FORT ST 2007 EXN 1

Total marks – 84
Attempt Questions 1 – 7
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1. (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Simplify $\dfrac{1+a^{-1}}{1+a^{-3}}$ **2**

(b) If $y = \sec x$

i) show that $\dfrac{dy}{dx} = \sec x \tan x$ **2**

ii) find $\dfrac{d^2y}{dx^2}$ in terms of $\sec x$ **2**

(c) Let $A(-3, 6)$ and $B(1, 10)$ be points on a number plane. **2**

Find the coordinates of point C, which divides the interval AB externally in the ratio $5:3$

(d) Evaluate $\displaystyle\int_{-3}^{3} \frac{1}{9+x^2}\,dx$ **2**

(e) If $\dfrac{dy}{dx} = 1 + y$ and when $x = 0$ $y = 2$, find an expression for y in terms of x **2**

Question 2. (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) i) Express $\sqrt{2}\sin x + \sqrt{2}\cos x$ in the form $R\sin(x+\alpha)$, **2**

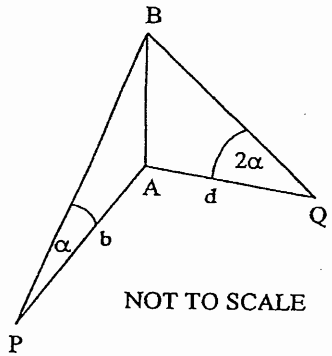
where $R > 0$ and $0 \le \alpha \le \dfrac{\pi}{2}$.

ii) Hence sketch $y = \sqrt{2}\sin x + \sqrt{2}\cos x$ for $0 \le x \le 2\pi$ **2**

(Show intercepts and endpoints clearly).

iii) Hence find the value(s) of k for which $\sqrt{2}\sin x + \sqrt{2}\cos x = k$ has 3 solutions in the domain of $0 \le x \le 2\pi$. **1**

(b) From a point P, a distance of b metres south of a tower AB, the angle of elevation to the top of the tower B is α. From point Q, a distance of d metres due east of the tower the angle of elevation to the top of the tower is 2α.



i) Show that $b\tan\alpha = d\tan 2\alpha$ **2**

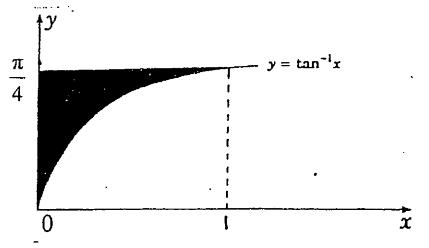
ii) Find the height of the tower in terms of d and b **3**

iii) If the distance PQ is $d\sqrt{10}$ metres find α **2**

Question 3. (12 marks) Use a SEPARATE writing booklet. **Marks**

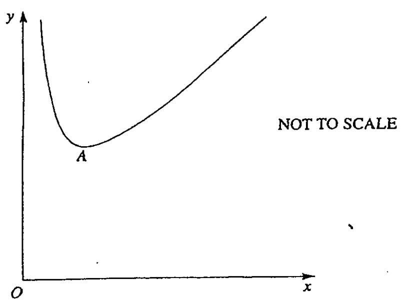
(a) Solve the inequation $\frac{x}{x-3} < 4$ **3**

(b) Show that the shaded area bounded by the curve $y = \tan^{-1} x$ the y-axis and the line $y = \frac{\pi}{4}$ is given by $A = \frac{1}{2}\log_e 2$ units^2. **3**



(c) Consider the function $f(x) = 4x + \frac{1}{x}$ for $x > 0$.

The diagram below shows the graph of the function and the stationary point $A\left(\frac{1}{2}, 4\right)$



i) What is the largest domain for which $f(x)$ has an inverse function $f^{-1}(x)$? **1**

ii) Copy or trace the graph of $y = f(x)$ into your Writing Booklet.
On the same set of axes, draw the graph of $y = f^{-1}(x)$. **2**

iii) Find the inverse function $f^{-1}(x)$. **3**

Question 4. (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Consider the equation $2x^3 + x^2 - 15x - 18 = 0$. **3**

One of the roots of this equation is positive and equals the product of the other two roots.

