V\cos\alpha}$, sub for y

$\therefore y = -\frac{1}{2}\frac{gx^2}{V^2\cos^2\alpha} + \frac{Vx\sin\alpha}{V\cos\alpha} + 2$

$= x\tan\alpha - \frac{gx^2}{2V^2\cos^2\alpha} + 2$

$= x\tan\alpha - \frac{gx^2}{2V^2}(1+\tan^2\alpha) + 2$

$V = 21$, $g = 9.8$

$y = x\tan\alpha - \frac{9.8}{2\times21^2}x^2(1+\tan^2\alpha) + 2$

$= x\tan\alpha - \frac{x^2}{90}(1+\tan^2\alpha) + 2$

(ii) Sub $x = 15$, $y = 12$ into equation of path

$12 = -\frac{15^2}{90}(1+\tan^2\alpha) + 15\tan\alpha + 2$

$900 = -225 - 225\tan^2\alpha + 1350\tan\alpha$

$1125 = -225\tan^2\alpha + 1350\tan\alpha$

$225(\tan^2\alpha - 6\tan\alpha + 5) = 0$

$\tan\alpha = 1$ or $\tan\alpha = 5$

$\alpha = 45°$ $\quad$ $\alpha = 78°41'$

But $\alpha < 50°$ (condition)

$\therefore \alpha = 45°$

(iii) For $\alpha = 45°$, $y = \frac{-x^2}{90}(1+1) + x + 2$

$= \frac{-x^2}{45} + x + 2$

For $x = 36$, $y = -\frac{36^2}{45} + 36 + 2 = 9.2$ m

$\therefore$ C is 9.2 m above the ground.

(iv) $\frac{dy}{dx} = 0$ for max height

i.e. $\frac{-2x}{45} + 1 = 0$, $x = 22\frac{1}{2}$

$\frac{d^2y}{dx^2} < 0$ for all x

$\therefore$ Max when $x = 22\frac{1}{2}$

i.e. Max. height $= 13.25$ m. (7)

7.(a)

$t(x) = \frac{\sqrt{x^2+9}}{80} + \frac{9-x}{100}$

(6)

(b) $t(x) = \frac{(x^2+9)^{\frac{1}{2}}}{80} + \frac{9-x}{100}$

$t'(x) = \frac{\frac{1}{2}(x^2+9)^{-\frac{1}{2}}\cdot 2x}{80} - \frac{1}{100}$

$= \frac{x}{80\sqrt{x^2+9}} - \frac{1}{100}$

Let $t'(x) = 0$, $\frac{x}{80\sqrt{x^2+9}} = \frac{1}{100}$

$10x = 8\sqrt{x^2+9}$

$5x = 4\sqrt{x^2+9}$

$25x^2 = 16x^2 + 144$

$9x^2 = 144$

$x^2 = \frac{144}{9}$

$x = 4$ $\quad$ $(x > 0)$

(6)

x	3	4	5
$t'(x)$	−	0	+
Min	when $x = 4$		

$t(x) = \frac{\sqrt{16+9}}{80} + \frac{9-4}{100}$

$= \frac{5}{80} + \frac{5}{100}$

$= \frac{9}{80}$ h

$= \frac{9}{80} \times 60$ ...