GIRRAWEEN 2006 EX 1
TRIAL

Total marks – 84
Attempt Questions 1 – 7
All questions are of equal value

Answer each question on a SEPARATE piece of paper clearly marked Question 1, Question 2, etc. Each piece of paper must show your name.

Marks

Question 1 ***(12 marks)*** Use a *separate* piece of paper

a) Find $\int_0^5 \frac{dx}{\sqrt{25-x^2}}$ 2

b) Find the coordinates of the point that divides the interval AB with $A(1,4)$ and $B(5,2)$ externally in the ratio 1:3. 2

c) Evaluate $\lim_{x \to 0} \frac{\sin\left(\frac{x}{3}\right)}{3x}$ 2

d) Solve $\frac{4}{5-x} \geq 1$ 3

e) Use the substitution $u = 1-x$ to evaluate $\int_{-1}^{0} \frac{dx}{\sqrt{1-x}}$ 3

Question 2 ***(12 marks)*** Use a *separate* piece of paper

a) Find $\frac{d}{dx}\left(x\cos^{-1}x\right)$ 2

b) How many ten letter arrangements can be made using the letters of the word PHENOMENON? 2

c) Write down the general solution of the equation $2\sin\theta = \sqrt{3}$ 2

d) State the domain and range of $y = 4\cos^{-1}\left(\frac{x}{3}\right)$ and sketch the curve. 3

e) Find the coefficient of x^7 in the expansion of $\left(x^2 - \frac{1}{x}\right)^{20}$ 3

Marks

Question 3 ***(12 marks)*** Use a *separate* piece of paper

a) Find $\int \cos^2 2x\,dx$ 2

b) Prove $\sin^{-1}\frac{1}{\sqrt{5}} + \sin^{-1}\frac{1}{\sqrt{10}} = \frac{\pi}{4}$ 3

c) Use $x = 0.5$ to find an approximation for the root of $\cos x = x$ using one application of Newton's Method, correct to 2 decimal places. 3

d) (i) Express $\sin x + \sqrt{3}\cos x$ in the form $A\sin(x+\alpha)$ 2

(ii) Hence solve $\sin x + \sqrt{3}\cos x = 1$ for $0 \leq x \leq 2\pi$ 2

Question 4 ***(12 marks)*** Use a *separate* piece of paper

a) Use the principle of mathematical induction to prove that; 3

$$\frac{1}{1\times 4} + \frac{1}{4\times 7} + \frac{1}{7\times 10} + \ldots + \frac{1}{(3n-2)(3n+1)} = \frac{n}{3n+1}$$

for all positive integers *n*.

b) The two points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are on the parabola $x^2 = 4ay$.

(i) The equation of the tangent to $x^2 = 4ay$ at an arbitrary point $T(2at, at^2)$ on the parabola is $y = tx - at^2$. (You do not need to prove this) 2

Show that the tangents at the points P and Q meet at R, where R is the point $\{a(p+q), apq\}$.

(ii) If R lies on the line $y = -x - 5a$ find the relationship between p and q. 1

(iii) Hence, or otherwise, find the locus of the midpoint of PQ. 2

c) A molten plastic at a temperature of 250°C is poured into moulds to form car parts. After 20 minutes the plastic has cooled to 150°C. If the temperature after t minutes is T°C, and if the temperature of the surroundings is 30°C, then the rate of cooling is approximately given by;

$$\frac{dT}{dt} = -k(T-30) \quad \text{, where } k \text{ is a positive constant}$$

(i) Verify that $T = 30 + 220e^{-kt}$ satisfies the above equation. 1

(ii) Show that $k = \frac{1}{20}\log\left(\frac{11}{6}\right)$. 2

(iii) The plastic can be taken out of the moulds when the temperature has dropped to 80°C. How long after the plastic has been poured will this temperature be reached? Give the answer to the nearest minute. 1

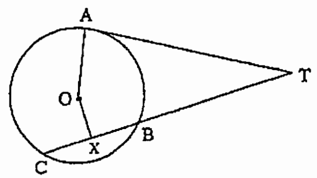
33

Marks

Question 5 *(12 marks)* Use a *separate* piece of paper

a) Solve the equation $3x^3 - 17x^2 - 8x + 12 = 0$ given that the product of two of the roots is 4. *3*

b) A, B and C are three points on a circle centre O. The tangent at A meets BC produced at T. X is the midpoint of BC.



