Total marks – 84

Attempt Questions 1 –7

All questions are of equal value.

Question 1 (12 marks). Start on a SEPARATE page.

Marks

(a) Find the acute angle between the lines $2x + y = 17$ and $3x - y = 3$. 2

(b) The point P(17,36) divides the line joining A(2,1) and B(5,8) externally in the ratio m : n. Find m and n. 2

(c) Solve for x: $\dfrac{2x-3}{x-2} \geq 1$ 2

(d) Differentiate $y = \tan^{-1}\sqrt{3x^2 - 1}$ 3

(e) Use the substitution $u = \cos x$ to evaluate $\int_0^{\frac{\pi}{3}} \cos^3 x \sin x \, dx$ 3

Question 2 (12 marks). Start on a SEPARATE page.

(a) Find the term independent of x in the expansion of $\left(2x^3-\frac{1}{x}\right)^{12}$ 2

(b) Evaluate: $\lim\limits_{x\to 0}\dfrac{\sin\frac{x}{3}}{2x}$ 2

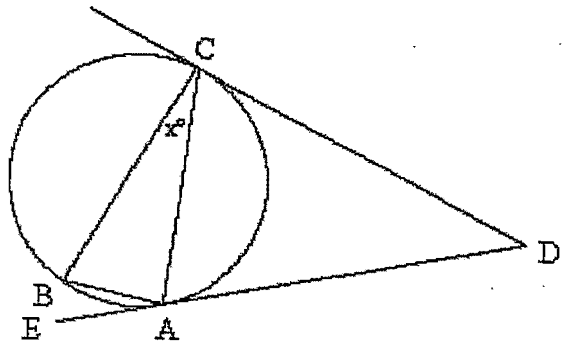
(c) If $f(x)=2\sin^{-1}3x$, find

(i) the domain and range of $f(x)$. 2

(ii) $f\left(\frac{1}{6}\right)$ 1

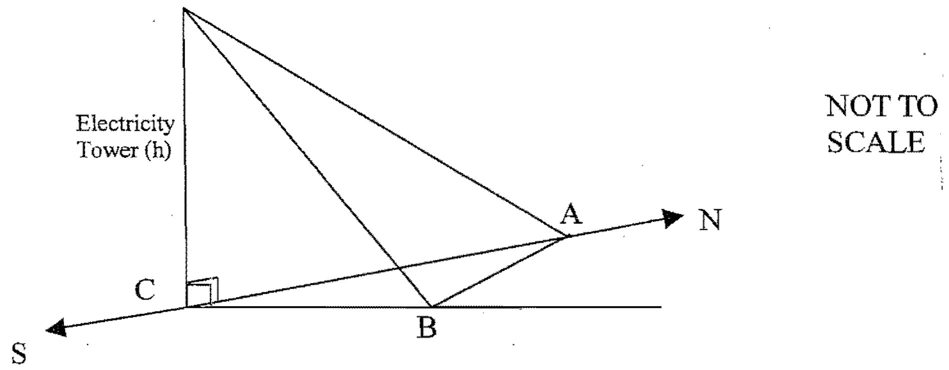
(iii) $f'\left(\frac{1}{6}\right)$ 2

(d) AD and CD are tangents to a circle. B is a point on the circle such that $\angle CBA$ and $\angle CDA$ are equal and are both double $\angle BCA$. Prove that BC is a diameter of the circle. 3



Question 3 (12 marks). Start on a SEPARATE page.

(a) Leon walks on level ground, in a northerly direction, away from an electricity tower. When he arrive at a point A, the angle of elevation to the top of the tower is $23°$. Luke walks on level ground on a bearing of $032°T$ from the same tower, until he reaches point B, and notices that the angle of elevation is $17°$. The distance between A and B is 55m. Let h be the height of the tower and assume that the tower base C, is perpendicular to the ground.



