Student Number: ____________________

Teacher: ____________________

Class: ____________________

FORT STREET HIGH SCHOOL

2019

HIGHER SCHOOL CERTIFICATE COURSE

ASSESSMENT TASK 4: TRIAL HSC

Mathematics Extension 1

Time allowed: 2 hours
(plus 5 minutes reading time)

Syllabus Outcomes	Assessment Area Description and Marking Guidelines
HE2	Uses inductive reasoning in the construction of proofs
HE3	Uses a variety of strategies to investigate mathematical models of situations involving projectiles, simple harmonic motion, exponential growth and decay or probability (permutations and combinations)
HE4	Uses the relationship between functions, inverse functions and their derivatives
HE5	Applies the chain rule to problems including those involving velocity and acceleration as functions of displacement
HE6	Determines integrals by reduction to a standard form through a given substitution.
HE7	Evaluates mathematical solutions to problems and communicates them in appropriate form

Total Marks 70

Section I 10 marks
Multiple Choice, attempt all questions
Allow about 15 minutes for this section

Section II 60 Marks
Attempt Questions 11-14
Allow about 1 hour 45 minutes for this section

Section I	Total 10	Marks
Q1-Q10	/10	
Section II	Total 60	Marks
Q11	/15	
Q12	/15	
Q13	/15	
Q14	/15	
	Percent	

General Instructions:

- Questions 11-14 are to be started in a new booklet.
- The marks allocated for each question are indicated.
- In Questions 11-14, show relevant mathematical reasoning and/or calculations.
- Marks may not be awarded for careless or poorly arranged work.
- NESA-approved calculators may be used.
- A reference sheet is provided with this paper.

SECTION I (1 mark each)

Answer Questions 1 to 10 on the multiple choice answer sheet.

1 What is the size of the acute angle between the lines whose equations are $y = 3x - 1$ and $x + 2y - 3 = 0$?

(A) $45°$

(B) $54°$

(C) $79°$

(D) $82°$

2 The point R divides the interval joining $P(a, 2b)$ and $Q(3a, -b)$ externally in the ratio $2:3$. What are the co-ordinates of R ?

(A) $(-3a, 8b)$

(B) $\left(\frac{11a}{5}, \frac{4b}{5}\right)$

(C) $(-3a, -7b)$

(D) $\left(\frac{9a}{5}, \frac{8b}{5}\right)$

3 The simplified form of the expression $\log_{\frac{1}{x}}\left(\frac{1}{x^2}\right)$ where $x > 1$ is :

(A) $\frac{1}{x}$

(B) 2

(C) $-\ln x$

(D) x

4 The equation $2x^3 - 8x^2 + 1 = 0$ has roots α, β and γ.
What is the value of $\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2$?

(A) 2

(B) -4

(C) -2

(D) 4

5 At a football club, a team of 11 players is to be chosen from a pool of 30 players consisting of 18 Australian-born players and 12 players born elsewhere. What is the probability that the team will consist of all Australian-born players?

(A) $\dfrac{{}^{18}C_{11}}{{}^{30}C_{11}}$

(B) $\dfrac{{}^{30}C_{11}}{{}^{18}C_{11}}$

(C) $\dfrac{{}^{18}C_{12}}{{}^{30}C_{12}}$

(D) $\dfrac{{}^{30}C_{12}}{{}^{18}C_{12}}$

6 A particle is moving along the x-axis. Its velocity v at position x is given by $v=\sqrt{8x-x^2}$. What is the acceleration when $x=3$?

(A) 1

(B) 2

(C) 3

(D) 4

7 What is the exact value of $\tan(\theta-180^\circ)$, if $4\cos\theta=-3$ and $\tan\theta>0$?

(A) $-\dfrac{\sqrt{7}}{3}$

(B) $-\dfrac{3}{\sqrt{7}}$

(C) $\dfrac{\sqrt{7}}{3}$

(D) $\dfrac{3}{\sqrt{7}}$

8 Consider the equation $\dfrac{\sin\theta\cos\theta}{2\cos^2\theta-1}=-\dfrac{\sqrt{3}}{2}$.

How many solutions does the above equation have in the domain $0\le\theta\le 2\pi$?

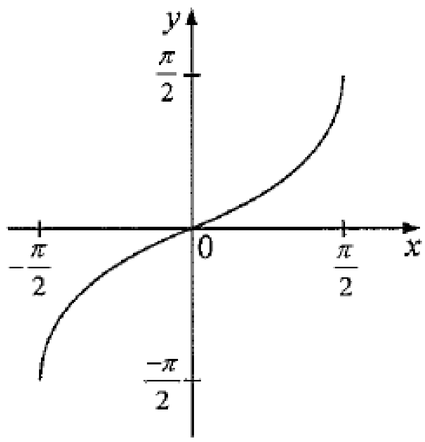
(A) Two

(B) Three

(C) Four

(D) Five

9 Which of the following could be the equation of the graph shown below?



(A) $y = \sin^{-1} \dfrac{\pi x}{2}$

(B) $y = \sin^{-1} \dfrac{2x}{\pi}$

(C) $y = \dfrac{\pi}{2} \sin^{-1} x$

(D) $y = \dfrac{2}{\pi} \sin^{-1} x$

10 A ball is thrown into the air from a point O, where $x = 0$, with an initial velocity of 25 m/s at an angle $\theta = \tan^{-1}\left(\dfrac{3}{4}\right)$ to the horizontal. If air resistance is neglected and the acceleration due to gravity is taken as 10 m/s^2, then the ball reaches the greatest height after:

(A) 1.5 seconds

(B) 15 seconds

(C) $\dfrac{2}{3}$ of a second

(D) 3 seconds

End of Section I

SECTION II (15 marks for each question)

Answer each question in the appropriate booklet. Extra writing booklets are available.