Find the roots of this equation.

(b) When the polynomial $P(x)$ is divided by $1 - x^2$ it gives $4 - x$ as the remainder. **2**

What is the remainder of $P(x)$ when divided by $1 + x$?

(c) The equation $\sin x = x^2 - 10$ has a root close to $x = \pi$. **2**

Use one application of Newton's Method to give a better approximation.

(correct your answer to 4 decimal places).

(d) Prove by induction that $\cos(x + n\pi) = (-1)^n \cos x$ for integer $n \geq 1$. **5**

Question 5. (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) AB and CD are two intersecting chords of a circle and CD is parallel to the tangent to the circle at B.

(i) Draw a neat sketch of the above information. **1**

(ii) Prove that AB bisects $\angle CAD$. **3**

(b) Two points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$.

(i) Derive the equation of the tangent to the parabola at the point P. **2**

(ii) Find the coordinates of the point of intersection T of the tangents to the parabola at P and Q. **2**

(iii) You are given that the tangents at P and Q intersect at 45°. **2**

Show that $p - q = 1 + pq$, where $p > q$.

(iv) Find the locus of T when the tangents at P and Q intersect as given in (iii). **2**

Question 6. (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) The rate at which a body cools in air is given by the difference between the air temperature, T°, at any time t minutes and the temperature, A°, of the surrounding air.

This rate is given by the differential equation:

$$\frac{dT}{dt} = k(T - A) \qquad \text{where } k \text{ is constant.}$$

i) Show by differentiation that $T = A + Pe^{kt}$, where P is constant, is a solution of the differential equation. **2**

ii) A hot cup of coffee cools from 90°C to 70°C in 8 minutes, the temperature of the air being 22°C. **2**

Find the time required for the cup of coffee to cool to a drinkable temperature of 60°C.

iii) Sketch the graph of T as a function of t and describe the behaviour of T as t becomes large. **2**

(b) A particle moves in a straight line in simple harmonic motion.

The acceleration in metres per second per second is given by $\ddot{x} = 2 - 3x$, where x metres is the displacement of the particle from the origin.

Initially the particle is at $x = 1$ moving with a velocity of $\sqrt{5}$ m/s.

i) Using integration show that the velocity v m/s of the particle is given by **2**

$$v^2 = 4 + 4x - 3x^2$$

ii) Find the amplitude of the motion. **1**

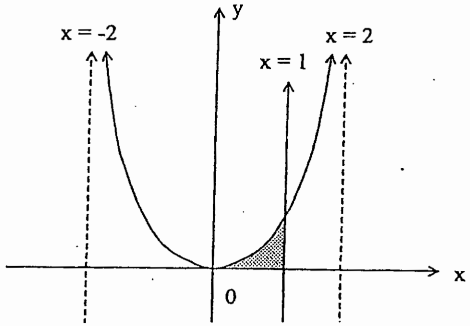
iii) Find the centre of motion. **1**

iv) Find the maximum speed of the particle. **1**

v) Find the period of the motion. **1**

Question 7. (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) The shaded area in the diagram below represents the area bounded by the curve $y = \dfrac{x^2}{\sqrt{4-x^2}}$, the x-axis and the line $x = 1$.



