Question 1:

a) Evaluate $\displaystyle\int_0^3 \frac{dx}{9+x^2}$ giving your answer in exact form. 2

b) Use the table of standard integrals to find the exact value of 2

$$\int_0^4 \frac{dx}{\sqrt{9+x^2}}$$

c) Show that $\dfrac{1-\cos 2\theta}{\sin 2\theta} = \tan\theta$. 4

Hence express $\tan\ 67\frac{1}{2}^0$ in simplest form.

d) Solve the inequality $x \geq \dfrac{1}{x}$ 4

Question 2:

Marks:

a) Find the acute angle between the lines $y = \frac{1}{3}x + 3$ and $y = \frac{-2}{3}x + 3$. 2

Give your answer in radians correct to two decimal places.

b) The polynomial $x^3 - 3x + 1 = 0$ has roots α, β and γ. Find the exact value of $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$ 3

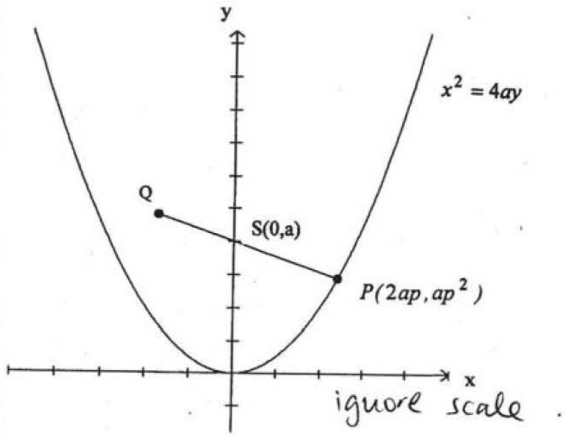
c) i) By using graphs or otherwise show that the curves $y = \ln x$ and $y = 2 - x$ have a point of intersection for which the x co-ordinate is close to 1.5. 1

ii) Use $x = 1.5$ and one application of Newton's method to find a better approximation of the x co-ordinate of this point of intersection. Give answer correct to two decimal places. 2

d) Use the substitution $u = x - 1$ to evaluate $\displaystyle\int_2^5 \frac{x+1}{\sqrt{x-1}}dx$ 4

Question 3:

a)



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In the diagram above $P(2ap, ap^2)$ is a point on the parabola $x^2 = 4ay$. The point Q lies on PS produced such that Q divides PS externally in the ratio 3:2.

i) Prove that Q has co-ordinates $(-4ap, a(3-2p^2))$ 2

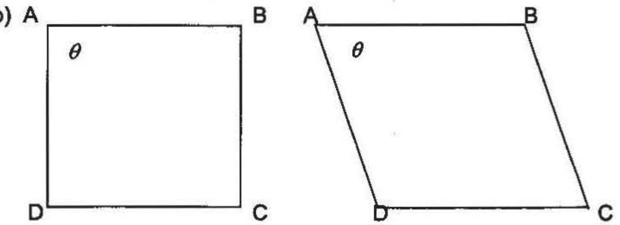
ii) Show that as P varies the locus of Q is another parabola. Find its equation and write down the co-ordinates of its vertex and focus in terms of a. 5

b) Prove by mathematical induction that $\sin(x+n\pi) = (-1)^n \sin x$ where n is a positive integer. 5

Question 4: **Marks:**

a) Solve for $x : \log_{\frac{1}{2}}\left(\frac{1}{x}\right) \geq \log_2(3x-1)$ 3

b)



A square ABCD of side 1 unit is gradually 'pushed over' to become a rhombus. The angle at A (θ) decreases at a constant rate of 0.1^c/ second.

i) Show that the area of the rhomubus is equal to $\sin\theta$ 1

ii) At what rate is the area of the rhombus ABCD decreasing when $\theta = \frac{\pi}{6}$? (Give answer correct to 2 decimal places). 2

iii) At what rate is the shorter diagonal of the rhombus ABCD decreasing when $\theta = \frac{\pi}{3}$? (Give answer correct to 2 decimal places). 3

c) Prove that $\frac{\sin 3\theta}{\sin\theta} - \frac{\cos 3\theta}{\cos\theta} = 2$ (for $\sin\theta \neq 0$, $\cos\theta \neq 0$) 3