Question 11: *Use a separate writing booklet.*

a) Evaluate $\lim_{x\to 0}\frac{\sin\frac{x}{2}}{4x}$ **2**

b) Solve $\frac{5}{x-4} \geq 1$ **2**

c) Evaluate $\int_0^{\frac{3}{4}} \frac{1}{\sqrt{9-4x^2}}dx$ **3**

d) Let $f(x)=\frac{1}{\sqrt{1+x^2}}$ for $x \leq 0$ **3**

Find an expression for the inverse function $f^{-1}(x)$ in terms of x.

e) Simplify $\frac{{}^{n+1}C_r}{{}^{n}C_{r-1}}$ **2**

f) Use the substitution $u = x^2 + 4x - 3$ to evaluate **3**

$$\int_1^2 \frac{x+2}{\sqrt{x^2+4x-3}}\,dx$$

End of Question 11

Question 12: *Use a separate writing booklet.*

a) The polynomial $P(x)$ is given by the equation $P(x) = x^3 + ax + b$ for some real numbers a and b. $x = 2$ is a zero of $P(x)$. When $P(x)$ is divided by $(x+1)$, the remainder is -15. Find the values of a and b. **2**

b) Evaluate $\int_0^{\frac{\pi}{2}} \cos^2 2x + \sin^2 \frac{x}{2}\, dx$ **3**

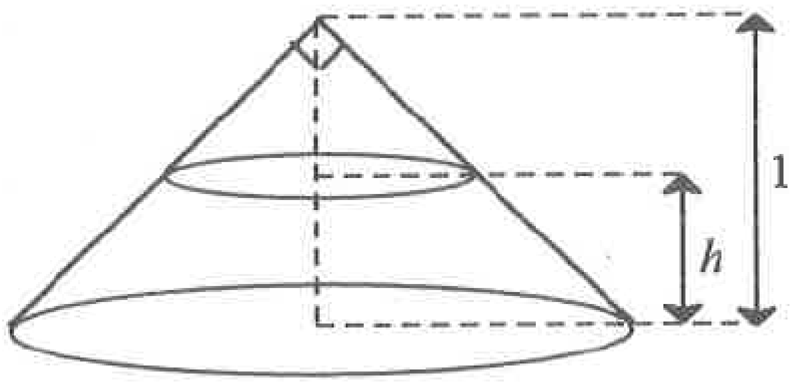
c) Use the method of mathematical induction to show that
$1 + 3 + 6 + ... + \frac{1}{2}n(n+1) = \frac{1}{6}n(n+1)(n+2)$ for all integers $n \geq 1$. **3**

d) Find the number of ways in which the letters of the word EXTENSION can be arranged in a straight line so that no two consonants are next to each other. **3**

Question 12 continues on page 7

Question 12 continued

e) A closed, right, hollow cone has a height of 1m and semi vertical angle $45°$.
The cone stands with its base on a horizontal surface. Water is poured into the cone through a hole in its apex at a constant rate of 0.1 m^3 per minute.



(i) Show that when the depth of water in the cone is h metres $(0 < h < 1)$,the volume of water V m^3 in the cone is given by

$$V = \frac{\pi}{3}(h^3 - 3h^2 + 3h)$$ **2**

(ii) Hence find the rate at which the depth of water in the cone is increasing when $h = 0.5$ **2**

End of Question 12

Question 13: *Use a separate writing booklet.*

a) The rate at which a body warms in air is proportional to the difference between its temperature T and the constant temperature A of the surrounding air. This rate can be expressed by the differential equation $\frac{dT}{dt} = k(T - A)$ where t is the time in minutes and k is a constant.

(i) Show that $T = A + Be^{kt}$, where B is a constant, is a solution of the differential equation. **1**

(ii) An object warms from $5°\text{C}$ to $15°\text{C}$ in 20 minutes. The temperature of the surrounding air is $25°\text{C}$. Find the temperature of the object after a further 50 minutes has elapsed. Give your answer to the nearest degree. **3**

b) A particle moves in a straight line and its position at time t is given by

$$x = 1 + \sin 4t + \sqrt{3}\cos 4t$$

(i) Prove that the particle is undergoing simple harmonic motion about $x = 1$. **2**

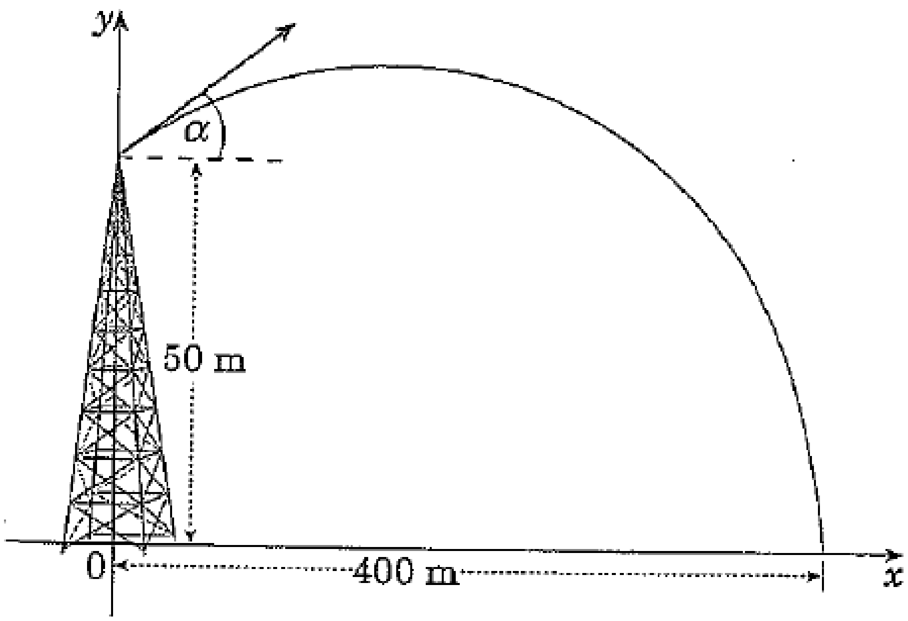
(ii) Find the period and amplitude of the motion. **2**