(i) Copy the diagram above onto your booklet and clearly mark on it all the information given. 1

(ii) Find expressions for AC and BC in terms of h. 2

(iii) Hence,or otherwise, find the height h of the tower to the nearest metre. 3

(b) Find how many arrangements can be made by taking all the letters of the word

(i) MATHEMATICS

(ii) In how many of them do the vowels occur together? 3

(c) An archer finds that in the long run, he scores a bull's eye on 3 out of 5 occasions. He fires 8 rounds at a target. Assuming that each trial is an independent event, find the probability of

(i) exactly 5 bull's eyes.

(ii) at least 7 bull's eyes. 3

Question 4 (12 marks). Start on a SEPARATE page.

(a) Prove the following by the Principle of mathematical induction.

$5^{2n} - 1$ is divisible by 24 for $n \geq 1$. **3**

(b) $P(2ap, ap^2)$ is a variable point on the parabola $x^2 = 4ay$. M is the foot of the perpendicular from P to the x-axis. Q is the point on MP such that $MP = PQ$. Find the equation of the locus of Q. **3**

(c) For the function $y = \dfrac{x^2}{x^2 - 9}$

(i) Write down the equations of horizontal and vertical asymptotes. **2**

(ii) Find any stationary points and determine their nature. **2**

(iii) Sketch the graph showing the above features. **2**

Question 5 (12 marks)

(a) (i) Express $3\sin x - \sqrt{3}\cos x$ in the form $A\sin(x-\alpha)$ 2

(ii) Hence find the general solution to $3\sin x - \sqrt{3}\cos x = \sqrt{3}$ 2

(b) N is the number of Kagaroos in a certain population at time t years.

The population size N satisfies the equation

$\frac{dN}{dt} = -k(N-500)$, for some constant k.

(i) Verify that $N = 500 + Ae^{-kt}$ where A is a constant, is a solution of the equation. 2

(ii) Initially, there are 3500 Kangaroos but after 3 years there are only 3300 left. Find the value of A and the exact value of k. 2

(iii) Find when the number of Kangaroos begin to fall below 2300. 2

(iv) Sketch the graph of the population size against time. 2

Question 6 (12 marks). Start on a SEPARATE page.

(a) Find the roots of the equation $x^3 - 15x + 4 = 0$, given that two of its roots are reciprocals. 3

(b) If $\frac{dx}{dt} = x + 6$ and $x = -5$ when $t = 0$, find an expression for x in terms of t. 2

(c) The speed $v\ m/s$ of a particle moving in a straight line is given by $v^2 = 64 - 16x - 8x^2$ where the displacement from a fixed point O is x metres.

(i) Find an expression for the acceleration and show that the motion is simple harmonic. 2

(ii) Between which two points is the particle oscillating? 2

(iii) Find the period and amplitude of the motion. 2

(iv) Find the maximum speed of the particle. 1

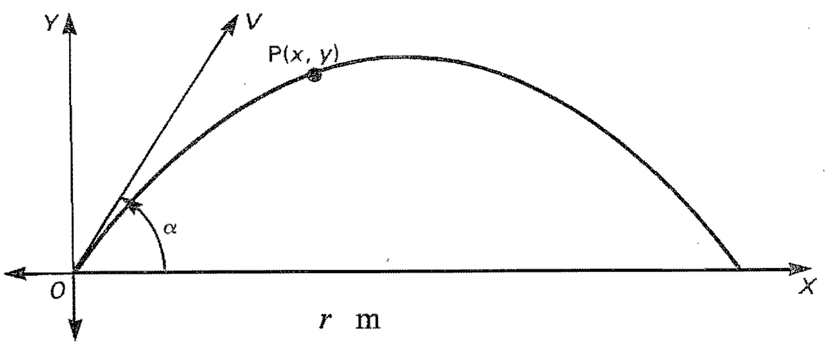
Question 7 (12 marks). Start on a SEPARATE page.