Question 5:

a)

i) Express $\sqrt{3}\ \cos 2t - \sin 2t$ in the form $R\cos(2t+\alpha)$ where 2

$0 < \alpha < \dfrac{\pi}{2}$

ii) Hence or otherwise find all positive solutions of 3

$\sqrt{3}\cos 2t - \sin 2t = 0$

b) A particle moves in a straight line and is x metres from a fixed point O after t seconds where:

$x = 5 + \sqrt{3}\cos 2t - \sin 2t$

i) Prove that the acceleration of the particle is $-4(x-5)$. 3

ii) Between which two points does the particle oscillate. 2

(You may use your answers from part (a))

iii) At what times does the particle first pass through the point x=5. 2

Question 6:

Marks:

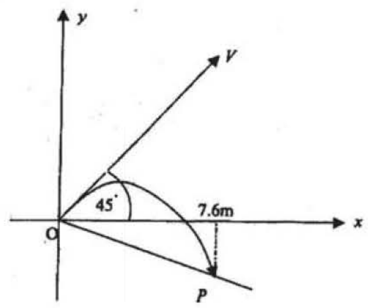
a) The acceleration at any time t of a body moving in a straight line is $-e^{-2x}$.

When t=0, x=0, and v=1.

i) Express its velocity v in terms of x. 3

ii) Express its displacement x in terms of time t. 3

b) A garden hose placed at the top of an incline releases a stream of water with a velocity of 8m/s at an angle of 45^0 with the horizontal. Assuming that $x = vt\cos\alpha$ and $y = vt\sin\alpha - \frac{1}{2}gt^2$ where x and y are the horizontal and vertical displacements of the stream of water from O at any time, $g = 10m/s^2$ and the coordinate axes are taken as shown.



i) Show that the equation of the path of the stream of water is given by $y = x - \dfrac{5x^2}{32}$ 3

ii) If the stream of water strikes the incline at the point P, 7.6m horizontally from O, find the equation of the incline. 3

Question 7:

a) During the early summer months the rate of increase of the population P of fruit flies is proportional to the excess of the population over 3000.

$\frac{dP}{dt} = k(P - 3000)$ where k is a constant. At the beginning of summer the population is 4000 and 1 month later it is 10 000.

i) Show that $P = 3000 + Ae^{kt}$ is a solution of the differential equation, A is a constant. 1

ii) Find the value of A and that of k. 2

iii) Find to the nearest 100, the population after 2 ½ months. 1

iv) After how many weeks does the population reach ½ million? (Give your answer to 1 decimal place). 1

b) Consider the function $y = x^3e^{-x}$

i) State the greatest possible domain of the function. 1

ii) Find the maximum value of the function in the domain. 2

iii) Show that there are 3 points of inflexion and that one of them has a horizontal tangent. 3

iv) Sketch the curve for $-1 \le x \le 6$ 1

END OF EXAMINATION

TRIAL HSC 2003 - Extension One
SOLUTIONS

COMMENTS

Question 1.

a) $\displaystyle\int_0^3 \frac{dx}{9+x^2} = \frac{1}{3}\tan^{-1}\frac{x}{3}\Big]_0^3$ ✓

$= \frac{1}{3}\tan^{-1}1 - \frac{1}{3}\tan^{-1}0$

$= \frac{1}{3}\times\frac{\pi}{4}$

$= \frac{\pi}{12}$ ✓

- some extension II students made this a difficult integral.
- some forgot the $\frac{1}{3}$ at the front
- answer needs to be in degrees

b) $\displaystyle\int_0^4 \frac{dx}{\sqrt{9+x^2}} = \ln\left(x+\sqrt{x^2+9}\right)\Big]_0^4$ ✓

$= \ln 9 - \ln 3$

$= \ln 3$ ✓

- Some students did not use table of integral
- another error $I = \ln(3+\sqrt{9+x^2})$

c) $\displaystyle\frac{1-\cos 2\theta}{\sin 2\theta} = \frac{1-(1-2\sin^2\theta)}{2\sin\theta\cos\theta}$ ✓

$= \frac{2\sin^2\theta}{2\sin\theta\cos\theta}$

$= \tan\theta$ ✓

$\tan 67\frac{1}{2}^\circ = \frac{1-\cos 135^\circ}{\sin 135^\circ}$

$= \frac{1+\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}$ ✓ ✓

$= \sqrt{2}+1$

- generally well done.
- some need to revise exact values.

d)

COMMENT.

