Question 1 *(12 Marks)* Use a separate piece of paper — Marks

a) Evaluate $\int_0^{\sqrt{3}} \frac{dx}{\sqrt{4-x^2}}$ — 2

b) Let A be the point (-8 , -3) and B the point (4 , 7). Find the coordinates of the point that divides AB externally in the ratio 1 : 2. — 2

c) Use the substitution $u = \tan x$ to evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \tan^3 x \sec^2 x dx$ — 3

d) State the domain and range of the function $f(x) = 2\sin^{-1}\frac{x}{3}$ — 2

e) Solve for x $\frac{3x}{x-1} \le 2$ — 3

Question 2 *(12Marks)* Use a separate piece of paper

a) Find $\frac{d}{dx} x\tan^{-1} x^2$ — 3

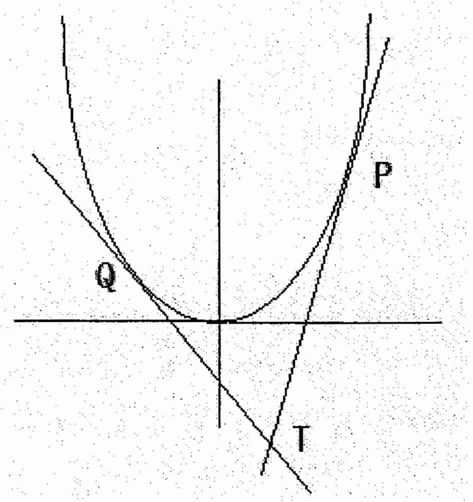
b) (i) Write $5\sin x + 3\cos x$ in the form $R\sin(x+\alpha)$ where $0 \le \alpha \le 90^0$ and $R \ge 0$ — 2

(ii) Hence or otherwise solve the equation $5\sin x + 3\cos x = 4$ for x to the nearest degree for $0 \le x \le 360^0$ — 2

c) Find the term independent of x in the expansion of $\left(2x - \frac{1}{x^2}\right)^9$ — 3

d) Evaluate $\int_0^{\frac{\pi}{4}} \sin^2 x dx$ — 2

Question 3 *(12 Marks)* Use a separate piece of paper Marks



a) The diagram above shows the parabola $x^2 = 4ay$ the points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are points on the parabola.

(i) Find the equation of the tangent at *P*. 3

(ii) State the equation of the tangent at *Q*. 1

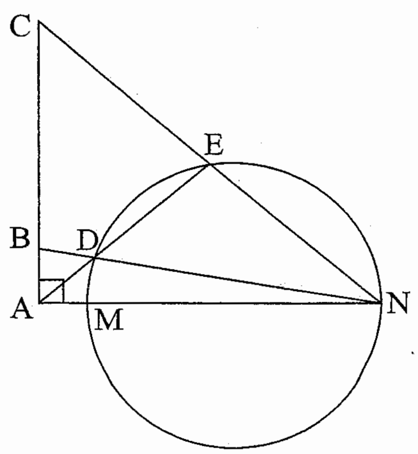
(iii) Find the coordinates of *T* the point of intersection of the two tangents 2

b) When *P(x)* is divided by $(x + 1)$ the remainder is 3, when *P(x)* is divided by $(x - 2)$ the remainder is -5. What is the remainder when *P(x)* is divided by $(x + 1)(x - 2)$. 3

c) In the figure, M, N, E and D are the points on the circle. MN is a diameter. NE is produced to C and NM is produced to A such that the CA ⊥ AN. AE meets the circle at D. ND is produced to meet CA at B.

(i) Prove ΔMEN is similar to ΔCAN 1

(ii) Hence or otherwise show that B, C, E and D are concyclic 2



Question 4 ***(12 Marks)*** Use a separate piece of paper — Marks

a) Assume that the rate at which a body cools in air is proportional to the difference between its temperature T and the constant temperature of the surrounding air A. This can be expressed as

$$\frac{dT}{dt} = -k(T-A) \text{ where } t \text{ is time in minutes and } k \text{ a constant}$$

(i) Show that $T = A + Ce^{-kt}$ where C is a constant is a solution to the differential equation. — 1

(ii) If molten steel cools from an initial temperature of 500^0 to 200^0 in 15 minutes with an air temperature of 20^0. Find the values of A, C and k. — 3