Using the substitution $x = 2\sin\theta$, find the shaded area. **4**

(b) Use the substitution $u = \tan 2x$ to show that $\displaystyle\int_0^{\frac{\pi}{8}} \frac{2\sec^2 2x}{\sqrt{2-\tan^2 2x}}\,dx = \frac{\pi}{4}$ **3**

(c) A projectile is fired horizontally with speed $v\,\text{ms}^{-1}$ from a point h metres above horizontal ground.

i) Prove that it will reach the ground after $\sqrt{\dfrac{2h}{g}}$ seconds. **2**

ii) If it does so at an angle of 60° to the horizontal, show that $3v^2 = 2gh$ **3**

Hint $\dfrac{dy}{dx} = \tan 120^\circ$

SOLUTIONS / MARKING SCHEME

Question 1.

a) $\dfrac{1+a^{-1}}{1+a^{-3}} = \dfrac{1+\frac{1}{a}}{1+\frac{1}{a^3}}$

$= \dfrac{a+1}{a} \times \dfrac{a^3}{a^3+1}$ ✓

$= \dfrac{a^2}{a^2-a+1}$ ✓

Many did not complete factorisation

b) (i) $y = \sec x$

$= (\cos x)^{-1}$

$\dfrac{dy}{dx} = -(\cos x)^{-2} \times -\sin x$ ✓

$= \dfrac{\sin x}{\cos^2 x}$

$= \tan x \sec x$ ✓

(ii) $\dfrac{d^2y}{dx^2} = \tan x(\tan x \sec x) + \sec^3 x$ ✓

$= \sec x(\tan^2 x + \sec^2 x)$

$= 2\sec^3 x - \sec x$. ✓

Show means more than repeat table of standard integrals

c) $C = \left(\dfrac{5\times1 - 3\times-3}{5-3}, \dfrac{5\times10 - 3\times6}{5-3}\right)$ ✓

$= (7, 16)$ ✓

Many still need to learn formula

d) $\displaystyle\int_{-3}^{3} \frac{1}{9+x^2}\,dx = \left[\frac{1}{3}\tan^{-1}\frac{x}{3}\right]_{-3}^{3}$ ✓

$= \frac{1}{3}(\tan^{-1}1 - \tan^{-1}-1)$

$= \frac{1}{3}\left(\frac{\pi}{4} - -\frac{\pi}{4}\right)$

$= \dfrac{\pi}{6}$ ✓

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Question 1.

e) $\dfrac{dy}{dx} = 1+y$

$\dfrac{dx}{dy} = \dfrac{1}{1+y}$

$x = \ln(1+y) + C$ ✓

$x = 0,\ y = 2 \therefore C = -\ln 3$

$x = \ln\left(\dfrac{1+y}{3}\right)$

$e^x = \dfrac{1+y}{3}$

$y = 3e^x - 1.$ ✓

SOLUTIONS / MARKING SCHEME

18

Question 1.

a) $\dfrac{1+a^{-1}}{1+a^{-3}} = \dfrac{1+\frac{1}{a}}{1+\frac{1}{a^3}}$

$= \dfrac{a+1}{a} \times \dfrac{a^3}{a^3+1}$ ✓

$= \dfrac{a^2}{a^2-a+1}$ ✓

Many did not complete calculation

b) (i) $y = \sec x$

$= (\cos x)^{-1}$

$\dfrac{dy}{dx} = -(\cos x)^{-2} \times -\sin x$ ✓

$= \dfrac{\sin x}{\cos^2 x}$

$= \tan x \sec x$ ✓

(ii) $\dfrac{d^2y}{dx^2} = \tan(\tan x \sec x) + \sec^3 x$ ✓

$= \sec x(\tan^2 x + \sec^2 x)$

$= 2\sec^3 x - \sec x$. ✓

Show means more than repeat table of standard integrals

c) $C = \left(\dfrac{5\times 1 - 3\times -3}{5-3}, \dfrac{5\times 10 - 3\times 6}{5-3}\right)$ ✓

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d) $\displaystyle\int_{-3}^{3} \frac{1}{9+x^2}\,dx = \left.\frac{1}{3}\tan^{-1}\frac{x}{3}\right]_{-3}^{3}$ ✓