(i) Prove that AOXT is a cyclic quadrilateral. *3*

(ii) Hence state why $\angle AOT = \angle AXT$ *1*

c) A particle moves in a straight line with an acceleration given by;

$$\frac{d^2x}{dt^2} = 9(x-2)$$

where x is the displacement in metres from the origin O after t seconds. Initially the particle is 4 metres to the right of O with a velocity $v = -6$

(i) Show that $v^2 = 9(x-2)^2$ *2*

(ii) Find an expression for v and hence find x as a function of t *3*

Marks

Question 6 *(12 marks)* Use a *separate* piece of paper

a) By considering both sides of the identity $(1+x)^m(1+x)^n \equiv (1+x)^{m+n}$ and comparing coefficients, show that; *3*

$$\binom{m+n}{3} = \binom{m}{3} + \binom{m}{2}\binom{n}{1} + \binom{m}{1}\binom{n}{2} + \binom{n}{3}$$

b) A particle moves in a straight line and its position at time t seconds is given by

$$x = 5 + 4\sin 2t$$

(i) Show that the particle undergoes Simple Harmonic Motion *2*

(ii) Find the centre and amplitude of the motion. *2*

(iii) Determine the particle's maximum speed. *1*

c) The probability that a vaccine succeeds is $\frac{29}{30}$. An experiment is conducted m times with white mice.

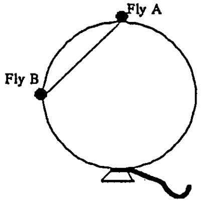
(i) What is the probability that the experiment will fail at least once? *1*

(ii) If the probability that the experiment will fail at least once in m trials is greater than 90%, find the minimum number of times the experiment was conducted. *3*

34

Question 7 *(12 marks)* Use a *separate* piece of paper **Marks**

a) Two flies are sitting on a spherical balloon of radius r cm, while it is being inflated at a constant rate of $5\ \text{cm}^3/\text{s}$.
Assume that the balloon has no air in it to begin with and that the two flies are located at the North Pole and the Equator of the balloon.



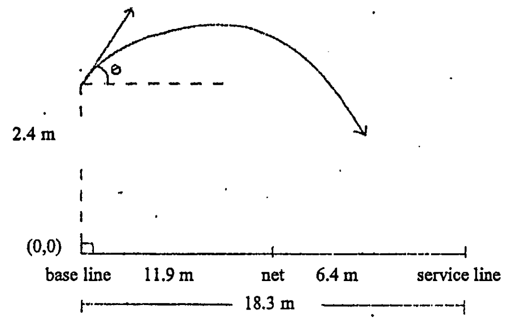
(i) Show that the distance between the two flies is $\sqrt{2}r$ cm. **1**

(ii) Hence show that the velocity of the two flies parting company is $\dfrac{5\sqrt{2}}{4\pi r^2}$ cm/s **2**

(iii) How fast are the two flies parting company after 3 seconds? Give your answer correct to two decimal places. **2**

b) In the 2006 Wimbledon Men's Final, Roger Federer's serve was measured to have an initial velocity of 180 km/h or 50 m/s.

Federer served the ball at the base line from a height of 2.4 metres at an angle of inclination of θ. In order not fault, the ball must land past the net and before the service line, that is a range between 11.9 metres and 18.3 metres.



Taking the origin as in the diagram and acceleration due to gravity as $9.8\ \text{m/s}^2$,

(i) Derive the equations of motion and show that the position of the ball after t seconds is given by; **3**

$$x = 50t\cos\theta \qquad \text{and} \qquad y = -4.9t^2 + 50t\sin\theta + 2.4$$

(ii) Hence show that $y = \dfrac{-4.9x^2\sec^2\theta}{50^2} + x\tan\theta + 2.4$ **2**

(iii) Calculate whether Federer will serve a fault if he serves the ball horizontally. **2**

Extension 1 Trial HSC 2006 Solutions

Question 1 (12)

a) $\displaystyle\int_0^5 \frac{dx}{\sqrt{25-x^2}} = \left[\sin^{-1}\frac{x}{5}\right]_0^5$

