2012

TRIAL

HIGHER SCHOOL CERTIFICATE

MATHEMATICS EXTENSION 1

General Instructions:

- Reading Time - 5 minutes

Working time - 2 hours

- Write using black or blue pen.
- Board - approved calculators may be used
- A table of standard integrals is provided at the back of this paper.
- All necessary working should be shown in every question

Total marks – 70

Section 1 **10 marks**

Attempt Questions 1 – 10

Section II **60 Marks**

Attempt Questions 11 – 14

All questions are of equal value

SECTION 1 **Multiple Choice Questions: Circle the correct answer**

1.The polynomial $P(x) = x^4 - kx^3 - 2x + 33$ has $(x-3)$ as a factor.

What is the value of k?

(A) -5

(B) -4

(C) 4

(D) 5

2.The velocity of a particle moving along the x axis is given by $v^2 = 24 + 2x - x^2$. Which of the following expressions is the correct equation for the acceleration of the particle in terms of x?

(A) $1 - x$

(B) $1 - 2x$

(C) $12x + \frac{x^2}{2} - \frac{x^3}{6}$

(D) $24x + x^2 - \frac{x^3}{3}$

3. If $f(x) = e^{x+2}$ what is the inverse function $f^{-1}(x)$?

(A) $f^{-1}(x) = e^{y-2}$

(B) $f^{-1}(x) = e^{y+2}$

(C) $f^{-1}(x) = \log_e x - 2$

(D) $f^{-1}(x) = \log_e x + 2$

4. At a football club a team of 11 players is to be chosen from a pool of 30 players consisting of 18 Australian-born players and 12 players born elsewhere. What is the probability that the team will consist of all Australian-born players?

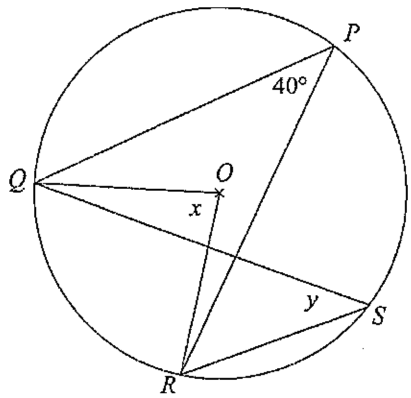
(A) $\dfrac{{}^{18}C_{11}}{{}^{30}C_{11}}$

(B) $\dfrac{{}^{30}C_{11}}{{}^{18}C_{11}}$

(C) $\dfrac{{}^{18}C_{12}}{{}^{30}C_{12}}$

(D) $\dfrac{{}^{30}C_{12}}{{}^{18}C_{12}}$

5. P, Q, R and S are points on a circle with centre O. $\angle QPR = 40^\circ$.



Why are the values of x and y?

(A) $x = 40^\circ$ and $y = 20^\circ$

(B) $x = 40^\circ$ and $y = 40^\circ$

(C) $x = 80^\circ$ and $y = 20^\circ$

(D) $x = 80^\circ$ and $y = 40^\circ$

6.A curve has parametric equations $x = \dfrac{2}{t}$ and $y = 2t^2$.

What is Cartesian equation of this curve?

(A) $y = \dfrac{4}{x}$

(B) $y = \dfrac{8}{x}$

(C) $y = \dfrac{4}{x^2}$

(D) $y = \dfrac{8}{x^2}$

7. What is the solution to the equation $|2x-5|=-3x$?

(A) $x=-5$

(B) $x=-1$

(C) $x=1$

(D) $x=5$

8. What is the exact value of $\tan 75^{\circ}$?

(A) $2-\sqrt{3}$

(B) $4-\sqrt{3}$

(C) $2+\sqrt{3}$

(D) $4+\sqrt{3}$

9. A parabola has the parametric equations $x=12t$ and $y=-6t^2$.
What are the coordinates of the focus?

(A) $(-6,0)$

(B) $(0,-6)$

(C) $(6,0)$

(D) $(0,6)$

10 How many four-digit numbers can be formed with the digits 1, 2, 3, 4 and 5 if no digit is repeated?

(A) 20

(B) 120

(C) 625

(D) 3125

SECTION II

Question11. (15 marks)

a)Solve $\dfrac{x}{x-2} \geq 2$ **2**

b) Find the coordinates of the point P which divides the interval AB ***externally***

in the ratio 1:3, given A= (1,4) and B = (5,2) **2**

c) Evaluate

i) $\displaystyle\int_1^2 \frac{1}{\sqrt{4-x^2}}dx$ **2**

ii) $\displaystyle\int_{-1}^0 x\sqrt{1+x}dx$, using the substitution $u = 1+x$ **2**

d) Find the acute angle between the lines $y = 2x - 1$ and $y = \dfrac{1}{3}x + 1$ **2**

e) Find the number of ways in which two consonants and three vowels

can be chosen from the letters of the word EQUATION? **1**

f) The polynomial $P(x) = x^3 + px^2 + qx + r$ has real roots

$\sqrt{k}, -\sqrt{k}$, and α .