$= \frac{1}{3}(\tan^{-1} 1 - \tan^{-1} -1)$

$= \frac{1}{3}\left(\frac{\pi}{4} - -\frac{\pi}{4}\right)$

$= \dfrac{\pi}{6}$ ✓

Question 1.

e) $\dfrac{dy}{dx} = 1+y$

$\dfrac{dx}{dy} = \dfrac{1}{1+y}$

$x = \ln(1+y) + c$ ✓

$x = 0,\ y = 2 \quad c = -\ln 3$

$x = \ln\left(\frac{1+y}{3}\right)$

$e^x = \dfrac{1+y}{3}$ ✓

$y = 3e^x - 1$.

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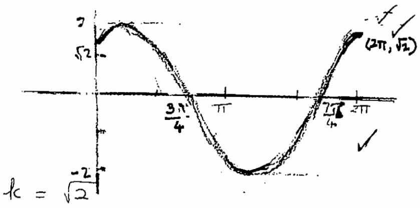
Question 2.

a) (i) $\sqrt{5}\sin x + \sqrt{2}\cos x = 2\sin\left(x + \frac{\pi}{4}\right)$ ✓✓

well done by majority

(ii)



✓

more care required in terms of intercep and endpoints for many students

(iii) $k = \sqrt{2}$

b) (i) $\tan\alpha = \frac{AB}{b}$, $\tan 2\alpha = \frac{AB}{d}$ ✓

$b\tan\alpha = d\tan 2\alpha$ ✓

generally well executed

(ii) Let $AB = h$

$h = d\tan 2\alpha$

$= d.\left(\frac{2\tan\alpha}{1-\tan^2\alpha}\right)$ ✓

$= d\left(\frac{2\frac{h}{b}}{1 - \frac{h^2}{b^2}}\right) = \frac{2hdb}{b^2 - h^2}$ ✓

$b^2 - h^2 = 2db$

$h^2 = b^2 - 2db$

$h = \sqrt{b^2 - 2db}$ ✓

most attempted the early part of question, bu some students could not achieve the desired result.

(iii) $b^2 + d^2 = 10d^2$ Pythagoras'

$\tan\alpha = \frac{h}{b} = \frac{\sqrt{b^2 - 2db}}{b} = \frac{\sqrt{b^2 - 2\frac{b}{3}b}}{b}$ ✓

$= \frac{1}{\sqrt{3}}$

$\frac{\pi}{\ }$ ✓

generally performed well by students who obtained the result in part (ii)

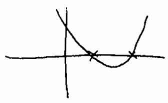
Question 3.

a) $\frac{x}{x-3} < 4$

$x(x-3) < 4(x-3)^2$ ✓✓

$0 < (x-3)[4(x-3) - x]$

$< (x-3)(3x-12)$



$\therefore 3(x-4)(x-3) > 0$ ✓

$\therefore x > 4$ or $x < 3$ ✓

Many students expand and fail to realise they can factorise $(x-3)$ from both sides.

b) $A = \int_0^{\pi/4} \tan y \, dy$ ✓

$= -\ln(\cos y)\Big]_0^{\pi/4}$ ✓

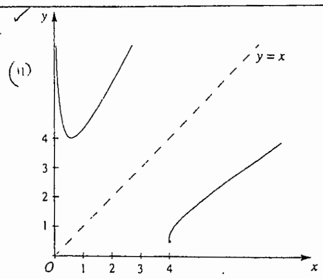
$= -\ln\frac{1}{\sqrt{2}}$ ✓

$= \frac{1}{2}\ln 2$

c) (i) $x \geq \frac{1}{2}$ ✓

largest Domain $x \geq 1$

(ii)



✓✓

Many sketched an inverse which not a function

(iii) $y = 4x + \frac{1}{x}$, for $x \geq \frac{1}{2}$

Inverse: $x = 4y + \frac{1}{y}$, for $y \geq \frac{1}{2}$ ✓

$xy = 4y^2 + 1$

$4y^2 - xy + 1 = 0$

$y = \frac{x \pm \sqrt{x^2 - 16}}{8}$

$y = \frac{x + \sqrt{x^2 - 16}}{8}$ or $\frac{x - \sqrt{x^2 - 16}}{8}$ ✓

For the appropriate range, $y \geq \frac{1}{2}$

Question 4

a) Let roots be $\alpha, \beta, \alpha\beta$.