$= \sin^{-1}1 - \sin^{-1}0$

$= \frac{\pi}{2}$ (2)

b) $A(1,4)$ $B(5,2)$

$-1:3$

$P = \left(\frac{3-5}{2}, \frac{12-2}{2}\right)$

$= (-1,5)$ (2)

c) $\displaystyle\lim_{x\to 0} \frac{\sin\left(\frac{x}{3}\right)}{3x}$

$= \displaystyle\lim_{x\to 0} \frac{1}{9}\,\frac{\sin\left(\frac{x}{3}\right)}{\frac{x}{3}}$

$= \frac{1}{9}$ (2)

d) $\frac{4}{5-x} \geq 1$

$5-x \neq 0$, $x \neq 5$

$\frac{4}{5-x} = 1$, $4 = 5-x$, $x = 1$

test $x=0$: $\frac{4}{5} \geq 1$ ✗

$1 \leq x < 5$ (3)

e) $\displaystyle\int_{-1}^{0} \frac{dx}{\sqrt{1-x}}$ $\quad u = 1-x$, $du = -dx$, $x=0, u=1$; $x=-1, u=2$

$= -\displaystyle\int_2^1 \frac{du}{\sqrt{u}}$

$= -\left[2\sqrt{u}\right]_2^1$

$= -2(\sqrt{1}-\sqrt{2})$

$= 2\sqrt{2} - 2$ (3)

Question 2 (12)

a) $\frac{d}{dx}(x\cos^{-1}x)$

$= (x)\left(\frac{-1}{\sqrt{1-x^2}}\right) + (\cos^{-1}x)(1)$

$= \frac{-x}{\sqrt{1-x^2}} + \cos^{-1}x$ (2)

b) Arrangements $= \frac{10!}{2!3!2!}$

$= 151\,200$ (2)

c) $2\sin\theta = \sqrt{3}$

$\sin\theta = \frac{\sqrt{3}}{2}$

$\theta = \pi k + (-1)^k \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$= \pi k + (-1)^k \frac{\pi}{3}$

where k is an integer (2)

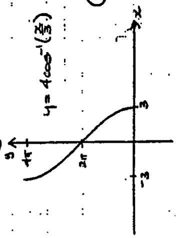
d) $y = 4\cos^{-1}\left(\frac{x}{3}\right)$

domain: $-1 \leq \frac{x}{3} \leq 1$

$-3 \leq x \leq 3$

range: $0 \leq \frac{y}{4} \leq \pi$

$0 \leq y \leq 4\pi$



(3)

e) $\left(x^2 - \frac{1}{x}\right)^{20}$

$T_{k+1} = {}^{20}C_k (x^2)^{20-k}\left(-\frac{1}{x}\right)^k$

$= {}^{20}C_k (-1)^k x^{40-3k}$

$40 - 3k = 7$

$3k = 33$

$k = 11$

$\therefore$ coefficient of x^7 is $-{}^{20}C_{11} = -167960$ (3)

Question 3 (12)

a) $\displaystyle\int \cos^2 2x\,dx$

$= \frac{1}{2}\displaystyle\int (1+\cos 4x)\,dx$

$= \frac{1}{2}\left(x + \frac{1}{4}\sin 4x\right) + c$

$= \frac{1}{2}x + \frac{1}{8}\sin 4x + c$ (2)

b) $\alpha = \sin^{-1}\frac{1}{\sqrt{5}}$ $\quad \beta = \sin^{-1}\frac{1}{\sqrt{10}}$

$\sqrt{5}$, 1, 2 $\qquad \sqrt{10}$, 1, 3

$\sin(\alpha+\beta)$

$= \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$= \left(\frac{1}{\sqrt{5}}\right)\left(\frac{3}{\sqrt{10}}\right) + \left(\frac{2}{\sqrt{5}}\right)\left(\frac{1}{\sqrt{10}}\right)$

$= \frac{5}{\sqrt{50}}$

$= \frac{1}{\sqrt{2}}$

$\therefore \alpha + \beta = \sin^{-1}\frac{1}{\sqrt{2}}$

$\sin^{-1}\frac{1}{\sqrt{5}} + \sin^{-1}\frac{1}{\sqrt{10}} = \frac{\pi}{4}$ (3)

c) $f(x) = \cos x - x$

$f'(x) = -\sin x - 1$

$f(0.5) = 0.3776$

$f'(0.5) = -1.4794$

$x_1 = 0.5 - \frac{f(0.5)}{f'(0.5)}$

$= 0.5 + \frac{0.3776}{1.4794}$

$= 0.76$ (to 2dp) (3)

d) $\sin x + \sqrt{3}\cos x$

$= 2\sin\left(x + \frac{\pi}{3}\right)$ (2)