i) Explain why $\alpha + p = 0$ **1**

ii) Show that $k\alpha = r$ **1**

iii) Show that $pq = r$ **2**

Question 12. (15 marks)

a) Find $\int \sin^2 3x dx$ **2**

b) Simplify $\sin 2\theta(\tan\theta + \cot\theta)$ **2**

c) Consider the function $f(x) = 3\cos^{-1}\left(\frac{x}{2}\right)$

i) Sketch the graph $y = f(x)$ **3**

ii) Find the gradient of the tangent to the curve at the point

where $x = \sqrt{3}$ **1**

d) If $\alpha = \sin^{-1}\left(\frac{8}{17}\right)$ and $\beta = \tan^{-1}\left(\frac{3}{4}\right)$, calculate the exact value of $\sin(\alpha - \beta)$ **3**

e) Solve $\cos 2\theta = \cos\theta$ for $0 \le \theta \le 2\pi$ **2**

f) Differentiate $e^x \cos^{-1} x$ **2**

Question 13. (15 marks)

a)) Prove by mathematical induction that

$$1\times 2^2 + 2\times 3^2 + 3\times 4^2 + \dots\dots\dots + n(n+1)^2 = \frac{1}{12}n(n+1)(n+2)(3n+5)$$ **4**

b)) Let $g(x) = 2x^3 + x + 4$

i) Show that $g(x) = 0$ has a root between integers -1 and -2 **1**

ii) Taking $x = -1.5$ as the first approximation to this root, use one application of Newton's method to obtain a better approximation for this root. **2**

Question 13 continued

c) The velocity, vms^{-1}, of a particle moving in simple harmonic motion along the x-axis is given by $v^2 = 8 - 2x - x^2$, where x is in metres.

i) Between which two points is the particle oscillating? **1**

ii) Find the centre of the motion. **1**

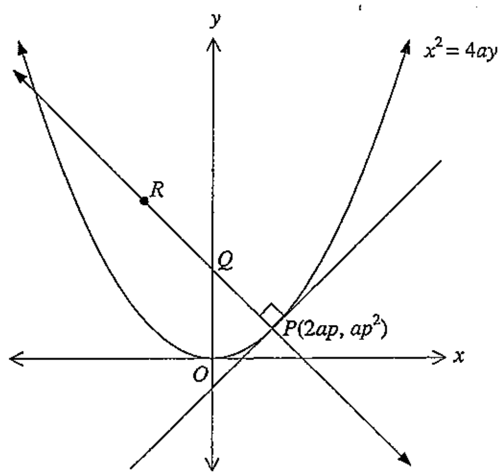
iii) Find the maximum speed. **1**

iv) Find an expression for the acceleration of the particle in terms of x. 2

d) i) Express $\cos x - \sin x$ in the form $A\cos(x+\alpha)$, where $0 \le \alpha \le \frac{\pi}{2}$ **1**

ii) Hence, or otherwise, solve $\cos x - \sin x = 1$ for $0 \le x \le 2\pi$ **2**

Question 1 4. 15 marks)



a) The diagram shows a variable point $P(2ap, ap^2)$ on the parabola $x^2 = 4ay$.

The normal to the parabola at P intersects the y axis at Q.The point Q is the mid point of PR.

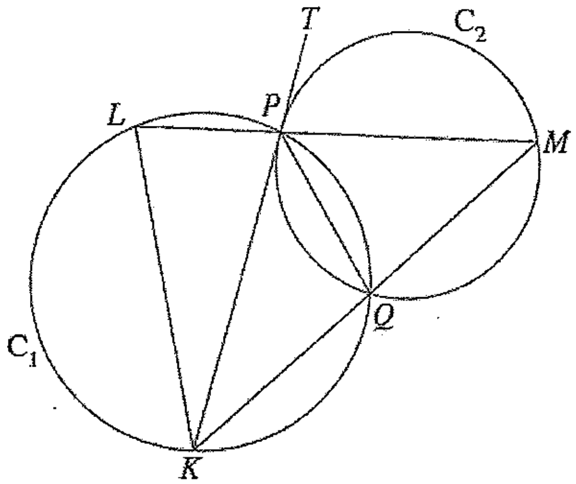
The equation of the normal is $x + py - 2ap - ap^3 = 0$.(Do Not prove this.)

i) Find the coordinates of the point Q. **1**

ii) The locus of the point R is a parabola. Find the equation of this parabola in Cartesian form and state its vertex. **2**

Question 14 continued

b)

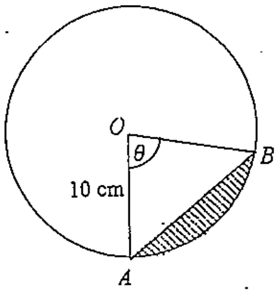


Two circles C_1 and C_2 intersect at P and Q as shown in the diagram. The tangent TP to C_2 at P meets C_1 at K. The line KQ meets C_2 at M. The line MP meets C_1 at L.

Copy the diagram into your writing booklet.