$\alpha + \beta + \alpha\beta = -\frac{1}{2}$ ①

$\alpha\beta + \alpha^2\beta + \alpha\beta^2 = -\frac{15}{2}$ ② ✓

$\alpha^2\beta^2 = 9$.

$\alpha\beta = \pm 3$ but from question

$\alpha\beta = 3$. ✓

$\therefore$ From ① $\alpha + \beta = -3\frac{1}{2}$, $\alpha\beta = 3$ solve simultaneously

$\therefore$ Roots are $-\frac{3}{2}, -2$ and 3. ✓

b) $P(x) = (1 - x^2)Q(x) + (4 - x)$

$= (1-x)(1+x)Q(x) + (4-x)$ ✓

$P(-1) = 4 - -1 = 5$.

$\therefore$ remainder is 5. ✓

c) $P(x) = \sin x - x^2 + 10$

$P'(x) = \cos x - 2x$ ✓

$x_2 = \pi - \left(\frac{10 - \pi^2}{-1 - 2\pi}\right)$ ✓

$= 3.1594$

d) If $n = 1$ LHS $= \cos(x + \pi)$

$= \cos x \cos\pi - \sin x \sin\pi$

$= -\cos x$

RHS $= -\cos x$

True for $n = 1$. ✓

Many students did not show LHS = RHS for $n=1$

Assume true for $n = k$.

$\cos(x + k\pi) = (-1)^k \cos x$ ✓

Consider $n = k+1$

$\cos(x + (k+1)\pi) = \cos((x + k\pi) + \pi)$ ✓

$= \cos(x + k\pi)\cos\pi - \sin(x + k\pi)\sin\pi$

$= (-1)^k \cos x . x - 1 - 0$

$= (-1)^{k+1}\cos x$. ✓

with trig. "think" expansions

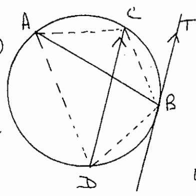
which is of the same form as for $x = k$

If true for $n = k$, also true for $n = k+1$. Since true for $n = 1$, true for $n = 2$ and hence all following positive integers. ✓

Concluding statement was poorly written by many students.

Question 5

a) (i)



(ii) $\angle CAB = \angle CDB$ angles in same segment ✓

$\angle CDB = \angle DBT$ alternate angles ✓

$\angle DBT = \angle DAB$ angle in alternate segment ✓

$\therefore \angle CAB = \angle DAB$

ie AB bisects $\angle CAD$.

b) (i) $y = \dfrac{x^2}{4a}$

$\dfrac{dy}{dx} = \dfrac{2x}{4a} = \dfrac{x}{2a} = p$ at $(2ap, ap^2)$ ✓

Eq^n is $y - ap^2 = p(x - 2ap)$.

$y = px - ap^2$. ✓

(ii) $px - ap^2 = qx - aq^2$.

$(p-q)x = ap^2 - aq^2$

$= a(p-q)(p+q)$

$x = a(p+q)$. ✓

$y = pa(p+q) - ap^2$

$= apq$.

$\therefore T = (a(p+q), apq)$ ✓

(iii) $\tan 45^\circ = \dfrac{p-q}{1+pq}$ ✓

$1 = \dfrac{p-q}{1+pq}$ ✓

Diagrams need to be neat, clear and ½ page. Use a ruler to draw straight lines. Make sure all information in question is placed on diagram.

* If you assumed chords passed through centre — zero marks for proof.