(iii) How long does it take, to the nearest minute, for the steel to cool to 100^0. — 1

b) Prove by Mathematical induction that

$$1^3 + 2^3 + 3^3 + \ldots\ldots + n^3 = \frac{1}{4}n^2(n+1)^2$$ — 3

c) The polynomial $P(x) = x^3 - x^2 - 2x + 5$ has a root between -1 and -2. Using $x = -1$ as the first value use Newton's Method once to find a better approximation. (Give your answer to two decimal places) — 2

d) Use the definition of the derivative $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

to find $f'(x)$ when $f(x) = \sqrt{x}$ — 2

Question 5 *(12 Marks)* Use a separate piece of paper

a) How many arrangements of the letters of the word WALLABY are possible. 1

b) For positive integers n and r with $r < n$ show that 3

$$^{n}C_{r} + {^{n}C_{r+1}} = {^{n+1}C_{r+1}}$$

$$where\ ^{n}C_{r} = \frac{n!}{r!(n-r)!}$$

c) A gambling game consists of three fair dice being rolled and betting on a particular number appearing on one of the uppermost faces. If such a game is played and the chosen number is 6

(i) What is the probability that no 6's will appear 1

(ii) What is the probability that at least one 6 will appear 1

(iii) If five such games are played, using a binomial expansion or otherwise, find the probability that exactly three turns will include at least one 6. (leave answer in index form) 2

(iv) Find the probability that at least one game will include at least one 6. (leave answer in index form) 2

d) Show that $\cos^{-1}\frac{1}{\sqrt{5}} + \cos^{-1}\frac{1}{\sqrt{10}} = \frac{3\pi}{4}$ 2

Question 6*(12 marks)*Use a separate piece of paper

Marks

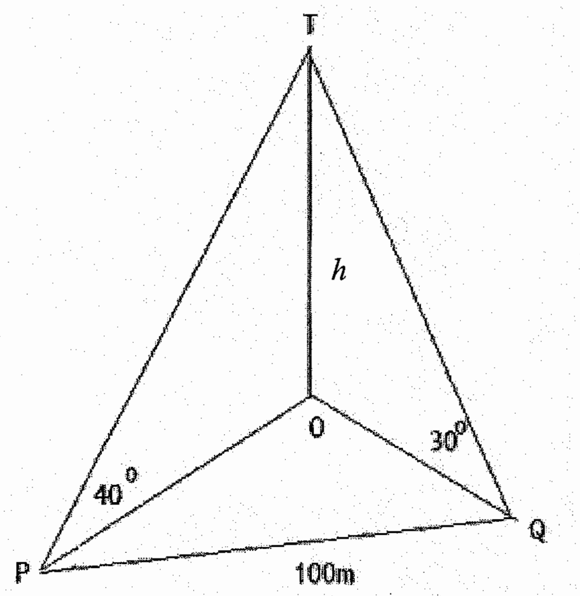
a) By putting $\frac{d^2x}{dt^2} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$ in the equation of S.H.M. $\frac{d^2x}{dt^2} = -n^2x$

(i) Show that $v^2 = n^2\left(a^2 - x^2\right)$ where a is the amplitude. 2

(ii) A particle P performing S.H.M. in a straight line about a point O has speeds of 5m/s and 3m/s at two points A and B which are 0.2 m and 0.6 m respectively from O.
Find the amplitude and frequency of the motion. 2

(iii) Find the length PO at the instant the velocity v the particle is 2/3 the maximum velocity of the motion. 2

b)



A surveyor stands at a point P due south of a tower OT of height h, and finds the angle of elevation of the top of the tower to be 40^0. And then walks 100m to a point Q, so that the angle POQ is 90°, and finds that the angle of elevation from Q is 30°

(i) Find expressions for OP and OQ in terms of *h*. 2

(ii) Show that $h = \frac{100(\tan 40^\circ \tan 30^\circ)}{\sqrt{\tan^2 40^\circ + \tan^2 30^\circ}}$ 2

(iii) Find the bearing of P from Q. 2

Question 7 *(12 Marks)* Use a separate piece of paper — Marks

a) A coal loader is stacking coal on a flat surface, in the shape of cone.