(a) A spherical bubble is expanding so that its volume is increasing at $10\,cm^3/s$. Find the rate of increase of its radius when the surface area is $500\,cm^2$. 3

(b) Given that the function $f(x) = \cos x - \log_e x$ has a root between 1.3 and 1.4. Using halving the interval method, find a better approximation to the root correct to one decimal place. 2

(c) A projectile is fired with initial speed $V\,m/s$ to strike a target on the level ground which is at a distance of r m from the origin. The position of the particle at any time t is given by

$x = Vt\cos\alpha$ and $y = Vt\sin\alpha - \frac{1}{2}gt^2$ (do not prove this)



(i) If α is a suitable angle of projection, prove that

$$\tan^2\alpha - \left(\frac{2v^2}{gr}\right)\tan\alpha + 1 = 0$$ 3

(ii) Prove that there are two angles of projection if $r < \frac{v^2}{g}$ 2

(iii) Show that the two angles of projection are complementary.
(Hint: consider the product of the roots of the equation in (i)) 2

End of paper

Trial HSC – Extension 1 – 2008 Solutions

Question 1 (12 marks)

(a) $2x + y = 17$

$y = -2x + 17$

$m_1 = -2$

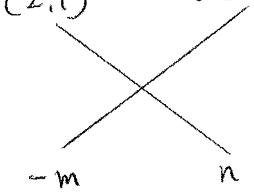
$y = 3x - 3 \quad \therefore m_2 = 3$

$\tan\theta = \left| \dfrac{m_1 - m_2}{1 + m_1 m_2} \right|$

$= \left| \dfrac{-2-3}{1-6} \right| = 1$ (2)

$\theta = 45^\circ$

(b) $A(2,1)$ $\qquad$ $B(5,8)$



$\dfrac{-5m + 2n}{n - m} = 17$

$-5m + 2n = 17n - 17m$

$12m = 15n$

$m = \dfrac{15}{12} n$

$\dfrac{m}{n} = \dfrac{15}{12}$ (2)

$\dfrac{m}{n} = \dfrac{5}{4}$

$m : n = 5 : 4$

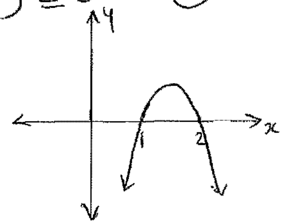
(c) $\dfrac{2x-3}{x-2} \geq 1$

$(x-2)^2 \times \dfrac{(2x-3)}{x-2} \geq (x-2)^2$

$(x-2)(2x-3) \geq (x-2)^2$

$(x-2)(x-2-2x+3) \leq 0$ (2)

$(x-2)(1-x) \leq 0$



From the graph we get

$x \leq 1$ or $x > 2$

(d) $y = \tan^{-1}\sqrt{3x^2 - 1}$

$y' = \dfrac{1}{1 + 3x^2 - 1} \times \dfrac{1}{2\sqrt{3x^2-1}} \times 6x$ (3)

$= \dfrac{1}{3x^2} \times \dfrac{3x}{\sqrt{3x^2-1}} = \dfrac{1}{x\sqrt{3x^2-1}}$

(e) $\displaystyle\int_0^{\frac{\pi}{3}} \cos^3 x \sin x \, dx$

Let $u = \cos x$; $\dfrac{du}{dx} = -\sin x$

$\sin x \, dx = -du$

When $x = 0$, $u = \cos 0 = 1$

When $x = \dfrac{\pi}{3}$, $u = \cos\dfrac{\pi}{3} = \dfrac{1}{2}$

$\displaystyle\int_1^{\frac{1}{2}} u^3 (-du) = -\int_1^{\frac{1}{2}} u^3 \, du$

$= \displaystyle\int_{\frac{1}{2}}^{1} u^3 \, du = \left[\dfrac{u^4}{4}\right]_{\frac{1}{2}}^{1}$