Parts (i), (ii), (iii) well done. Must state tan 45° = 1. We do not assume you knew this – especially with 'SHOW' questions.

(iv) $(p-q)^2 = (1+pq)^2$

$= 1 + (pq)^2 + 2pq$ ✓

$(p+q)^2 - 4pq = 1 + (pq)^2 + 2pq$

$\dfrac{x^2}{a^2} - 4\dfrac{y}{a} = 1 + \dfrac{y^2}{a^2} + 2\dfrac{y}{a}$ ✓

$x^2 - 4ay = (a+y)^2$.

To do better with locus problems, you need to practice a variety of problem involving different techniques.

An alternative way to do (iv)

$x = a(p+q)$ $\quad y = apq$

$\dfrac{x}{a} = (p+q)$

$\dfrac{x^2}{a^2} = (p^2 + q^2 + 2pq)$

$= (p-q)^2 + 2pq + 2pq$

$= (p-q)^2 + 4pq$

$= (1+pq)^2 + 4pq$ $\quad$ use $1+pq = p-q$

$= \left(1 + \dfrac{y}{a}\right)^2 + \dfrac{4y}{a}$ $\quad$ use $\dfrac{y}{a} = pq$

$\dfrac{x^2}{a^2} = 1 + \dfrac{2y}{a} + \dfrac{y^2}{a^2} + \dfrac{4y}{a}$

$x^2 = a^2 + 2ay + y^2 + 4ay$

$x^2 = a^2 + y^2 + 6ay$.

Question 6

(a) (i) $T = A + Pe^{kt}$

$\dfrac{dT}{dt} = kPe^{kt}$ ✓

$= k(T - A)$ ✓

• generally well done

(ii) $t = 0$, $T = 90^\circ$ $\quad 90 = 22 + P$

$P = 68$ ✓

$t = 8$ $\quad T = 70^\circ$ $\quad 70 = 22 + 68e^{8k}$

$e^{8k} = \dfrac{48}{68}$

$= 0.0435383336$

$k \doteq -0.04$ ✓

$60 = 22 + 68e^{kt}$ ✓

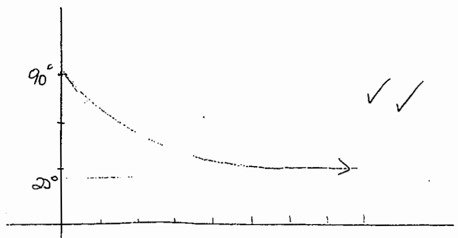
$\therefore t = 13$ mins

• some did not interpret initial conditions correctly.

• $k = \dfrac{\ln\left(\frac{12}{17}\right)}{8}$ also gain the mark.

• ignored seconds as units in some answers

(iii)



✓✓

$T \to 22^\circ$.

• many did not show 90° y-intercept, nor the asymptote as $t \to \infty$

• also some strange interpretations of what an exponential graph looks like!

Question 6(b).

$\ddot{x} = 2 - 3x \qquad x = 1, v = \sqrt{5}$

(i) $\dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right) = 2 - 3x$

$\dfrac{1}{2}v^2 = 2x - \dfrac{3}{2}x^2 + c$ ✓

When $x = 1$, $v = \sqrt{5}$

$\dfrac{1}{2} \times 5 = 2 - \dfrac{3}{2} + c$

$c = 2$ ✓

$\therefore \dfrac{1}{2}v^2 = 2x - \dfrac{3}{2}x^2 + 2$

$v^2 = 4x - 3x^2 + 4$

$v^2 = 4 + 4x - 3x^2$ ✓

• generally well done.

(ii) When $v = 0$, $\quad 4 + 4x - 3x^2 = 0$

$3x^2 - 4x - 4 = 0$

$(3x+2)(x-2) = 0$

$x = -\dfrac{2}{3}, 2$

Amplitude $= \dfrac{1}{2} \times 2\dfrac{2}{3}$

$= 1\dfrac{1}{3}$ m. ✓

(iii) $a = -3\left(x - \dfrac{2}{3}\right)$

Centre of motion is $x = \dfrac{2}{3}$ ✓

• many found extremes but did not halve the distance between these extremes correctly!