The cone has a semi vertical angle of 30^0. If the coal is being deposited at the rate of $1\text{m}^3/\text{min}$, find

(i) An expression for the volume of the cone in terms of the radius only — 2

(ii) The rate at which the radius is changing when the radius is 2m. — 2

b) (i) Find the largest positive domain of the function $f(x) = x^2 - 4x + 5$ for which $f(x)$ has an inverse function $f^{-1}(x)$ — 1

(ii) Find $f^{-1}(x)$ and hence sketch the graphs of $f(x)$ and $f^{-1}(x)$ on the same set of axes. — 2

c) A particle is projected with a velocity of V m/s at an angle of θ^0. Using

$$x = V\cos\theta t$$

$$y = V\sin\theta t - \frac{1}{2}gt^2$$

$$\dot{y} = V\sin\theta - gt$$

(There is no need to prove these results)

(i) Find an expression for the maximum height reached by the projectile — 1

(ii) Prove that the Cartesian equation of the particle is

$$y = \tan\theta x - \frac{g\sec^2\theta x^2}{2V^2}$$ — 2

(iii) If the particle passes through a point at height ***b***, and horizontal distance ***a*** from the origin, prove that the maximum height reached is given by

$$\frac{1}{4}\left[\frac{a^2\tan^2\theta}{a\tan\theta - b}\right]$$ — 2

Solutions Trial HSC Ext 1 Aug. 2007

Question 1

(a) $I = \int_0^{\sqrt{3}} \frac{dx}{\sqrt{4-x^2}}$

$= \left[\sin^{-1}\frac{x}{2}\right]_0^{\sqrt{3}}$

$= \sin^{-1}\frac{\sqrt{3}}{2} - \sin^{-1}0$

$I = \pi/3$ (2m)

(b) $A(-8,-3)$ $B(4,7)$ $1:-2$

$P_x = \frac{-2(-8)+1\times4}{-1}$ $P_y = \frac{-2(-3)+1\times7}{-1}$

$P(-20,-13)$ (2m)

(c) $I = \int_{\frac{\pi}{4}}^{\pi/3} \tan^3 x \sec^2 x\, dx$

let $u = \tan x$

$\frac{du}{dx} = \sec^2 x$

$I = \int_{\pi/4}^{\pi/3} u^3\, du$

$= \left[\frac{u^4}{4}\right]_{\pi/4}^{\pi/3}$

$= \left[\frac{\tan^4 x}{4}\right]_{\pi/4}^{\pi/3}$

$= \frac{9}{4} - \frac{1}{4}$

$I = 2u^2$ (3m)

(d) $f(x) = 2\sin^{-1} x/3$

Domain $-1 \le x/3 \le 1$

$-3 \le x \le 3$ (1m)

Range $-\pi/2 \le \sin^{-1} x/3 \le \pi/2$

$-\pi \le y \le \pi$ (1m)

(e) $\frac{3x}{x-1} \le 2$ $x \ne 1$

$\frac{3x\,(x-1)^2}{x-1} \le 2(x-1)^2$

$3x(x-1) \le 2(x-1)^2$

$0 \le 2(x-1)^2 - 3x(x-1)$

$0 \le (x-1)\{2x-2-3x\}$

$0 \le (-x-2)(x-1)$

$0 \ge (x+2)(x-1)$

$\therefore -2 \le x \le 1$

But $x \ne 1$ $\therefore -2 \le x < 1$ (3m)

Question 2

(a) $\frac{d}{dx}\, x\tan^{-1}x$

$= \tan^{-1}x^2 \cdot 1 + \frac{2x}{1+x^4} \cdot x$

$= \tan^{-1}x^2 + \frac{2x^2}{1+x^4}$ (3m)

(b)(i) $5\sin x + 3\cos x$

$= R(\sin x\cos\alpha + \cos x\sin\alpha)$

$R\cos\alpha = 5$ $R\sin\alpha = 3$

$\therefore \tan\alpha = 3/5$ $\alpha = 31^\circ$ (1m)

$R^2 = 25+9$ $R = \sqrt{34}$ (1m)

(ii) $5\sin x + 3\cos x = 4$

$\sqrt{34}\,(\sin[x+\alpha]) = 4$

$\sin(x+\alpha) = 4/\sqrt{34}$

$x+\alpha = 43^\circ, 137^\circ$

$x = 12^\circ, 106^\circ$ (2m)