2, $\sqrt{3}$, 1, α

$\alpha = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$

(ii) $\sin x + \sqrt{3}\cos x = 1$

$\sin\left(x + \frac{\pi}{3}\right) = \frac{1}{2}$

Q1, Q2

$\sin\beta = \frac{1}{2}$

$\beta = \frac{\pi}{6}$

$x + \frac{\pi}{3} = \frac{\pi}{6}, \frac{5\pi}{6}$

$x = -\frac{\pi}{6}, \frac{\pi}{2}$

$x = \frac{11\pi}{6}, \frac{\pi}{2}$ (2)

Question 4 (12)

a) Step 1 Prove true for $n=1$

LHS $= \frac{1}{(3-2)(3+1)} = \frac{1}{1\times 4} = \frac{1}{4}$ $\qquad$ RHS $= \frac{1}{3+1} = \frac{1}{4}$

$\therefore$ LHS = RHS

Hence the result is true for $n=1$.

Step 2 Assume true for $n=k$, where k is a positive integer.

i.e. $\frac{1}{1\times4} + \frac{1}{4\times7} + \frac{1}{7\times10} + \ldots + \frac{1}{(3k-2)(3k+1)} = \frac{k}{3k+1}$

Step 3 Prove true for $n=k+1$

i.e. Prove $\frac{1}{1\times4} + \frac{1}{4\times7} + \frac{1}{7\times10} + \ldots + \frac{1}{(3k+1)(3k+4)} = \frac{k+1}{3k+4}$

Proof:

$\frac{1}{1\times4} + \frac{1}{4\times7} + \frac{1}{7\times10} + \ldots + \frac{1}{(3k-2)(3k+1)} + \frac{1}{(3k+1)(3k+4)}$

$= \frac{k}{3k+1} + \frac{1}{(3k+1)(3k+4)}$

$= \frac{k(3k+4)+1}{(3k+1)(3k+4)}$

$= \frac{3k^2+4k+1}{(3k+1)(3k+4)}$

$= \frac{(3k+1)(k+1)}{(3k+1)(3k+4)}$

$= \frac{k+1}{3k+4}$

Hence the result is true for $n=k+1$ if it is also true for $n=k$.

Step 4 Since the result is true for $n=1$ then it is also true for $n=1+1$ i.e. $n=2$ and since its true for $n=2$ then it is also true for $n=2+1$ i.e. $n=3$ and soon for all positive integral values of n. (3)

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$P = \left(\frac{3-5}{2}, \frac{12-2}{2}\right)$

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$= \lim_{x\to 0} \frac{1}{9} \times \frac{\sin\left(\frac{x}{3}\right)}{\frac{x}{3}}$

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d) $\frac{4}{5-x} \geq 1$

$5-x \neq 0$, $x \neq 5$

$\frac{4}{5-x} = 1$

$4 = 5-x$

$x = 1$

test $x=0$

$\frac{4}{5} \geq 1$ x

$1 \leq x < 5$ (3)

e) $\int_{-1}^{0} \frac{dx}{\sqrt{1-x}}$ $u = 1-x$, $du = -dx$, $x=0, u=1$, $x=-1, u=2$

$= -\int_2^1 \frac{du}{\sqrt{u}}$

$= -\left[2\sqrt{u}\right]_2^1$

$= -2(\sqrt{1} - \sqrt{2})$

$= 2\sqrt{2} - 2$ (3)

Question 2 (12)

a) $\frac{d}{dx}(x\cos^{-1}x)$

$= (x)\left(\frac{-1}{\sqrt{1-x^2}}\right) + (\cos^{-1}x)(1)$

$= \frac{-x}{\sqrt{1-x^2}} + \cos^{-1}x$ (2)

b) Arrangements $= \frac{10!}{2!3!2!}$

$= 151\,200$ (2)

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$\sin\theta = \frac{\sqrt{3}}{2}$