$= \dfrac{1}{4}\left[u^4\right]_{\frac{1}{2}}^{1}$ (3)

$= \dfrac{1}{4}\left(1 - \dfrac{1}{16}\right) = \dfrac{15}{64}$

Question 2 (12 marks)

(a)

$T_{r+1} = (-1)^r \, {}^nC_r \, a^{n-r} b^r$

$T_{r+1} = (-1)^r \, {}^{12}C_r (2x^3)^{12-r} \left(\dfrac{1}{x}\right)^r$

$= (-1)^r \, {}^{12}C_r \, 2^{12-r} x^{36-3r} \dfrac{1}{x^r}$

$= (-1)^r \, {}^{12}C_r \, 2^{12-r} x^{36-4r}$

For term independent of x

$36 - 4r = 0$

$r = 9$

$T_{10} = (-1)^9 \, {}^{12}C_9 \, 2^3$ (2)

$= -{}^{12}C_9 \, 2^3$

$= -1760$

(b) $\displaystyle\lim_{x \to 0} \dfrac{\sin\frac{x}{3}}{2x}$

$= \displaystyle\lim_{x \to 0} \dfrac{\sin\frac{x}{3}}{\frac{x}{3}} \times \dfrac{\frac{x}{3}}{2x}$ (2)

$= \dfrac{1}{6}$ $\left(\because \displaystyle\lim_{x \to 0} \dfrac{\sin\frac{x}{3}}{\frac{x}{3}} = 1 \right)$

(c) $f(x) = 2\sin^{-1} 3x$

(i) $D: -1 \leq 3x \leq 1$

$\dfrac{-1}{3} \leq x \leq \dfrac{1}{3}$ (2)

$R: 2 \times \dfrac{-\pi}{2} \leq y \leq 2 \times \dfrac{\pi}{2}$

$-\pi \leq y \leq \pi$

(ii) $f\left(\frac{1}{6}\right) = 2\sin^{-1}\left(3 \times \frac{1}{6}\right)$

$= 2\sin^{-1}\frac{1}{2}$ (1)

$= 2 \times \dfrac{\pi}{6} = \dfrac{\pi}{3}$

(iii) $f'(x) = 2 \times \dfrac{1}{\sqrt{1 - 9x^2}} \times 3$

$= \dfrac{6}{\sqrt{1-9x^2}}$

$f'\left(\frac{1}{6}\right) = \dfrac{6}{\sqrt{1 - 9 \times \frac{1}{36}}}$

$= \dfrac{6}{\sqrt{1 - \frac{1}{4}}} = \dfrac{6}{\frac{\sqrt{3}}{2}}$

$= 4\sqrt{3}$ (2)

(d) $\angle CBA = \angle CDA = 2x$ (given)

$\angle DAC = \angle CBA = 2x$ (angle between tangent and chord is equal to angle in the alternate segment)

$\angle DCA = \angle CBA = 2x$ (" " ")

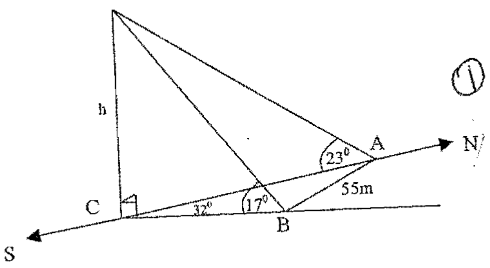
In ΔCDA, $2x + 2x + 2x = 180$ (angle sums of triangle)

$x = 30°$

$\angle BAC = 180 - 3x$
$= 90$ ③

$\therefore$ BC is diameter (If the angle subtended at the circumference of a circle by a chord is equal to 90°, then that chord is a diameter)

Question 3 (12 marks)