(c) $\left(2x - \frac{1}{x^2}\right)^9 = {}^9C_0(2x)^9 + {}^9C_1(2x)^8\frac{-1}{x^2}\ldots$

required term ${}^9C_3(2x)^6\left(\frac{-1}{x^2}\right)^3$

$= (-1)\,{}^9C_3\, 64$

$= -5376$ (3m)

(d) $I = \int_0^{\pi/4} \sin^2 x\, dx$

$= \frac{1}{2}\int_0^{\pi/4} 1 - \cos 2x\, dx$

$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi/4}$

$= \frac{1}{2}\left[(\pi/4 - 1/2) - 0\right]$

$= \frac{1}{8}(\pi - 2)$ u^2

Question 3.

(a) (i) $P(2ap, ap^2)$

$x^2 = 4ay$

$y = \frac{x^2}{4a}$ $\frac{dy}{dx} = \frac{2x}{4a}$.

Question 3 cont.

Equation of tangent

$y - y_1 = m(x - x_1)$

$y - ap^2 = p(x - 2ap)$

$y - ap^2 = px - 2ap^2$

$y = px - ap^2$ (3m)

(ii) Equation of tangent at Q

$y = qx - aq^2$ (1m)

(iii) $\therefore px - ap^2 = qx - aq^2$

$(p-q)x = ap^2 - aq^2$

$x = \frac{a(p-q)(p+q)}{(p-q)}$

$\therefore x = a(p+q)$

$y = px - ap^2$

$y = ap(p+q) - ap^2$

$y = apq$

$T(a(p+q), apq)$ (2m)

(b) $P(x) \div (x+1)(x-2) = Q(x) + \frac{Rx}{(x+1)(x-2)}$

$\therefore P(x) = Q(x)(x+1)(x-2) + R(x)$

where $R(x) = mx + b$.

$P(-1) = Q(-1)(0)(-3) + m(-1) + b$

$3 = -m + b$ (A)

$P(2) = Q(2)(3)(0) + m(2) + b$

$-5 = 2m + b$ (B)

(B) − (A) $-8 = 3m$

$m = -8/3$

From (A) $3 = 8/3 + b$ $\therefore b = 1/3$

$R(x) = \frac{1}{3}(-8x+1)$ (3m)

(c)(i) Join M to E

In ΔMEN, ΔCAN

$\angle MEN = 90^\circ$ ($\angle$ standing on diametre)

$\angle CAN = 90^\circ$ (given)

$\angle MNE$ is common

$\therefore \Delta MEN \,|||\, \Delta CAN$ (AAA)

$\therefore \angle EMN = \angle NCA$ (3rd $\angle$ of $|||$ Δ's)

(ii) $\angle EMN = \angle EDN$ ($\angle$ on chord EN)

$\therefore \angle NCA = \angle EDN$

$\therefore$ Exterior angle = interior opposite angle

$\therefore$ BCED is a cyclic quad (2m)

Question 4.

(a) (i) $T = A + Ce^{-kt}$

$\frac{dT}{dt} = -kCe^{-kt}$

$= -k(A + Ce^{-kt} - A)$

$= -k(T - A)$ (1m)

(ii) $T = 20 + Ce^{-kt}$

when $t=0$ $T = 500$

$500 = 20 + C$

$C = 480^\circ$ $A = 20^\circ$ (air temp) (2m)

$\therefore T = 20 + 480e^{-kt}$

$t = 15$ $T = 200^\circ$

$200 = 20 + 480e^{-15k}$

$\frac{180}{480} = e^{-15k}$ $\left(-15k = \ln\frac{180}{480}\right)$

$k = 0.0654$. (1m)

(iii) $T = 20 + 480e^{-0.0654t}$

$100 = 20 + 480e^{-0.0654t}$

$\frac{80}{480} = e^{-0.0654t}$

$t = \frac{-\ln 80/480}{0.0654}$ (1m)