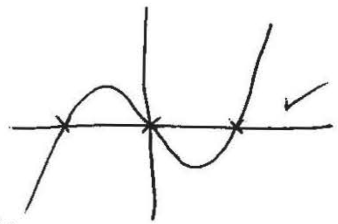
1d) $x \geq \frac{1}{x}$ $\quad x \neq 0.$

$x^3 \geq x$

$x^3 - x \geq 0$ ✓

$x(x^2-1) \geq 0$

$x(x-1)(x+1) \geq 0.$ ✓



$-1 \leq x < 0$ or $x \geq 1$ ✓

OR

If $x > 0$

$x^2 \geq 1$ ✓

$\therefore x \geq 1$ or $x \leq -1$

$\therefore$ Soln $x \geq 1.$ ✓

If $x < 0$

$x^2 \leq 1$ ✓

$-1 \leq x \leq 1.$

$\therefore$ Soln $-1 \leq x < 0.$ ✓

- generally poorly done.

Need to solve by

a) examining critical pts

or

b) mult. b.s. by x^2.

or

c) use two cases - give 2 partial solns and then an o'all soln.

SOLUTIONS

COMMENTS

Question 2.

a) $m_1 = \frac{1}{3}$ $m_2 = -\frac{2}{3}$

$$\tan\theta = \frac{\frac{1}{3} + \frac{2}{3}}{1 + \frac{1}{3} \times -\frac{2}{3}}$$

$$= \frac{1}{1 - \frac{2}{9}}$$

$$= \frac{9}{7}$$ ✓

$\theta = 0.91^c$ to two decimal places. ✓

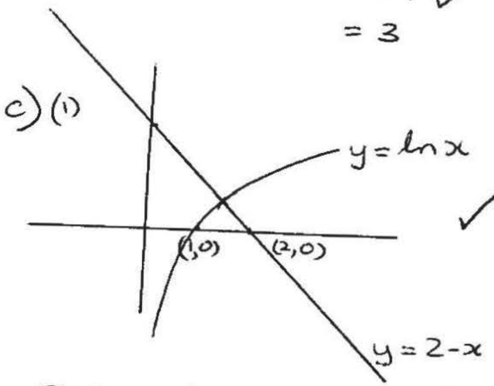
$-\frac{1}{2}$ for error finding approximate value of $\tan^{-1}\frac{9}{7}$. No penalty if in degrees.

b) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$ ✓

$$= \frac{-3}{-1}$$ ✓ ✓

$$= 3$$

c) (i)



✓

Intersection point close to $x = 1.5$

(ii) $P(x) = \ln x - 2 + x$

$P'(x) = \frac{1}{x} + 1$

$\therefore x_2 = 1.5 - \left(\frac{\ln 1.5 - 0.5}{1\frac{2}{3}}\right)$ ✓

$= 1.56$ correct to 2 dec. places ✓

N.

d) $\int_2^5 \frac{x+1}{\sqrt{x-1}}\,dx$

$u = x - 1$

$du = dx$

$= \int_1^4 \frac{u+2}{\sqrt{u}}\,du$

① - correct substitution ✓

① - correct limits

$= \int_1^4 \left(\sqrt{u} + 2u^{-1/2}\right) du$ ✓

$= \left[\frac{2}{3}u^{3/2} + 4u^{1/2}\right]_1^4$ ✓

$= \left(\frac{16}{3} + 8\right) - \left(\frac{2}{3} + 4\right)$

$= 8\frac{2}{3}$ ✓

No penalty for small arithmetic error to get $8\frac{2}{3}$

SOLUTIONS.