• for (ii) and (iii) many tried (poorly) to complete the square to put into form $v^2 = n^2(a^2 - x^2)$

(iv) Maximum speed occurs at centre of motion.

When $x = \dfrac{2}{3}$, $v^2 = 4 + 4 \times \dfrac{2}{3} - 3 \times \dfrac{4}{9}$

$= 5\dfrac{1}{3}$

$v = \pm\dfrac{4}{\sqrt{3}}$

Max. speed is $\dfrac{4}{\sqrt{3}}$ or $\dfrac{4\sqrt{3}}{3}$ m/s. ✓ $\quad 2.3094$

• generally well done.

• leave as a surd unless a number of d.p.s is asked for.

(v) $n^2 = 3$, $\quad n = \sqrt{3}$

$\therefore$ Period $= \dfrac{2\pi}{\sqrt{3}}$ ✓

• many had trouble deducing $n^2 = 3$.

Question 7

a) Area = $\int_0^1 \frac{x^2\,dx}{\sqrt{4-x^2}}$ $\qquad x = 2\sin\theta$, $\quad \frac{dx}{d\theta} = 2\cos\theta$.

students did not replace dx with $d\theta$

$= \int_0^{\pi/6} \frac{4\sin^2\theta \cdot 2\cos\theta\, d\theta}{\sqrt{4-4\sin^2\theta}}$ ✓

students did not cancel. $2\cos\theta$

$= \int_0^{\pi/6} \frac{4\sin^2\theta \cdot \cancel{2\cos\theta}\, d\theta}{\cancel{2\cos\theta}}$ ✓

$= 2\int_0^{\pi/6} 1-\cos 2\theta \, d\theta$ ✓

$= 2\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/6}$

$= 2\left[\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right]$

$= \frac{2\pi - 3\sqrt{3}}{6}$ ✓

b) $\int_0^{\pi/8} \frac{2\sec^2 2x}{\sqrt{2-\tan^2 2x}}\,dx$ $\qquad u = \tan 2x$, $\quad \frac{du}{dx} = 2\sec^2 2x\,dx$.

Well done

$= \int_0^1 \frac{du}{\sqrt{2-u^2}}$ ✓

$= \left.\sin^{-1}\frac{u}{\sqrt{2}}\right]_0^1$ ✓

students did not change integration boundaries to 1 & 0.

$= \frac{\pi}{4}$ ✓

Question 7

c) (i) $\ddot{x} = 0$ $\qquad \ddot{y} = -g$

$\dot{x} = V\cos\alpha$ $\qquad \dot{y} = -gt$

$= V$ (speed $= 0$)

$x = Vt$ $\qquad y = h - \frac{g}{2}t^2$ ✓

Well done!

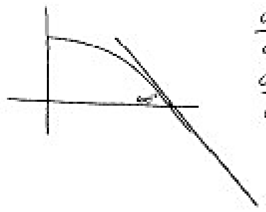
It reaches ground when $y = 0$.

$\therefore 0 = h - \frac{g}{2}t^2$

$t^2 = \frac{2h}{g}$

$t = \sqrt{\frac{2h}{g}}$ ✓

(ii)



$\frac{dy}{dx} = \tan 120°$

$= -\sqrt{3}$.

$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$

$= -gt \cdot \frac{1}{V}$ ✓

$= -\frac{gt}{V}$

$= -\frac{g}{V}\sqrt{\frac{2h}{g}}$ ✓

$-\sqrt{3} = -\frac{g}{V}\sqrt{\frac{2h}{g}}$

$3V^2 = g^2\,\frac{2h}{g}$

$3V^2 = 2gh$ ✓