$\theta = \pi k + (-1)^k \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$= \pi k + (-1)^k \frac{\pi}{3}$ (2)

where k is an integer

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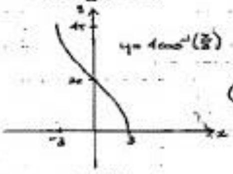

(3)

e) $\left(x^2 - \frac{1}{x}\right)^{20}$

$T_{k+1} = {}^{20}C_k (x^2)^{20-k}\left(-\frac{1}{x}\right)^k$

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a) $\int \cos^2 2x \, dx$

$= \frac{1}{2}\int (1+\cos 4x)\,dx$

$= \frac{1}{2}\left(x + \frac{1}{4}\sin 4x\right) + c$

$= \frac{1}{2}x + \frac{1}{8}\sin 4x + c$ (2)

b) $\alpha = \sin^{-1}\frac{1}{\sqrt{5}}$ $\beta = \sin^{-1}\frac{1}{\sqrt{10}}$

$\sin(\alpha+\beta)$

$= \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$= \left(\frac{1}{\sqrt{5}}\right)\left(\frac{3}{\sqrt{10}}\right) + \left(\frac{2}{\sqrt{5}}\right)\left(\frac{1}{\sqrt{10}}\right)$

$= \frac{5}{\sqrt{50}}$

$= \frac{1}{\sqrt{2}}$

$\therefore \alpha + \beta = \sin^{-1}\frac{1}{\sqrt{2}}$

$\sin^{-1}\frac{1}{\sqrt{5}} + \sin^{-1}\frac{1}{\sqrt{10}} = \frac{\pi}{4}$ (2)

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$x_1 = 0.5 - \frac{f(0.5)}{f'(0.5)}$

$= 0.5 + \frac{0.3776}{1.4794}$

$= 0.76$ (to 2dp) (3)

d) $\sin x + \sqrt{3}\cos x$

$= 2\sin\left(x + \frac{\pi}{3}\right)$ (2) $\alpha = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$

(ii) $\sin x + \sqrt{3}\cos x = 1$

$\sin\left(x + \frac{\pi}{3}\right) = \frac{1}{2}$

Q1, Q2

$\sin\alpha = \frac{1}{2}$

$\alpha = \frac{\pi}{6}$

$x + \frac{\pi}{3} = \frac{\pi}{6}, \frac{5\pi}{6}$

$x = -\frac{\pi}{6}, \frac{\pi}{2}$

$x = \frac{11\pi}{6}, \frac{\pi}{2}$ (2)

Question 4 (12)

a) Step 1 Prove true for $n=1$

LHS $= \frac{1}{(3-2)(3+1)}$ RHS $= \frac{1}{3+1}$

$= \frac{1}{1\times4}$ $= \frac{1}{4}$

$= \frac{1}{4}$ $\therefore$ LHS = RHS

Hence the result is true for $n=1$.

Step 2 Assume true for $n=k$, where k is a positive integer.

ie $\frac{1}{1\times4} + \frac{1}{4\times7} + \frac{1}{7\times10} + \dots + \frac{1}{(3k-2)(3k+1)} = \frac{k}{3k+1}$

Step 3 Prove true for $n = k+1$

ie Prove

$\frac{1}{1\times4} + \frac{1}{4\times7} + \frac{1}{7\times10} + \dots + \frac{1}{(3k+1)(3k+4)} = \frac{k+1}{3k+4}$

Proof:

$\frac{1}{1\times4} + \frac{1}{4\times7} + \frac{1}{7\times10} + \dots + \frac{1}{(3k-2)(3k+1)} + \frac{1}{(3k+1)(3k+4)}$

$= \frac{k}{3k+1} + \frac{1}{(3k+1)(3k+4)}$

$= \frac{k(3k+4)+1}{(3k+1)(3k+4)}$

$= \frac{3k^2+4k+1}{(3k+1)(3k+4)}$

$= \frac{(3k+1)(k+1)}{(3k+1)(3k+4)}$

$= \frac{k+1}{3k+4}$

Hence the result is true for $n=k+1$ if it is also true for $n=k$

Step 4

Since the result is true for $n=1$ then it is also true for $n=1+1$ ie $n=2$ and since it's true for $n=2$ then it is also true for $n=2+1$ ie $n=3$ and soon for all positive integral values of n. (3)

b) $px - y = ap^2$ (−)