COMMENTS

Question 3

a) (i) $P(2ap, ap^2)$ $\quad S(0,a)$

$$\therefore Q = \left(\frac{-2\times 2ap + 0}{1}, \frac{-2\times ap^2 + 3a}{1}\right)$$

$$= \left(-4ap,\ a(3-2p^2)\right)$$

(ii) From $x = -4ap$

$$p = \frac{x}{-4a}$$

$$y = a\left(3 - 2\left(\frac{x^2}{16a^2}\right)\right)$$

$$= 3a - \frac{x^2}{8a}$$

$$\frac{x^2}{8a} = -y + 3a$$

$$x^2 = -8a(y-3a)$$

Any variation of this quadratic was marked correct.

This is another parabola

Vertex $(0, 3a)$

Focal length is $2a$. $\therefore$ Focus $(0,a)$

Many Students did not find focal length $4l = -8a$, $l = -2a$

b) If $n=1$. $\sin(x+\pi) = \sin x\cos\pi + \cos\pi\sin x$

$$= -\sin x + 0$$

$$= (-1)^1 \sin x$$

$\therefore$ True for $n=1$

Assume true for $n=k$

ie $\sin(x+k\pi) = (-1)^k \sin x$

Consider $n=k+1$

$$\sin(x+(k+1)\pi) = \sin(x+k\pi+\pi)$$

$$= \sin(x+k\pi)\cos\pi + \cos(x+k\pi)\sin\pi$$

$$= (-1)^k \sin x \times -1 + 0$$

$$= (-1)^{k+1}\sin x$$

which is of same form as for $n=k$.

$\therefore$ If true for $n=k$, it is also true for $n=k+1$. Since it is true for $n=1$, it is true for $n=2$ and hence all following positive integers.

One mark for both Step 1 and Step 2.
3 marks for Step 3

SOLUTIONS — COMMENTS

Question 4.

a) $\log_{\frac{1}{2}} \frac{1}{x} \geq \log_2 (3x-1)$

$\frac{\log_2 \frac{1}{x}}{\log_2 \frac{1}{2}} \geq \log_2 (3x-1)$ ✓

$\frac{\log_2 \frac{1}{x}}{-1} \geq \log_2 (3x-1)$

$\log_2 x \geq \log_2 (3x-1)$ ✓

$x \geq 3x-1$

$2x-1 \leq 0$

$x \leq \frac{1}{2}$

But $x > \frac{1}{3}$ since $3x-1>0$.

$\therefore$ soln $\frac{1}{3} \leq x \leq \frac{1}{2}$ ✓

Many Problems Not Many people realised

$3x-1>0$

$\therefore x > \frac{1}{3}$

Straight away

$\boxed{/3}$

b)(i) Area $= 2 \times \left(\frac{1}{2} \times 1 \times 1 \times \sin\theta\right)$

$= \sin\theta$ ✓

Good

(ii) $\frac{dA}{dt} = \frac{dA}{d\theta} \cdot \frac{d\theta}{dt}$

$= \cos\theta \times -0.1^c$ ✓

when $\theta = \frac{\pi}{6}$ $\frac{dA}{dt} = \frac{\sqrt{3}}{2} \times -0.1$

$= -0.09\ u^2/s.$ ✓

Ok

Substitution caused problems.

(iii) $l^2 = 1^2 + 1^2 - 2 \times 1 \times 1 \times \cos\theta$

$= 2 - 2\cos\theta$

$l = \sqrt{2}\,(1-\cos\theta)^{1/2}$ ✓

Ok

$\frac{dl}{dt} = \frac{dl}{d\theta} \cdot \frac{d\theta}{dt}$

$= \left(\sqrt{2} \times \frac{1}{2} \times (1-\cos\theta)^{-1/2} \times \sin\theta\right) \times -0.1$ ✓

At $x = \frac{\pi}{3}$ $\frac{dl}{dt} = \sqrt{2} \times \frac{1}{2} \times \sqrt{2} \times \frac{\sqrt{3}}{2} \times -0.1$

$= -0.09\ u/s.$ ✓

Differentiation of l caused problems

A lot of people left off the $\sin\theta$ from chain rule

$\boxed{/6}$

c) $\frac{\sin 3\theta}{\sin\theta} - \frac{\cos 3\theta}{\cos\theta}$

$= \frac{\sin\theta\cos 2\theta + \cos\theta\sin 2\theta}{\sin\theta} - \frac{\cos\theta\cos 2\theta - \sin\theta\sin 2\theta}{\cos\theta}$ ✓

$= \cos 2\theta + \frac{\cos\theta}{\sin\theta}(\sin 2\theta) - \cos 2\theta + \frac{\sin\theta}{\cos\theta} \cdot \sin 2\theta$ ✓

$= 2\sin\theta\cos\theta\left(\frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta}\right)$ ✓

$= 2.$

Well done Overall.