$t = 27.4$ minutes $t = 27$ minutes

(b) Step 1. Prove true for $n=1$

LHS $= 1^3 = 1$ RHS $= \frac{1}{4}(1)^2(2)^2 = 1$

LHS = RHS True for $n=1$

Step 2. Assume true for $n=k$

$1^3 + 2^3 + 3^3 + \ldots + k^3 = \frac{1}{4}(k^2)(k+1)^2$

Prove true for $n = k+1$

RHS $= \frac{1}{4}(k+1)^2(k+2)^2$

LHS $= 1^3 + 2^3 + 3^3 \ldots + k^3 + (k+1)^3$

$= \frac{1}{4}(k)^2(k+1)^2 + (k+1)^3$

$= \frac{1}{4}(k+1)^2\{k^2 + 4(k+1)\}$

$= \frac{1}{4}(k+1)^2(k+2)^2$

$=$ RHS. True for $n = k+1$

Step 3. Proved true for $n=1$

Step 2 implies true for $n = 2, 3, 4 \ldots$

$\therefore$ By the principal of mathematical induction true for all n (3m)

Question 4 cont.

(c) $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

$= -1 - \frac{5}{3}$

$= -2.67$ (2M)

(d) $f'(x) = \lim\limits_{h\to 0} \dfrac{\sqrt{x+h} - \sqrt{x}}{h}$

$= \lim\limits_{h\to 0} \dfrac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})}$

$= \lim\limits_{h\to 0} \dfrac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} = \lim\limits_{h\to 0} \dfrac{1}{\sqrt{x+h} + \sqrt{x}}$

$= \dfrac{1}{2\sqrt{x}}$ (2M)

Question 5.

(a) WALLABY $= \dfrac{7!}{2!\,2!}$ ← 2 L's

2 A's →

$= 1260.$ (1M)

(b) ${}^{n}C_{r} + {}^{n}C_{r+1}$

$= \dfrac{n!}{(n-r)!\,r!} + \dfrac{n!}{(n-(r+1))!\,(r+1)!}$

$= \dfrac{n!(r+1)}{(n-r)!(r+1)!} + \dfrac{n!\,(n-r)}{(n-r)!\,(r+1)!}$

$= \dfrac{r\,n! + n! + n(n!) - r\,n!}{(n-r)!\,(r+1)!}$

$= \dfrac{n+1\,!}{(n-r)!\,(r+1)!} = {}^{n+1}C_{r+1}$ (3M)

(c) (i) $\frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} = \frac{125}{216}$ (1M)

(ii) $1 - \frac{125}{216} = \frac{91}{216}$ (1M)

(iii) $\left(\frac{125}{216} + \frac{91}{216}\right)^5 = {}^{5}C_0\left(\frac{125}{216}\right)^5 + \dots$

$+ {}^{5}C_3\left(\frac{125}{216}\right)^2\left(\frac{91}{216}\right)^3$

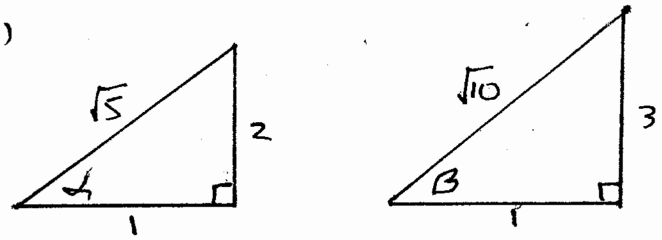
$\therefore P(\text{exactly 3 TURNS}) = 10\,\dfrac{125^2 \cdot 91^3}{(216)^5}$ (2M)

(iv) $P(\text{AT LEAST ONE 6}) = 1 - P(\text{NO 6's})$

$= 1 - {}^{5}C_0\left(\frac{125}{216}\right)^5$

$= 1 - \left(\frac{125}{216}\right)^5$ (2M)

(d)



let $\cos\alpha = \frac{1}{\sqrt{5}}$ let $\cos\beta = \frac{1}{\sqrt{10}}$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$= \frac{1}{\sqrt{5}} \cdot \frac{1}{\sqrt{10}} - \frac{2}{\sqrt{5}} \cdot \frac{3}{\sqrt{10}}$

$= \dfrac{-5}{5\sqrt{2}} = \dfrac{-1}{\sqrt{2}}$

$\therefore \cos(\alpha+\beta) = -1/\sqrt{2}$

$\therefore \alpha + \beta = 3\pi/4$

$\therefore \cos^{-1}\frac{1}{\sqrt{5}} + \cos^{-1}\frac{1}{\sqrt{10}} = \frac{3\pi}{4}$ 2M

Question 6.