(iii) When does the particle first reach maximum speed after time $t = 0$ **2**

Question 13 continues on page 9

Question 13 continued

c) A projectile is fired from the top of a 50 m tower at an angle of elevation α, with an initial speed of 80 m/s . The acceleration due to the gravity is assumed to be 10 m/s^2.



(i) Show that

$$x = 80t\cos\alpha$$
$$y = -5t^2 + 80t\sin\alpha + 50$$

where x and y are the horizontal and vertical distances of the projectile in metres from O after t seconds after launching. **2**

(ii) The projectile lands on the ground 400 m away from O, the base of the tower. Find the possible values for α, giving your answer to the nearest degree. **3**

End of Question 13

Question 14: *Use a separate writing booklet.*

a) Use one application of Newton's method to solve $x^2 - \sqrt{x} - 2 = 0$, by using a first approximation of $x = 2$. Answer correct to two decimal places. **2**

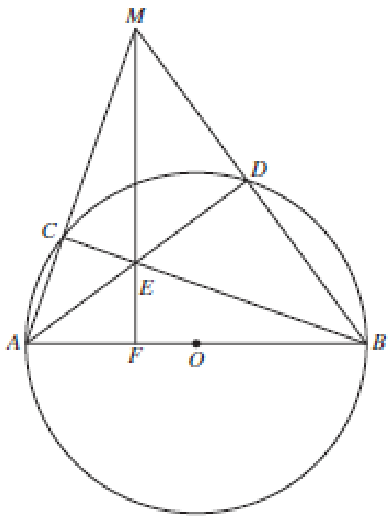
b)

(i) Find $\dfrac{d}{dx}\left(x\cos^{-1}x - \sqrt{1-x^2}\right)$ **2**

(ii) Hence , find the area between the curve $y = \cos x$, the y-axis and the lines $y = \dfrac{1}{2}$ and $y = 1$. **2**

c) The circle centred at O has a diameter AB . From the point M outside the circle the line segments MA and MB are drawn meeting the circle at C and D respectively, as shown in the diagram. The chords AD and BC meet at E . The line segment ME produced meet the diameter AB at F .

NOT TO SCALE



Copy or trace the diagram into your writing booklet.

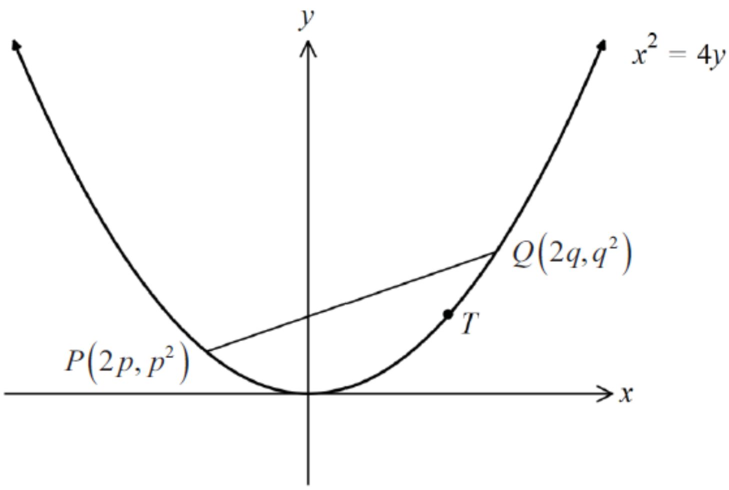
(i) Show that $CMDE$ is a cyclic quadrilateral **2**

(ii) Hence, or otherwise , prove that MF is perpendicular to AB **2**

Question 14 continues on page 11

Question 14 continued

d) Consider the parabola $x^2 = 4y$. $P(2p, p^2)$ and $Q(2q, q^2)$ lie on the parabola .



(i) Show that the equation of the chord PQ is $y = \left(\frac{p+q}{2}\right)x - pq$ **2**

(ii) Given that PQ is a focal chord .
$T(2t, t^2)$, $t > 0$ and $R(2r, r^2)$ are two other points on the parabola distinct from P and Q. If TR is also a focal chord and P, T, Q and R are concyclic, show that $p^2 + q^2 = t^2 + r^2$. **3**

End of Examination.