$\boxed{/3}$

SOLUTIONS | COMMENTS.

Question 5.

a) (i) $\sqrt{3}\cos 2t - \sin 2t = 2\cos(2t+\alpha)$ ✓

$$\begin{bmatrix} \tan\alpha = \frac{1}{\sqrt{3}} \\ \alpha = \frac{\pi}{6} \end{bmatrix}$$

$$= 2\cos\left(2t+\frac{\pi}{6}\right)$$ ✓

(ii) $\cos\left(2t+\frac{\pi}{6}\right) = 0.$

$$2t+\frac{\pi}{6} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

$$2t = \frac{\pi}{3}, \frac{4\pi}{3}, \frac{7\pi}{3}, \ldots$$

$$t = \frac{\pi}{6}, \frac{4\pi}{6}, \frac{7\pi}{6}, \ldots$$

$$= \frac{(3n+1)\pi}{6} \quad \text{for } n = 0, 1, 2, \ldots$$

Many did not know general solution Either this way OR

$2t+\frac{\pi}{6} = 2n\pi \pm \frac{\pi}{2}$

$\therefore 2t = 2n\pi - \frac{\pi}{6} \pm \frac{\pi}{2}$

$t = n\pi - \frac{\pi}{12} \pm \frac{\pi}{4}$

b) (i) $x = 5 + \sqrt{3}\cos 2t - \sin 2t$

$$\dot{x} = -2\sqrt{3}\sin 2t - 2\cos 2t$$ ✓

$$\ddot{x} = -4\sqrt{3}\cos 2t + 4\sin 2t$$ ✓

$$= -4(\sqrt{3}\cos 2t - \sin 2t)$$

$$= -4(x-5).$$ ✓

Well done.

(ii) Motion is Simple Harmonic Motion ✓ centre of motion is 5, amplitude is 2 from part (a).

$\therefore$ Particle oscillates between 3 and 7

Many did not use part (a) hence made extra work for themselves

(iii) At $x=5$: $5 = 5 + \sqrt{3}\cos 2t - \sin 2t$ ✓

$$0 = \sqrt{3}\cos 2t - \sin 2t$$ ✓

$$\therefore t = \frac{\pi}{6} \text{ (from (ii))}$$

SOLUTIONS | COMMENT.

Question 6.

a) (i) $\ddot{x} = -e^{-2x}$

$$\frac{d}{dx}\frac{1}{2}v^2 = -e^{-2x}$$

$$\frac{1}{2}v^2 = \frac{-e^{-2x}}{-2} + C$$ ✓

$$\frac{v^2}{2} = \frac{e^{-2x}}{2} + C$$

when $v=1$, $x=0$

$$\frac{1}{2} = \frac{1}{2} + C \quad \therefore \quad C = 0.$$ ✓

$$\therefore v^2 = e^{-2x}$$

$$v = \pm\sqrt{e^{-2x}}$$

Since $v=1$ when $x=0$

$$v = \sqrt{e^{-2x}} = e^{-x}$$ ✓

Most students used $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$. performed well on this question. However, many did not explain why v is the positive square root.

(ii) $v = e^{-x}$

$$\frac{dx}{dt} = e^{-x}$$

$$\frac{dt}{dx} = e^{x}$$ ✓

$$t = e^{x} + C$$

when $t=0$, $x=0$ $\therefore$ $C=-1$

$$t = e^{x} - 1$$ ✓

$$e^{x} = t+1$$

$$x = \ln(t+1)$$ ✓

Some students used $\frac{dx}{dt}$ correctly but did not make x the subject. Some used an incorrect formula from (i) and could not achieve a final result.