(a) (i) $\dfrac{d^2x}{dt^2} = -n^2 x$

$\dfrac{d\frac{1}{2}v^2}{dx} = -n^2 x$

$\frac{1}{2}v^2 = -n^2\int x\,dx$

$= -\dfrac{n^2x^2}{2} + c$

when $x = a$ $v = 0$ $\therefore c = \dfrac{n^2a^2}{2}$

$\therefore \frac{1}{2}v^2 = \dfrac{n^2a^2}{2} - \dfrac{n^2x^2}{2}$

$v^2 = n^2(a^2 - x^2)$ (2M)

(ii) Using $v^2 = n^2(a^2 - x^2)$

$25 = n^2(a^2 - (0.2)^2)$ (A)

$9 = n^2(a^2 - (0.6)^2)$ (B)

$\dfrac{(A)}{(B)}$ $\dfrac{25}{9} = \dfrac{a^2 - 0.04}{a^2 - 0.36}$

$25a^2 - 9 = 9a^2 - 0.36$

$16a^2 = 8.64$

$a^2 = 0.54$

$a = 0.735$ (1M)

(A) − (B) $16 = 0.32n^2$

$n^2 = 50$

$n = \sqrt{50}$

Frequency $= n/2\pi$

$= \dfrac{\sqrt{50}}{2\pi}$ (1M)

Question 6 cont.

(iii) MAX VELOCITY OCCURS AT $x=0$

$v^2 = n^2a^2$

$v^2 = 27 \qquad v = 3\sqrt{3}$.

$\frac{2}{3}v_{MAX} = 2\sqrt{3}$.

$v^2 = n^2(a^2-x^2)$

$12 = 50(0.54 - x^2)$

$x^2 = 0.3$

$x = 0.548\text{ m}$ (2m)

(b)(i) $\tan 40 = \frac{h}{OP}$

$OP = \frac{h}{\tan 40}$

Similarly $OQ = \frac{h}{\tan 30}$ (2m)

(ii) As $\triangle POQ$ IS RIGHT ANGLED

$OP^2 + OQ^2 = 100^2$

$\frac{h^2}{\tan^2 40} + \frac{h^2}{\tan^2 30} = 100^2$

$\frac{h^2\tan^2 30 + h^2\tan^2 40}{\tan^2 30 \tan^2 40} = 100^2$

$h^2 = \frac{100^2(\tan^2 30 \tan^2 40)}{\tan^2 30 + \tan^2 40}$

$h = \frac{100(\tan 30^\circ \tan 40)}{\sqrt{\tan^2 30 + \tan^2 40}}$ (2m)

(iii) $h = 47.57\text{m}$.

$OP = \frac{h}{\tan 40}$

$= 56.69\text{ m}$

$\therefore \cos\angle OPQ = \frac{OP}{100}$

$= 0.5669$

$\angle OPQ = 55.47^\circ$

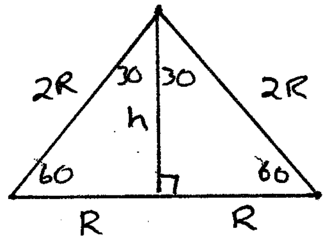
Bearing Q FROM P $= 055^\circ$

$\therefore$ Bearing P FROM Q $= 235^\circ$

2M

Question 7.

(a)(i) IF SEMI VERTICAL ANGLE IS 30°



By PYTHAGORAS $h = \sqrt{3}R$

$V_{CONE} = \frac{1}{3}\pi R^2 h$

$= \frac{1}{3}\pi R^2\sqrt{3}R$

$= \frac{\sqrt{3}}{3}\pi R^3$ (2m)

(ii) $\frac{dv}{dt} = \frac{dv}{dr}\cdot\frac{dr}{dt}$

$\frac{dv}{dr} = \sqrt{3}\pi R^2 = 4\sqrt{3}\pi \quad (R=2)$

$\frac{dv}{dt} = \frac{dv}{dr}\cdot\frac{dr}{dt}$

$1 = 4\sqrt{3}\pi\cdot\frac{dr}{dt}$

$\frac{dr}{dt} = \frac{1}{4\sqrt{3}\pi}$ M/min (2m)

(b)(i) $f(x) = x^2 - 4x + 5$.