Fort Street High School

2019
Assessment Task 4

Mathematics Extension 1

Syllabus	Assessment Area Description and Marking Guidelines	Questions
HE4	Uses the relationship between functions, inverse functions & their derivatives	2, 3
HE6	Determines integrals by reduction to a standard form through a given substitution	1, 3
HE7	Evaluates mathematical solutions to problems & communicates them in an appropriate form	3, 4

Solutions

Multiple Choice:

1	$\tan\theta = \left\|\frac{m_1 - m_2}{1 + m_1 m_2}\right\|$ $\tan\theta = \left\|\frac{3 - \left(-\frac{1}{2}\right)}{1 + 3\times\left(-\frac{1}{2}\right)}\right\|$ $\theta \approx 82^\circ$	1 Mark: D
2	$x = \frac{-3a + 6a}{2 + (-3)} = -3a$ $y = \frac{-6b - 2b}{2 + (-3)} = 8b$ $\therefore R(-3a, 8b)$	1 Mark: A
3	$\log_{\frac{1}{x}}\left(\frac{1}{x^2}\right) = \frac{\ln\left(\frac{1}{x^2}\right)}{\ln\frac{1}{x}} = \frac{2\ln\frac{1}{x}}{\ln\frac{1}{x}} = 2$	1 Mark: B
4	$\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2$ $= \alpha\beta\gamma(\alpha + \beta + \gamma)$ $= -\frac{1}{2}\times 4$ $= -2$	1 Mark: C
5	$\frac{^{18}C_{11}}{^{30}C_{11}}$	1 Mark: A

6	$v=\sqrt{8x-x^2}$ $v^2=8x-x^2$ $a=\frac{d}{dx}\left(\frac{1}{2}v^2\right)=\frac{1}{2}(8-2x)=4-x$ $x=3, a=4-3=1$	1 Mark: A
7	$\tan(\theta-180^\circ)=-\tan(180^\circ-\theta)=\tan\theta=\frac{\sqrt{7}}{3}$	1 Mark: C
8	$\frac{\sin\theta\cos\theta}{2\cos^2\theta-1}=-\frac{\sqrt{3}}{2}$ $\frac{2\sin\theta\cos\theta}{2\cos^2\theta-1}=-\sqrt{3}$ $\frac{\sin 2\theta}{\cos 2\theta}=-\sqrt{3}$ $\tan 2\theta=-\sqrt{3}$ $2\theta=\frac{2\pi}{3},\frac{5\pi}{3},\frac{8\pi}{3},\frac{11\pi}{3}$ $\theta=\frac{\pi}{3},\frac{5\pi}{6},\frac{4\pi}{3},\frac{11\pi}{6}$ $\therefore$ 4 solutions	1 Mark: C
9	$y=\sin^{-1}\frac{2x}{\pi}$ gives an inverse Sine graph with range $-\frac{\pi}{2}\le y\le\frac{\pi}{2}$ And domain $-1\le\frac{2x}{\pi}\le 1$ $\Rightarrow -\frac{\pi}{2}\le x\le\frac{\pi}{2}$	1 Mark: B
10	$\ddot{y}=-10$ $\dot{y}=-10t+C_1$ $C_1=15$ $\dot{y}=-10t+15$ $\dot{y}=0$ $-10t+15=0$ $\therefore t=1.5\sec onds$	1 Mark: A

Question 11. (15 marks)

(a) Evaluate $\lim_{x \to 0} \dfrac{\sin \frac{x}{2}}{4x}$

Solution

$$\lim_{\frac{x}{2} \to 0} \frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}} \times \frac{\frac{x}{2}}{4x}$$

$$= 1 \times \frac{1}{8}$$

$$= \frac{1}{8}$$

Marking guideline

2 Correct solution

1 Partial correct answer

Marker's comments

(b) Solve $\dfrac{5}{x-4} \geq 1$

Solution

$$\frac{5}{x-4} \times (x-4)^2 \geq 1 \times (x-4)^2$$

$$5(x-4) \geq (x-4)^2$$

$$(x-4)^2 - 5(x-4) \leq 0$$

$$(x-4)(x-9) \leq 0$$

$$4 < x \leq 9$$

Marking guideline

2 Correct solution

1 Partial correct answer

Marker's comments

(c) Evaluate $\int_0^{\frac{3}{4}} \frac{1}{\sqrt{9-4x^2}}dx$

Solution

$$\int_0^{\frac{3}{4}} \frac{1}{2\sqrt{\frac{9}{4}-x^2}}dx$$

$$\frac{1}{2}\int_0^{\frac{3}{4}} \frac{1}{\sqrt{\left(\frac{3}{2}\right)^2-x^2}}dx$$

$$=\frac{1}{2}\left[\sin^{-1}\frac{x}{\frac{3}{2}}\right]_0^{\frac{3}{4}}$$

$$=\frac{1}{2}\left[\sin^{-1}\frac{1}{2}-\sin^{-1}0\right]$$

$$=\frac{1}{2}\times\frac{\pi}{6}$$

$$=\frac{\pi}{12}$$

Marking guideline

3 **Correct solution**

2 **Getting** $\frac{1}{2}\left[\sin^{-1}\frac{x}{\frac{3}{2}}\right]_0^{\frac{3}{4}}$

1 **Getting** $\frac{1}{2}\int_0^{\frac{3}{4}} \frac{1}{\sqrt{\left(\frac{3}{2}\right)^2-x^2}}dx$ correct