SOLUTIONS	COMMENTS
c) (i) $x = Vt\cos\alpha$, $y = Vt\sin\alpha - \frac{1}{2}gt^2$	
$g = 10$, $V = 8$, $\alpha = 45^\circ$.	
$\therefore x = 4\sqrt{2}t$, $y = 4\sqrt{2}t - 5t^2$ ✓	Generally performed well
$y = 4\sqrt{2}\left(\frac{x}{4\sqrt{2}}\right) - 5 \cdot \frac{x^2}{32}$ ✓	
$= x - \frac{5x^2}{32}$. ✓	
(ii) Equation of line of incline is $y = mx$. ✓	Some students made this question much more complicated than it was meant to be
$\therefore mx = x - \frac{5x^2}{32}$	
when $x = 7.6$	
$7.6m = 7.6 - \frac{5\times 7.6^2}{32}$ ✓	
$m = 1 - \frac{5\times 7.6}{32}$	
$= -\frac{3}{16}$	
$\therefore y = -\frac{3x}{16}$ is required equation ✓	
$= 0.1875x$	

QUESTION 7 : SOLUTIONS	COMMENTS
a) (i) $P = 3000 + Ae^{kt}$ so $Ae^{kt} = P - 3000$	many students were unclear on their starting point, and what substitutions they were making.
$\frac{dP}{dt} = kAe^{kt}$	
$= k(P - 3000)$ ✓	
(ii) $t = 0$, $P = 4000$	
$4000 = 3000 + Ae^0$	
$\therefore A = 1000$ ✓	
$t = 1$, $P = 10{,}000$	
$\therefore 7000 = 1000e^k$	
$e^k = 7$	
$k = \ln 7$ ✓	
$\doteqdot 1.946$ to 3 dec places.	
(iii) $t = 2.5$	
$P = 3000 + 1000e^{2.5k}$ ✓	
$= 132600$ to nearest 100	
(iv) $500{,}000 = 3000 + 1000e^{kt}$	
$497000 = 1000e^{kt}$	
$e^{kt} = 497$	• many could not convert months to weeks! (and thus lost the mar
$kt = \ln 497$	
$t = 3.19\ldots$ months	
$= 12.8$ weeks ✓	
correct to 1 dec. place.	

SOLUTIONS | COMMENTS

b) (i) $D = \{x : x \in R\}$. ✓

(ii) $y' = -x^3e^{-x} + 3x^2e^{-x}$

$= x^2e^{-x}(-x+3)$

$= 0$ if $x = 0$ or $x = 3$.

$y'' = x^3e^{-x} + e^{-x}\cdot -3x^2 + -3x^2e^{-x} + 6xe^{-x}$

$= xe^{x}(x^2 - 6x + 6)$ ✓

At $x = 0$ $y'' = 0$ ∴ horizontal inflection

$x = 3$ $y'' < 0$ ∴ max at $x = 3$

Maximum value is $3^3e^{-3} = \frac{27}{e^3}$ ✓

(iii) Possible pts of inflection at

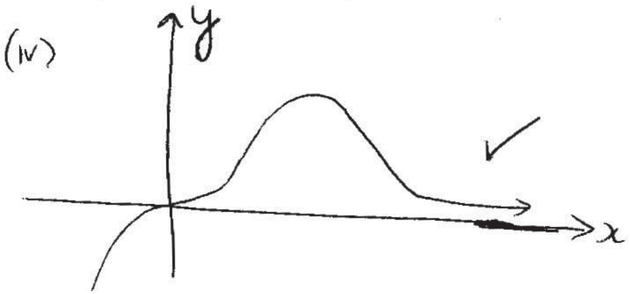
$y'' = 0$ ie $x = 0$ or $x = \frac{6 \pm \sqrt{12}}{2}$ ✓

$= 3 \pm \sqrt{3}$. ✓

At $x = 0$ horizontal inflection

Since changes concavity at each ✓ point - all are points of inflection

(iv)



✓

Comments:

① showing $x = 3$ is a max (any method) - this was often left out.

① max value - many did not read the question and left this out.

① 3 points from $\frac{d^2y}{dx^2} = 0$

① 3 points are correct

① show $x = 0$ is horizontal ⟹ indicate $y' = 0$.

Many lost the graph mark because horizontal inflection at $x = 0$ was not clear!