①

(ii) $\tan 67° = \frac{AC}{h}$

$AC = h \tan 67°$

$\tan 73° = \frac{BC}{h}$

$BC = h \tan 73°$ ②

$55^2 = h^2 \tan^2 67° + h^2 \tan^2 73° - 2h \tan 67 \times h \tan 73 \cos 32°$

$= h^2(\tan^2 67 + \tan^2 73 - 2 \tan 67 \cdot \tan 73 \cos 32)$

$h^2 = \frac{55^2}{\tan^2 67 + \tan^2 73 - 2 \tan 67 \tan 73 \cdot \cos 32°}$

$h = 31m$ ③

(b) (i) $\frac{11!}{2!\,2!\,2!} = 4989600$ ①

(ii) $\frac{8!}{2!\times 2!} \times \frac{4!}{2!}$

$= 10080 \times 12$ ②

$= 120960$

(c) (i)

$^8C_5 \left(\frac{3}{5}\right)^5 \left(\frac{2}{5}\right)^3$

$= \frac{108864}{390625} = 0.279$

(ii) $^8C_7 \left(\frac{3}{5}\right)^7 \left(\frac{2}{5}\right)^1 +$ ③

$^8C_8 \left(\frac{3}{5}\right)^8$

$= \frac{41553}{390625} = 0.106$

Question 4 (12 marks)

(a) Testing $n=1$

$5^{2\times 1} - 1 = 25 - 1 = 24$ is divisible by 24

$\therefore$ the result is true for $n=1$

Assume the result is true for $n=k$

ie $5^{2k} - 1$ is divisible by 24

$5^{2k} - 1 = 24P$ where P is an integer —— ①

To prove that the result is true for $n = k+1$

ie to prove that $5^{2(k+1)} - 1$ is divisible by 24

ie $5^{2(k+1)} - 1 = 24Q$ where Q is an integer —— ②

Now

$5^{2(k+1)} - 1 = 5^{2k+2} - 1$

$= 5^2 \times 5^{2k} - 1$

$= 25(24P + 1) - 1$ (by assumpt ①)

$= 600P + 24$

$= 24(25P + 1)$ ③

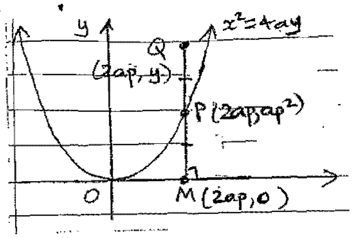
$= 24Q$

where $Q = 25P + 1$ is an integer.

Thus the result is true for $n = k+1$

$\therefore$ by the Principle of mathematical induction, the result is true for all $n \geq$

(b)



Given that $MP = PQ$

$\sqrt{(ap^2)^2} = \sqrt{(y-ap^2)^2}$

$ap^2 = y - ap^2$

$y = 2ap^2$

The coordinates of Q are

$x = 2ap$ —— (1)

$y = 2ap^2$ —— (2)

squaring (1) we get

$x^2 = 4a^2p^2$

$= 2a \times 2ap^2$

$= 2ay$ (3)

$\therefore$ locus of Q is $x^2 = 2ay$

(c) (i) $\lim_{x \to \infty} \frac{x^2}{x^2-9}$

$= \lim_{x \to \infty} \frac{1}{1-\frac{9}{x^2}} = 1$

Horizontal asymptote : $y = 1$

Vertical asymptotes are given by $x^2 - 9 = 0$

$x = \pm 3$

(2)