Marker's comments

(d) Let $f(x)=\frac{1}{\sqrt{1+x^2}}$ for $x\le 0$

Find an expression for the inverse function $f^{-1}(x)$ in terms of x

Solution

Let $y=\frac{1}{\sqrt{1+x^2}}$, $x\le 0$

$$x=\frac{1}{\sqrt{1+y^2}}$$

$$x^2=\frac{1}{1+y^2}$$

$$y^2=\frac{1-x^2}{x^2}$$

$$y=\pm\sqrt{\frac{1-x^2}{x^2}} \quad \text{but } y\le 0$$

$$\therefore f^{-1}(x)=-\sqrt{\frac{1-x^2}{x^2}}$$

Marking guideline

3 **Correct, fully worked solution**

2 **Correct expression for** $y=\pm\sqrt{\frac{1-x^2}{x^2}}$

1 **showing some progress**

Marker's comments

(e) Simplify $\dfrac{{}^{n+1}C_r}{{}^{n}C_{r-1}}$

Solution

$$\frac{{}^{n+1}C_r}{{}^{n}C_{r-1}}$$
$$=\frac{(n+1)!}{r!(n+1-r)!}\times\frac{(r-1)!(n-r+1)!}{n!}$$
$$=\frac{(n+1)!}{r!}\times\frac{(r-1)!}{n!}$$
$$=\frac{(n+1)n!}{r(r-1)!}\times\frac{(r-1)!}{n!}$$
$$=\frac{n+1}{r}$$

Marking guideline

3 Correct, fully worked solution

2 Correct expression for $y=\pm\sqrt{\dfrac{1-x^2}{x^2}}$

1 showing some progress

Marker's comments

(f) Use the substitution $u=x^2+4x-3$ to evaluate

$$\int_1^2 \frac{x+2}{\sqrt{x^2+4x-3}}\,dx$$

Solution

Let

$$u=x^2+4x-3$$
$$du=(2x+4)dx$$
$$x=1, u=2$$
$$x=2, u=9$$
$$\int_1^2 \frac{x+2}{\sqrt{x^2+4x-3}}\,dx$$
$$=\frac{1}{2}\int_2^9 \frac{du}{\sqrt{u}}$$
$$=\frac{1}{2}\left[\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right]_2^9$$
$$=3-\sqrt{2}$$

Marking guideline

3 Correct, fully worked solution

2 Correct expression for u

1 showing some progress

Marker's comments

Question 12. (15 marks)

(a) The polynomial $P(x)$ is given by the equation $P\left(x\right)=x^3+ax+b$ for some real numbers a and b. $x=2$ is a zero of $P(x)$. When $P(x)$ is divided by $(x+1)$, the remainder is -15. Find the values of a and b.

Solution

$P(x)=x^3+ax+b$

$P(2)=0$
$8+2a+b=0$
$2a+b=-8$

$P(-1)=-15$
$-1-a+b=-15$
$-a+b=-14$

Solving $2a+b=-8$ and $-a+b=-14$

$a=2$ and $b=-12$

Marking guideline

2 Correct response
1 Partial correct answer

Marker's comments
Overall well done

(b) Evaluate $\int_0^{\frac{\pi}{2}} \cos^2 2x+\sin^2 \frac{x}{2}\,dx$

Solution

$$\int_0^{\frac{\pi}{2}} \cos^2 2x+\sin^2 \frac{x}{2}\,dx$$

$$=\int_0^{\frac{\pi}{2}} \frac{1+\cos 4x}{2}+\frac{1-\cos x}{2}\,dx$$

$$=\frac{1}{2}\left[2x+\frac{\sin 4x}{4}-\sin x\right]_0^{\frac{\pi}{2}}$$

$$=\frac{1}{2}\left[\pi-1\right]$$

Marking guideline

3 Correct solution
2 correct integration
1 some progress

Marker's comments

(c) Use the method of mathematical induction to show that

$$1+3+6+...+\frac{1}{2}n\left(n+1\right)=\frac{1}{6}n(n+1)(n+2) \text{ for all integers } n\geq 1.$$

Step 1: show true for $n=1$

$LHS=1$

$RHS=\frac{1}{6}\times 1\times 2\times 3=1$

Hence true for $n=1$.

Step 2: Assume the statement is true for $n=k$: where $k\geq 1$

i.e. assume $1+3+6+...+\frac{1}{2}k\left(k+1\right)=\frac{1}{6}k(k+1)(k+2)$

Then show true for $n=k+1$

i.e. show $1+3+6+...+\frac{1}{2}(k+1)\left(k+2\right)=\frac{1}{6}(k+1)(k+2)(k+3)$

❶ Step 3:

$$\begin{aligned} LHS &= 1+3+6+...+\frac{1}{2}k\left(k+1\right)+\frac{1}{2}(k+1)(k+2) \\ &= \frac{k\left(k+1\right)(k+2)}{6}+\frac{1}{2}(k+1)(k+2) \\ &= \frac{(k+1)(k+2)}{6}\left(k+3\right) \\ &= \frac{1}{6}(k+1)(k+2)\left(k+3\right) \end{aligned}$$

as reqd. ❶ (correct algebra to result)

Hence, by Mathematical Induction,

$$1+3+6+...+\frac{1}{2}n\left(n+1\right)=\frac{1}{6}n(n+1)(n+2) \text{ for all integers } n\geq 1.$$

Marking guideline	
3	**Correct solution**
2	**Correct induction**
1	**Setting up correctly**

Marker's comments

Some students still try to prove LHS = RHS rather than start from LHS

(d) Find the number of ways in which the letters of the word EXTENSION can be arranged in a straight line so that no two consonants are next to each other

Solution

Number of ways for arranging consonants

$=\frac{5!}{2!}=60$

Number of ways vowels can be arranged

$=\frac{4!}{2!}=12$

Total number of ways

$=60\times 12=720$

Marking guideline	
3	**Correct solution**
2	**Correct number of ways for arranging consonants and vowels**
1	**Correct progress**

Marker's comments

Many forgot to divide by 2! 2!

(e) A closed, right, hollow cone has a height of 1m and semi vertical angle $45°$.