$qx - y = aq^2$

$(p-q)x = a(p^2-q^2)$

$= a(p-q)(p+q)$

$x = a(p+q)$

$y = ap(p+q) - ap^2$

$= ap^2 + apq - ap^2$

$= apq$

$\therefore R$ is $\{a(p+q), apq\}$ (2)

(ii) $y = -x - 5a$

$apq = -a(p+q) - 5a$

$pq = -(p+q) - 5$ (1)

(iii) $M = \left(\frac{2ap+2aq}{2}, \frac{ap^2+aq^2}{2}\right)$

$= \{a(p+q), \frac{1}{2}a(p^2+q^2)\}$

$y = \frac{1}{2}a(p+q)^2 - apq$

$= \frac{1}{2}a(p+q)^2 + a(p+q) + 5a$

$= \frac{1}{2}a\left(\frac{x}{a}\right)^2 + x + 5a$

$y = \frac{1}{2a}x^2 + x + 5a$ (2)

c)(i) $T = 30 + 220e^{-kt}$

$\frac{dT}{dt} = -220ke^{-kt}$

$= -k(220e^{-kt})$

$= -k(T-30)$ (1)

(ii) when $t=20$, $T=150$

$150 = 30 + 220e^{-20k}$

$120 = 220e^{-20k}$

$e^{-20k} = \frac{6}{11}$

$-20k = \log\frac{6}{11}$

$k = -\frac{1}{20}\log\frac{6}{11}$

$k = \frac{1}{20}\log\frac{11}{6}$ (2)

(iii) when $T = 80$,

$80 = 30 + 220e^{-kt}$

$50 = 220e^{-kt}$

$e^{-kt} = \frac{5}{22}$

$-kt = \log\frac{5}{22}$

$t = -\frac{1}{k}\log\frac{5}{22}$

$t = 48.9$ (1)

$\therefore$ After 49 minutes temperature will be 80°C

Question 5 (12)

a) $3x^3 - 17x^2 - 8x + 12 = 0$

$\alpha\beta\gamma = -4$

$4\gamma = -4$

$\gamma = -1$

$(x+1)(3x^2-20x+12) = 0$

$(x+1)(3x-2)(x-6) = 0$

$x = -1, \frac{2}{3}, 6$ (3)

b)(i) $BY = CY$ (given)

$\therefore \angle OYB = 90°$ (⊥ centre bisects chord)

$\angle OAT = 90°$ (radius ⊥ tangent)

$\therefore AOYT$ is cyclic quadrilateral (opposite ∠s are supplementary) (2)

(ii) $\angle AOT = \angle AYT$ (∠'s in same segment are =) (1)

c)(i) $\frac{d^2x}{dt^2} = 9(x-2)$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 9(x-2)$

$\frac{1}{2}v^2 = \frac{9}{2}(x-2)^2 + c$

when $x=4$, $v=-6$

$\frac{1}{2}(-6)^2 = \frac{9}{2}(4-2)^2 + c$

$18 = 18 + c$

$c = 0$

$\therefore \frac{1}{2}v^2 = \frac{9}{2}(x-2)^2$

$v^2 = 9(x-2)^2$ (2)

(ii) $v = \pm 3(x-2)$

$\frac{dx}{dt} = -3(x-2)$ ($v<0$ to match initial condition)

$t = -\frac{1}{3}\int\frac{dx}{x-2}$

$= -\frac{1}{3}\log(x-2) + c$

when $t = 0$, $x = 4$

$0 = -\frac{1}{3}\log 2 + c$

$c = \frac{1}{3}\log 2$

$t = \frac{1}{3}\log\left(\frac{2}{x-2}\right)$

$3t = \log\left(\frac{2}{x-2}\right)$

$\frac{2}{x-2} = e^{3t}$

$x - 2 = 2e^{-3t}$

$x = 2e^{-3t} + 2$ (3)

Question 6 (12)

a) $(1+x)^m(1+x)^n = (1+x)^{m+n}$

$(1+x)^m = \binom{m}{0} + \binom{m}{1}x + \binom{m}{2}x^2 + \binom{m}{3}x^3 \dots$

$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 \dots$

coefficient of x^3 in $(1+x)^m(1+x)^n$

$\binom{m}{0}\binom{n}{3} + \binom{m}{1}\binom{n}{2} + \binom{m}{2}\binom{n}{1} + \binom{m}{3}\binom{n}{0}$