$= (x-2)^2 + 1$

$\therefore$ vertex $(2,1)$ (1m)

$\therefore$ largest positive domain $x \geq$ 2

(ii) FOR INVERSE

Let $x = (y-2)^2 + 1$

$x - 1 = (y-2)^2$

$y - 2 = \pm\sqrt{x-1}$

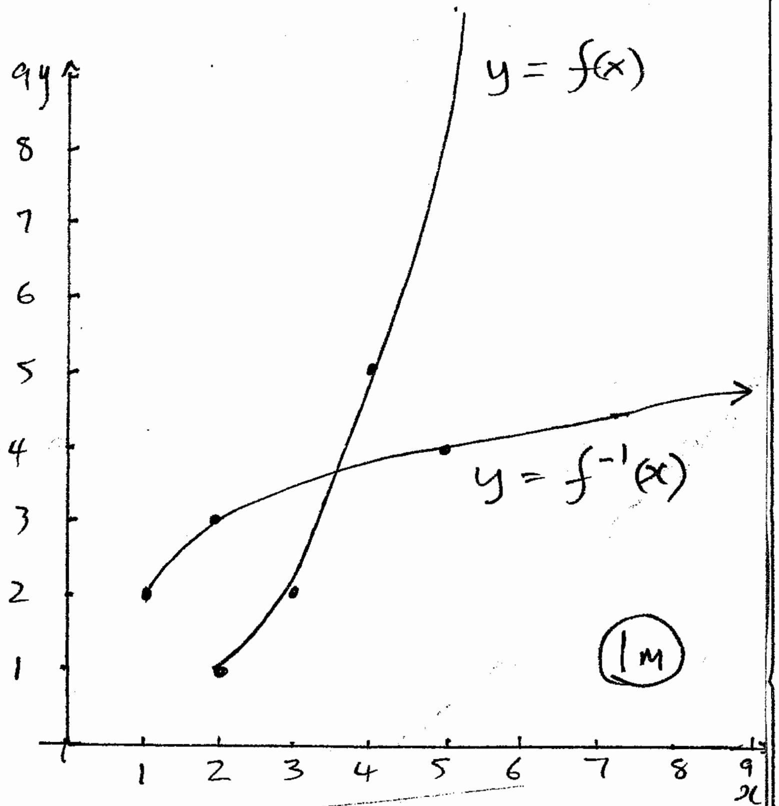
$y = 2 \pm \sqrt{x-1}$

$f^{-1}(x) = 2 + \sqrt{x-1}$

as $y \geq 2$

$\therefore f^{-1}(x) \geq 2$.

Question 7 cont.



(1M)

c) i) ~~For max height~~ $\dot{y} = 0$

$$0 = V\sin\theta - gt$$

$$t = \frac{V\sin\theta}{g}$$

sub $y = V\sin\theta\left(\frac{V\sin\theta}{g}\right) - \frac{1}{2}g\left(\frac{V\sin\theta}{g}\right)^2$

(HM) $y_M = \frac{V^2\sin^2\theta}{2g}$. (1M)

ii) From $x = V\cos\theta t$

$$t = \frac{x}{V\cos\theta}$$

$$y = \frac{V\sin\theta\ x}{V\cos\theta} - \frac{1}{2}g\left(\frac{x}{V\cos\theta}\right)^2$$

$$y = \tan\theta x - \frac{gx^2}{2V^2\cos^2\theta}$$

$$y = \tan\theta x - \frac{gx^2\sec^2\theta}{2V^2}$$ (2M)

c) iii) Put a, b into eqⁿ of path

$$b = a\tan\theta - \frac{ga^2\sec^2\theta}{2V^2}$$

$$\frac{ga^2\sec^2\theta}{2V^2} = a\tan\theta - b$$

$$\frac{ga^2\sec^2\theta}{a\tan\theta - b} = 2V^2$$

$$V^2 = \frac{ga^2\sec^2\theta}{2(a\tan\theta - b)}$$

subst V^2 into H

$$H = \frac{V^2\sin^2\theta}{2g}$$

$$H = \frac{ga^2\sec^2\theta\ \sin^2\theta}{2g \cdot 2(a\tan\theta - b)}$$

$$H = \frac{a^2\tan^2\theta}{4(a\tan\theta - b)}$$ (2M)