The cone stands with its base on a horizontal surface. Water is poured into the cone through a hole in its apex at a constant rate of $0.1\ \text{m}^3$ per minute.

Solution

(i) Show that when the depth of water in the cone is h metres $(0 < h < 1)$,the volume of water V m^3 in the cone is given by $V = \frac{\pi}{3}(h^3 - 3h^2 + 3h)$

Marking guideline
2 Correct solution
1 Correct progress
Marker's comments

Radius of bigger cone = 1

Radius of smaller cone $= 1-h$

$$V = \frac{1}{3}\pi \times 1^2 \times 1 - \frac{1}{3}\pi \times (1-h)^2 \times (1-h)$$
$$= \frac{\pi}{3}\left[1 - (1-h)^3\right]$$
$$= \frac{\pi}{3}\left(1 - 1 + h^3 + 3h - 3h^2\right)$$
$$= \frac{\pi}{3}\left(h^3 - 3h^2 + 3h\right)$$

(ii) Hence find the rate at which the depth of water in the cone is increasing when $h = 0.5$

$$\frac{dV}{dh} = \frac{\pi}{3}\left(3h^2 - 6h + 3\right)$$

Marking guideline
2 Correct solution
1 Correct progress
Marker's comments

$$= \pi\left(h^2 - 2h + 1\right)$$
$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt} = \frac{0.1}{\pi\left(h^2 - 2h + 1\right)}$$
$$When h = 0.5$$
$$\frac{dh}{dt} = \frac{0.1}{\pi\left(0.5^2 - 2 \times 0.5 + 1\right)} = \frac{0.1}{\pi \times 0.25} = \frac{2}{5\pi} \approx 0.13 m / \min$$

Question 13. (10 marks)

(a) The rate at which a body warms in air is proportional to the difference between its temperature T and the constant temperature A of the surrounding air. This rate can be expressed by the differential equation $\frac{dT}{dt} = k(T - A)$ where t is the time in minutes and k is a constant

(i) Show that $T = A + Be^{kt}$, where B is a constant, is a solution of the differential equation.

Solution

$T = A + Be^{kt}$

$\frac{dT}{dt} = Bke^{kt} = k \times Be^{kt}$

$\frac{dT}{dt} = k(T - A)$

$\therefore T = A + Be^{kt}$

is a solution of the differential equation.

> *Marking guideline*
>
> **1 Correct solution**
>
> *Marker's comments*
> Well done.

(ii)
An object warms from $5°C$ to $15°C$ in 20 minutes. The temperature of the surrounding air is $25°C$. Find the temperature of the object after a further 50 minutes has elapsed. Give your answer to the nearest degree

Solution

$A = 25^{\circ}C$

When

$t = 0, T = 5$

$\therefore 5 = 25 + B$

$B = -20$

$15 = 25 - 20e^{20k}$

$e^{20k} = \frac{1}{2}$

$k = \frac{1}{20}\ln\frac{1}{2}$

When

$t = 70$

$T = 25 - 20e^{(\frac{1}{20}\ln\frac{1}{2})\times 70}$

$T \approx 23^{\circ}$

> *Marking guideline*
>
> **3 Correct solution**
> **2 Correct B and K**
> **1 Correct B or K**
>
> *Marker's comments*
> Some careless errors but overall, well done.

(b) A particle moves in a straight line and its position at time t is given by

$$x = 1 + \sin 4t + \sqrt{3}\cos 4t$$

Marking guideline

2 Correct solution
1 Some progress

Marker's comments
Well done.

(i) Prove that the particle is undergoing simple harmonic motion about $x = 1$.

Solution

$$x = 1 + \sin 4t + \sqrt{3}\cos 4t$$

$$\dot{x} = 1 + 4\cos 4t - 4\sqrt{3}\sin 4t$$

$$\ddot{x} = -16\sin 4t - 16\sqrt{3}\cos 4t$$

$$\ddot{x} = -16(\sin 4t + \sqrt{3}\cos 4t)$$

$$\ddot{x} = -16(x-1)$$

In form $\ddot{x} = -n^2(x-b)$ where $n = 4, b = 1$.

$\therefore$ The particle is undergoing simple harmonic motion about $x = 1$.

OR…

$$x = 1 + \sin 4t + \sqrt{3}\cos 4t$$
$$x - 1 = \sin 4t + \sqrt{3}\cos 4t$$

$$x - 1 = 2(\frac{1}{2}\sin 4t + \frac{\sqrt{3}}{2}\cos 4t)$$

$$x - 1 = 2(\cos\frac{\pi}{3}\sin 4t + \sin\frac{\pi}{3}\cos 4t)$$

$$x - 1 = 2\sin(4t + \frac{\pi}{3})$$

$$x = 1 + 2\sin(4t + \frac{\pi}{3})$$

$$\dot{x} = 8\cos(4t + \frac{\pi}{3})$$

$$\ddot{x} = -32\sin(4t + \frac{\pi}{3})$$

$$= -16(2\sin(4t + \frac{\pi}{3}))$$

$$\ddot{x} = -16(x-1)$$

(ii) Find the period and amplitude of the motion

Solution

$Amplitude = 2$

$TimePeriod = \frac{2\pi}{4} = \frac{\pi}{2}$

Marking guideline

2 Amplitude and Time Period Correct
1 Amplitude or Time Period Correct

Marker's comments
Some students could not state the amplitude as they did not transform x correctly.