(ii) $y = \frac{x^2}{x^2-9}$

$\frac{dy}{dx} = \frac{(x^2-9) \times 2x - x^2 \times 2x}{(x^2-9)^2}$

$= \frac{2x^3 - 18x - 2x^3}{(x^2-9)^2}$

$= \frac{-18x}{(x^2-9)^2}$

$\frac{dy}{dx} = 0 \Longrightarrow -18x = 0 \Longrightarrow x = 0$

when $x = 0$, $y = 0$

$\frac{dy}{dx} = \frac{-18x}{(x^2-9)^2}$

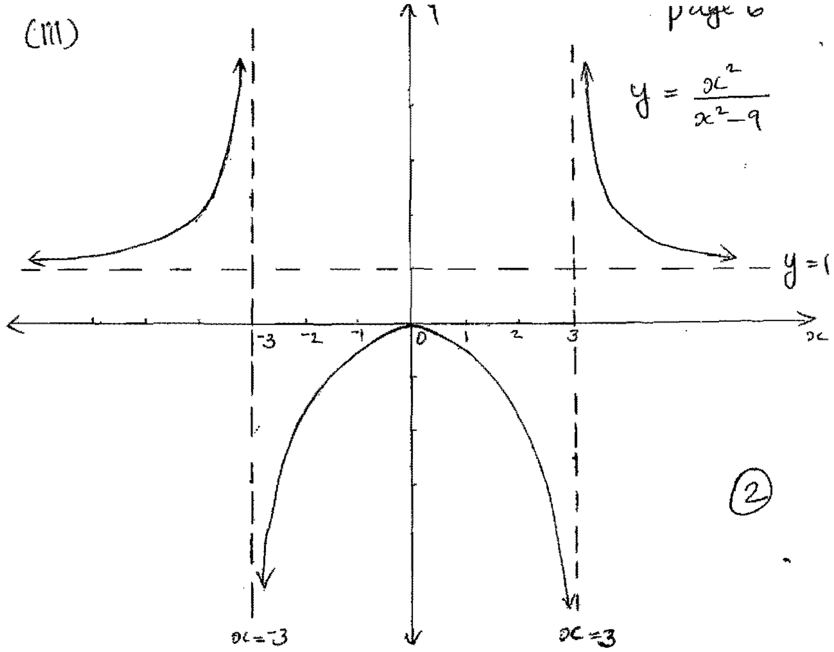
when $x = -1$, $\frac{dy}{dx} = \frac{18}{(1-9)^2} > 0$

when $x = 1$, $\frac{dy}{dx} = \frac{-18}{(1-9)^2} < 0$

$\therefore (0, 0)$ is a maximum turning point.

(2)

(iii)



(2)

Question 5 (12 marks)

(a) (i)

Let $3\sin x - \sqrt{3}\cos x = A\sin(x - \alpha)$

$= A(\sin x \cos\alpha - \cos x \sin\alpha)$

$= A\sin x\cos\alpha - A\cos x\sin\alpha$

Equating coefficients of $\sin x$ and $\cos x$ we get

$3 = A\cos\alpha$ —— (1)

$\sqrt{3} = A\sin\alpha$ —— (2)

squaring and adding (1) and (2) we get

$A^2(\sin^2\alpha + \cos^2\alpha) = 12$

$A^2 = 12$; $A = 2\sqrt{3}$

substitute the value of A in (1) and (2)

$\sin\alpha = \frac{1}{2}$

$\cos\alpha = \frac{\sqrt{3}}{2}$ $\therefore \alpha = \frac{\pi}{6}$

$\therefore 3\sin x - \sqrt{3}\cos x$

$= 2\sqrt{3}\sin\left(x - \frac{\pi}{6}\right)$ (2)

(ii) $2\sqrt{3}\sin\left(x - \frac{\pi}{6}\right) = \sqrt{3}$

$\sin\left(x - \frac{\pi}{6}\right) = \frac{1}{2}$

General solution is

$x - \frac{\pi}{6} = n\pi + (-1)^n \frac{\pi}{6}$

$x = \frac{\pi}{6} + n\pi + (-1)^n \frac{\pi}{6}$

(2)

b)(i) $N = 500 + Ae^{-kt}$

$\frac{dN}{dt} = Ae^{-kt} \times -k$

$= -kAe^{-kt}$

But $Ae^{-kt} = N - 500$ (2)