$= \binom{n}{3} + \binom{m}{1}\binom{n}{2} + \binom{m}{2}\binom{n}{1} + \binom{m}{3}$

coefficient of x^3 in $(1+x)^{m+n}$

$\binom{m+n}{3}$

equating coefficients (3)

$\therefore \binom{m+n}{3} = \binom{n}{3} + \binom{m}{1}\binom{n}{2} + \binom{m}{2}\binom{n}{1} + \binom{m}{3}$

b) $x = 5 + 4\sin 2t$

$\dot{x} = 8\cos 2t$

$\ddot{x} = -16\sin 2t$

$= -4(4\sin 2t)$

$\ddot{x} = -4(x-5)$ (2)

Hence particle undergoes SHM as $\ddot{x} = -n^2x$.

(ii) centre $x = 5$

amplitude = 4 units (2)

(iii) max speed = 8 units/s (1)

c) $P(\geq 1 \text{ failure}) = 1 - P(\text{never fail})$

$= 1 - \left(\frac{29}{30}\right)^m$ (1)

(ii) $P(\geq 1 \text{ failure}) > \frac{9}{10}$

$1 - \left(\frac{29}{30}\right)^m > \frac{9}{10}$

$\left(\frac{29}{30}\right)^m < \frac{1}{10}$

$\log\left(\frac{29}{30}\right)^m < \log\frac{1}{10}$

$m\log\left(\frac{29}{30}\right) < \log\frac{1}{10}$

$m > \frac{\log\frac{1}{10}}{\log\left(\frac{29}{30}\right)}$

$m > \frac{\log 10}{\log\left(\frac{30}{29}\right)}$

$m > 67.9$ (3)

$\therefore$ 68 trials are required

Question 7 (12)

a)(i) $x^2 = r^2 + r^2$

$x^2 = 2r^2$

$x = \sqrt{2}r$ (1)

(ii) $\frac{dx}{dt} = \frac{dV}{dt} \times \frac{dr}{dV} \times \frac{dx}{dr}$ $V = \frac{4}{3}\pi r^3$ $\frac{dV}{dr} = 4\pi r^2$

$= (5)\left(\frac{1}{4\pi r^2}\right)(\sqrt{2})$

$= \frac{5\sqrt{2}}{4\pi r^2}$ (2)

(iii) when $t = 3$, $V = 15\text{ cm}^3$

$\frac{4}{3}\pi r^3 = 45$

$r^3 = \frac{45}{4\pi}$

$r = \sqrt[3]{\frac{45}{4\pi}}$

$\frac{dx}{dt} = \frac{5\sqrt{2}}{4\pi\left(\sqrt[3]{\frac{45}{4\pi}}\right)^2}$

$= 0.24\text{ cm/s}$ (2)

b) $\ddot{x} = 0$ $\ddot{y} = -9.8$

$\dot{x} = c_1$ $\dot{y} = -9.8t + c_2$

when $t=0$, $\dot{x} = 50\cos\theta$, $\dot{y} = 50\sin\theta$

$c_1 = 50\cos\theta$ $c_2 = 50\sin\theta$

$\dot{x} = 50\cos\theta$ $\dot{y} = -9.8t + 50\sin\theta$

$x = 50t\cos\theta + c_3$ $y = -4.9t^2 + 50t\sin\theta + c_4$

when $t = 0$, $x = 0$, $y = 2.4$

$c_3 = 0$ $c_4 = 2.4$

$x = 50t\cos\theta$ $y = -4.9t^2 + 50t\sin\theta + 2.4$ (3)

(ii) $t = \frac{x}{50\cos\theta}$

$y = -4.9\left(\frac{x}{50\cos\theta}\right)^2 + \left(\frac{x}{50\cos\theta}\right)50\sin\theta + 2.4$

$y = \frac{-4.9x^2\sec^2\theta}{50^2} + x\tan\theta + 2.4$ (2)

(iii) when $\theta = 0$, $y = 0$

$0 = \frac{-4.9x^2}{50^2} + 2.4$

$\frac{4.9x^2}{50^2} = 2.4$

$x^2 = \frac{2.4(50)^2}{4.9}$

$x = \sqrt{\frac{2.4}{4.9}}(50)$

$x = 35m$

$\therefore$ Fielder will fault (2)