(iii) When does the particle first reach maximum speed after time $t = 0$

Solution

The particle is at max speed at the centre of its motion at $x = 1$

$$x = 1 + 2\sin\left(4t + \frac{\pi}{3}\right)$$

$$1 = 1 + 2\sin\left(4t + \frac{\pi}{3}\right)$$

From $2\sin\left(4t + \frac{\pi}{3}\right) = 0$

$$\left(4t + \frac{\pi}{3}\right) = 0, \pm\pi, \pm 2\pi\ldots\ldots\ldots\ldots$$

$$4t + \frac{\pi}{3} = n\pi$$

Where n is an integer.

The particle first reach maximum speed when $n = 1$

$$4t + \frac{\pi}{3} = \pi$$

$$4t = \frac{2\pi}{3}$$

$$t = \frac{\pi}{6}$$

$\therefore$ The particle first reaches the maximum speed at $t = \frac{\pi}{6}$

Marking guideline

2 Correct solution
1 Some progress

Marker's comments
Some students did not know where maximum speed occurs.

(c) A projectile is fired from the top of a 50 m tower at an angle of elevation α, with an initial speed of 80 m/s . The acceleration due to the gravity is assumed to be 10 m/s^2.

(i) Show that

$$x = 80t\cos\alpha$$
$$y = -5t^2 + 80t\sin\alpha + 50$$

where x and y are the horizontal and vertical distances of the projectile in metres from O after t seconds after launching

Solution

$\ddot{x} = 0$

$\dot{x} = C_1$

$T = 0, \dot{x} = 80\cos\alpha$

$\therefore C_1 = 80\cos\alpha$

$\dot{x} = 80\cos\alpha$

$x = (80\cos\alpha)t + C_2$

$t = 0, x = 0$

$\therefore C_2 = 0$

$x = 80t\cos\alpha$

$\ddot{y} = -10$

$\dot{y} = -10t + C_3$

$t = 0, \dot{y} = 80\sin\alpha$

$\therefore C_3 = 80\sin\alpha$

$\dot{y} = -10t + 80\sin\alpha$

$x = -5t^2 + (80\sin\alpha)t + C_4$

$t = 0, y = 50$

$\therefore C_4 = 50$

$y = -5t^2 + 80t\sin\alpha + 50$

Marking guideline

2 x **and** y **Correct**

1 x **or** y **Correct**

Marker's comments

Too many students did not consider that there are constants of integration. It should be clearly stated what the constants are equal to and why.

(ii) The projectile lands on the ground 400 m away from O, the base of the tower. Find the possible values for α, giving your answer to the nearest degree.

Solution

$x = 80t\cos\alpha$

$x = 400, \quad t = \dfrac{400}{80\cos\alpha} = \dfrac{5}{\cos\alpha}$

$y = -5t^2 + 80t\sin\alpha + 50$

$0 = -5 \times \dfrac{25}{\cos^2\alpha} + 80 \times \dfrac{5}{\cos\alpha} \times \sin\alpha + 50$

$0 = -125\sec^2\alpha + 400\tan\alpha + 50$

$0 = -5\sec^2\alpha + 16\tan\alpha + 2$

$0 = -5(1 + \tan^2\alpha) + 16\tan\alpha + 2$

$0 = 5\tan^2\alpha - 16\tan\alpha + 3$

$(5\tan\alpha - 1)(\tan\alpha - 3) = 0$

$\tan\alpha = \dfrac{1}{5} \text{ or } \tan\alpha = 3$

$\alpha = 11^\circ$ or $\alpha = 72^\circ$ (nrst deg)

Marking guideline

3 **Correct solution**

2 **Correct equation in** $\tan\alpha$

1 **Correct value of** t **in terms of** α

Marker's comments

Many students could not obtain the correct equation in terms of $\tan\alpha$ to solve.

Question 14. (15 marks)

(a) Use one application of Newton's method to solve $x^2-\sqrt{x}-2=0$, by using a first approximation of $x=2$. Answer correct to two decimal places.

Solution

$$f(x)=x^2-\sqrt{x}-2$$
$$f'(x)=2x-\frac{1}{2\sqrt{x}}$$
$$x_0=2$$
$$x_1=x_0-\frac{f(x_0)}{f'(x_0)}$$
$$x_1=2-\frac{2^2-\sqrt{2}-2}{2\times 2-\frac{1}{2\sqrt{2}}}$$
$$x_1=1.839354171...$$
$$x_1\approx 1.84$$

Marking guideline

2 Correct solution
1 Some progress

Marker's comments

Care needs to be exercised when attempting questions. Many students showed minimal working and made mistakes when finding the derivative of $f(x)$.

(b) (i) Find $\frac{d}{dx}\left(x\cos^{-1}x-\sqrt{1-x^2}\right)$

Solution

$$y=x\cos^{-1}x-\sqrt{1-x^2}$$
$$y'=\cos^{-1}x+x\times\frac{-1}{\sqrt{1-x^2}}-\frac{1}{2\sqrt{1-x^2}}\times(-2x)$$
$$y'=\cos^{-1}x-\frac{x}{\sqrt{1-x^2}}+\frac{x}{\sqrt{1-x^2}}$$
$$y'=\cos^{-1}x$$

Marking guideline

2 Correct solution
1 Correct differentiation

Marker's comments

Some students did not think to simplify the expression or incorrectly applied the chain rule.

Again, greater care should be exercised.

(ii) Hence , find the area between the curve $y=\cos x$, the y-axis and the lines $y=\frac{1}{2}$ and $y=1$.