$\therefore \frac{dN}{dt} = -k(N-500)$

$\therefore N = 500 + Ae^{-kt}$ is a solution of $\frac{dN}{dt} = -k(N-500)$

(ii) when $t=0$, $N = 3500$

$3500 = 500 + A$

$A = 3000$

$3300 = 500 + 3000e^{-3k}$

$e^{-3k} = \frac{2800}{3000} = \frac{14}{15}$

$-3k = \log \frac{14}{15}$ (2)

$k = -\frac{1}{3} \log \frac{14}{15}$

(iii) $N < 2300$

$500 + 3000e^{-kt} < 2300$

$e^{-kt} < \frac{3}{5}$

$e^{\frac{1}{3}\log\frac{14}{15}t} < \frac{3}{5}$

$\frac{1}{3}\log\frac{14}{15} t < \log\frac{3}{5}$

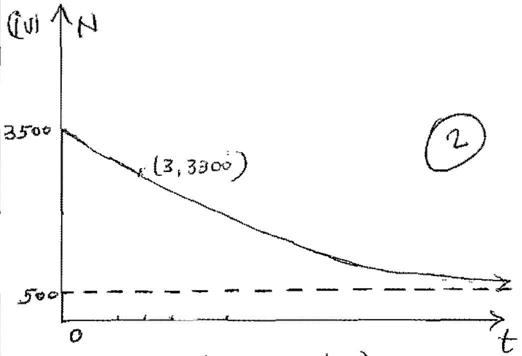
$t > \frac{\log \frac{3}{5}}{\frac{1}{3}\log\frac{14}{15}}$

$t > 22.21$ years

Number of Kangaroos will begin to fall below 2300 when (2)

$t > 22.21$ years

(iv)



(2)

Question 6 (12 marks)

(a) Let the roots be $\alpha, \frac{1}{\alpha}$ and β

$\alpha + \frac{1}{\alpha} + \beta = 0$ —①

$\alpha \times \frac{1}{\alpha} \times \beta = -4$ —② $\therefore \beta = -4$

substitute in ①

$\alpha + \frac{1}{\alpha} - 4 = 0$

$\alpha^2 + 1 - 4\alpha = 0$

$\alpha^2 - 4\alpha + 1 = 0$

$\alpha = \frac{4 \pm \sqrt{16-}}{2} = \frac{4 \pm \sqrt{12}}{2}$

$= \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$

The roots are (3)

$2+\sqrt{3},\ 2-\sqrt{3}$ and -4

(b) $\frac{dx}{dt} = x + 6$

$dt = \frac{1}{x+6} dx$

$\int dt = \int \frac{1}{x+6} dx$

$t = \log(x+6) + C$

when $t=0$, $x=-5$

$0 = \log_e(-5+6) + C$

$0 = \log(1) + C$ (2)

$C = 0$

$t = \log_e(x+6)$

$e^t = x + 6$

$x = e^t - 6$

(c)(i) $V^2 = 64 - 16x - 8x^2$

$\frac{1}{2}V^2 = 32 - 8x - 4x^2$

$\frac{d}{dx}\left(\frac{1}{2}V^2\right) = -8 - 8x$

$\ddot{x} = -8(x+1)$

This is of the form $\ddot{x} = -n^2(x-b)$ where $n = \sqrt{8}$ and $b = -1$

$\therefore$ the motion is (2) simple harmonic

(ii) Let $V^2 = 0$

$64 - 16x - 8x^2 = 0$

$8x^2 + 16x - 64 = 0$

$8(x^2 + 2x - 8) = 0$

$x^2 + 2x - 8 = 0$

$(x+4)(x-2) = 0$ (2)

$x = -4$ or 2

The particle oscillates between $x = -4$ and $x = 2$

(iii) $\ddot{x} = -8(x+1)$

$n = \sqrt{8}$

Period $T = \frac{2\pi}{n} = \frac{2\pi}{\sqrt{8}}$

$= \frac{\pi}{\sqrt{2}}$ seconds

-4 -1 2 (2)

Amplitude $= 3m$

(iv) Maximum speed occurs at the centre ie when $x = -1$

Substitute $x=-1$ in $V^2 = 64 - 16x - 8x^2$ we get

$V^2 = 64 + 16 - 8$

$= 72$ (1)

$V = \pm\sqrt{72}$

Max. speed $= \sqrt{72} = 6\sqrt{2}$ m/s

Question 7 (12 marks)

(a) $\frac{dV}{dt} = 10 \text{ cm}^3/\text{s}$

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

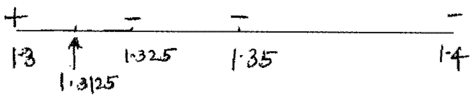
$= 4\pi r^2 \frac{dr}{dt}$

$10 = 500 \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{10}{500}$ (3)

$= \frac{1}{50}$ cm/s

(b)



$\frac{1.3 + 1.4}{2} = 1.35$

$f(1.35) = \cos 1.35 - \log 1.35$

$= -0.08$

The root lies between 1·3 and 1·35

$\frac{1.3 + 1.35}{2} = 1.325$

$f(1.325) = \cos 1.325 - \log 1.325$

$= -0.04$

page 9

The root lies between 1·3 and 1·325

$\frac{1.3 + 1.325}{2} = 1.3125$ (2)

The approximations to the root are 1·35, 1·325, 1·3125

∴ the root is 1·3 correct to one decimal place.

(c) (i) when the projectile strikes the target we have

$Vt\cos\alpha = r$ —(1)

$Vt\sin\alpha - \frac{gt^2}{2} = 0$ —(2)

From (1)

$t = \frac{r}{V\cos\alpha}$

substitute in (2)

$V \times \frac{r}{V\cos\alpha}\sin\alpha - \frac{g}{2} \times \frac{r^2}{V^2\cos^2\alpha} = 0$

$r\tan\alpha - \frac{gr^2}{2V^2}\sec^2\alpha = 0$

$r\tan\alpha - \frac{gr^2}{2V^2}(1 + \tan^2\alpha) = 0$

$\frac{gr^2}{2V^2}(1 + \tan^2\alpha) - r\tan\alpha = 0$

$1 + \tan^2\alpha - \frac{2V^2}{gr^2} r\tan\alpha = 0$

$1 + \tan^2\alpha - \frac{2V^2}{gr}\tan\alpha = 0$ (3)

$\tan^2\alpha - \frac{2V^2}{gr}\tan\alpha + 1 = 0$

(ii) The quadratic equation

$\tan^2\alpha - \frac{2V^2}{gr}\tan\alpha + 1 = 0$

has real and distinct solutions if $\Delta > 0$

ie $\left(\frac{2V^2}{gr}\right)^2 - 4 > 0$

$\left(\frac{2V^2}{gr}\right)^2 > 4$

$\frac{2V^2}{gr} > 2$

$2V^2 > 2gr$ (2)

$V^2 > gr$

$\frac{V^2}{g} > r$

ie $r < \frac{V^2}{g}$

(iii) Let $\tan\alpha_1$ and $\tan\alpha_2$ be the roots of the equation

$\tan^2\alpha - \frac{2V^2}{gr}\tan\alpha + 1 = 0$

$\tan\alpha_1 \times \tan\alpha_2 = 1$

$\frac{\sin\alpha_1}{\cos\alpha_1} \times \frac{\sin\alpha_2}{\cos\alpha_2} = 1$

$\sin\alpha_1 \sin\alpha_2 = \cos\alpha_1 \cos\alpha_2$

$\cos\alpha_1 \cos\alpha_2 - \sin\alpha_1 \sin\alpha_2 = 0$

$\cos(\alpha_1 + \alpha_2) = 0$

$\alpha_1 + \alpha_2 = 90°$ (2)

∴ α_1 and α_2 are complementary angles.