Solution

Required Area

$$=\int_{\frac{1}{2}}^{1} x dy=\int_{\frac{1}{2}}^{1}\cos^{-1} y dy$$

$$=\left[y\cos^{-1} y-\sqrt{1-y^2}\right]_{\frac{1}{2}}^{1}$$

$$=\left[(0-0)-\left(\frac{1}{2}\times\frac{\pi}{3}-\frac{\sqrt{3}}{2}\right)\right]$$

$$=\frac{\sqrt{3}}{2}-\frac{\pi}{6}$$

Marking guideline

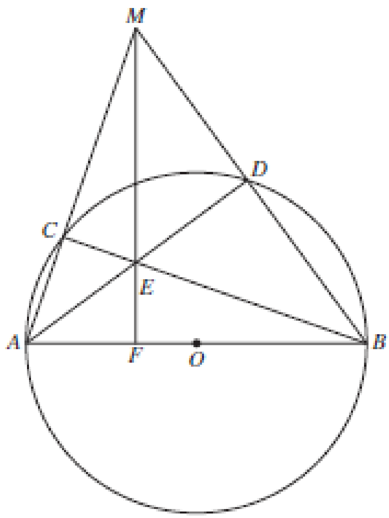
2 Correct solution
1 Correct integration

Marker's comments
Many students could not make the link with the previous part. To help, think,

1) draw a diagram, shading the area required
2) how do you find the shaded area?
3) can I use a previous part?

Many students retained the same pronumeral used in part 1 not recognising the integral would be in terms of *y*.

(a) The circle centred at O has a diameter AB. From the point M outside the circle the line segments MA and MB are drawn meeting the circle at C and D respectively, as shown in the diagram. The chords AD and BC meet at E. The line segment ME produced meet the diameter AB at F.



Copy or trace the diagram into your writing booklet

(i) Show that $CMDE$ is a cyclic quadrilateral
Solution

$\angle ADB=90^\circ$, $\angle BCA=90^\circ$ (angle in a semicircle)
$\angle MDE=90^\circ$ (straight angle)
$\angle ACB=\angle MDE$
$\therefore CMDE$ is a cyclic quadrilateral (exterior $\angle ACB$ is equal to the interior opposite $\angle MDE$)

Marking guideline

2 Correct solution
1 some progress

Marker's comments
Generally well done, although a small number of students over complicated their solution.

(ii) Hence, or otherwise , prove that MF is perpendicular to AB

Solution

Join CD.
Let

$\angle ABC = \theta$
$\therefore \angle ADC = \theta$

(Angle in the same segment)

$\angle CME = \theta \left(\text{Angle in the same segment} \right)$

$\angle CAB = 90^\circ - \theta$

In

ΔAMF,
$\angle MFA = 180^\circ - \theta - (90^\circ - \theta) = 90^\circ$

$\angle MFA = 90^\circ$

$\therefore MF \perp AB$

Marking guideline

2 Correct solution
1 some progress

Marker's comments

There were alternate solutions to this, including proving similar triangles.

Many students had trouble with this question.

To improve, students should

1) copy the diagram into their booklets
2) label an angle θ
3) find all other angles in terms of θ
4) use the relevant angles to answer the question

(d) Consider the parabola $x^2 = 4y$. $P(2p, p^2)$ and $Q(2q, q^2)$ lie on the parabola .

(i) Show that the equation of the chord PQ is $y = \left(\frac{p+q}{2} \right) x - pq$

Solution

$$
\begin{aligned}
y - p^2 &= \frac{q^2 - p^2}{2q - 2p}(x - 2p) \\
y - p^2 &= \frac{(q-p)(q+p)}{2(q-p)}(x - 2p) \\
y - p^2 &= \frac{(q+p)}{2}(x - 2p) \\
y &= \frac{(q+p)}{2}x - 2p\frac{(q+p)}{2} + p^2 \\
y &= \frac{(q+p)}{2}x - pq - p^2 + p^2 \\
y &= \frac{(q+p)}{2}x - pq
\end{aligned}
$$

Marking guideline

2 Correct solution
1 some progress

Marker's comments

Well answered.

(ii) If PQ is a focal chord, then

$$1 = \left(\frac{p+q}{2}\right) \times 0 - pq$$

$$pq = -1$$

TR is also a focal chord.

$\therefore rt = -1$

Now, P, T, Q and R are concyclic.

$\therefore PS \times SQ = RS \times ST$

$$PS^2 = (2p-0)^2 + (p^2-1)^2$$

$$PS^2 = p^4 + 2p^2 + 1 = (p^2+1)^2$$

$$PS = p^2 + 1$$

Similarly

$$SQ = q^2 + 1$$

$$RS = r^2 + 1$$

$$ST = t^2 + 1$$

$$(p^2+1)(q^2+1) = (r^2+1)(t^2+1)$$

$$p^2q^2 + p^2 + q^2 + 1 = r^2t^2 + r^2 + t^2 + 1$$

$$(-1)^2 + p^2 + q^2 = (-1)^2 + r^2 + t^2$$

$$\therefore p^2 + q^2 = r^2 + t^2$$

Hence proved.

Marking guideline

3 **Correct solution**

2 **Correct result for focal chord and recognition of** $PS \times SQ = RS \times ST$

1 **Finding the equation of the focal chords with the correct result for** pq *and* tr

Marker's comments

Many students experienced difficulty with this question.

Incorrect answers included

* Assumptions about the chords being diameters.
* Stating $pq = tr$ without proof
* Incorrectly squaring both sides, for example many students incorrectly wrote

$$p + q = t + r$$

$$p^2 + q^2 = t^2 + r^2$$

But $p + q \neq p^2 + q^2$

$t + r \neq t^2 + r^2$