Prove that ΔPKL is isosceles. **3**

c)



A circle has centre O and radius $10cm$. OA is a fixed radius of the circle. OB is a variable radius which moves so that $\angle AOB = \theta$ is increasing at a constant rate of 0.01 radians per second. The minor segment of the circle cut off by the chord AB has area $S\ cm^2$

Find the rate at which S is increasing when $\theta = \frac{\pi}{3}$. **2**

Question1 4 continued.

d) Molten metal at a temperature of 1400^0 C is poured into moulds to

form machine parts. After 15 minutes the metal has cooled to 995^0 C.

If the temperature of the surroundings is 35^0 C, then the rate of cooling

is approximately given by;

$$\frac{dT}{dt} = -k(T-35) \text{ where } k \text{ is a positive constant.}$$

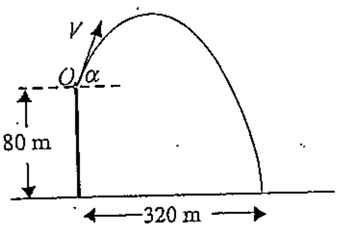
Show that a solution of this equation is $T = 35 + Ae^{-kt}$

where A is a constant. **1**

I) Find the value of k, correct tp three decimal places. **1**

iii) The metal can be taken out of the moulds when its temperature has dropped to 200^0 C. How long after the metal has been poured will this temperature be reached.? (answer correct to the nearest minute.) **2**

e)



A particle is projected with speed Vms^{-1} at an angle α above the horizontal from a point O at the edge of a vertical cliff which is $80m$ above horizontal ground. The particle moves in a vertical plane under gravity wherethe acceleration due to gravity is $10ms^{-2}$. It reaches its greatest height after 3 secondsand hits the ground at a horizontal distance 320m from the foot of the cliff.

The horizontal and vertical displacements, x and y metres respectively, of the particle from the point O after t seconds are given by $x = Vt\cos\alpha$ and $y = -5t^2 + Vt\sin\alpha$.(Do not prove this)

i) Show that $V\sin\alpha = 30$ **1**

ii) Show that the particle hits the ground after 8 seconds. **2**

TRIAL HSC EXTENSION 1 2012 SOLUTIONS

Section I

1. C
2. A
3. C
4. A
5. D
6. D
7. A
8. C
9. B
10. B

Section II

Question 11

a) $\dfrac{x}{x-2} \geq 2$ $\qquad x \neq 2$

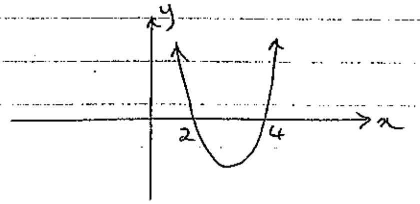
$\dfrac{x}{x-2} \times (x-2)^2 \geq 2(x-2)^2$

$x(x-2) \geq 2(x-2)^2$

$0 \geq 2(x-2)^2 - x(x-2)$

$0 \geq (x-2)(2x-4-x)$

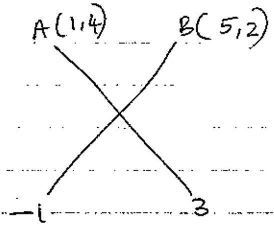
$0 \geq (x-2)(x-4)$ ①



$2 < x \leq 4$ ①

Question 11 continued.

b)



$x = \dfrac{3-5}{2} = -1$

$y = \dfrac{12-2}{2} = 5$

$\therefore P = (-1, 5)$ ②

c) (i) $\displaystyle\int_1^2 \frac{1}{\sqrt{4-x^2}}\,dx = \left[\sin^{-1}\frac{x}{2}\right]_1^2 = \sin^{-1}1 - \sin^{-1}\frac{1}{2}$

$= \dfrac{\pi}{2} - \dfrac{\pi}{6} = \dfrac{\pi}{3}$ ②

(ii) $\displaystyle\int_{-1}^{0} x\sqrt{1+x}\,dx$

$u = 1+x$, $du = dx$

Limits: if $x=-1$, $u=0$; $x=0$, $u=1$

$= \displaystyle\int_0^1 (u-1)u^{\frac{1}{2}}\,du$

$= \displaystyle\int_0^1 \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right) du = \left[\frac{2u^{\frac{5}{2}}}{5} - \frac{2u^{\frac{3}{2}}}{3}\right]_0^1$

$= \left(\dfrac{2}{5} - \dfrac{2}{3}\right) - (0-0)$

$= \dfrac{-4}{15}$ ②

d) $y = 2x-1$ $\qquad y = \frac{1}{3}x+1$

$m_1 = 2$ $\qquad m_2 = \frac{1}{3}$

$\tan\alpha = \left|\dfrac{2-\frac{1}{3}}{1+\frac{2}{3}}\right| = 1 \quad \therefore \alpha = 45^\circ$ ②

e) Vowels $= 5$; consonants $= 3$

$\therefore {}^3C_2 \times {}^5C_3 = 30$ ways. ①

Question 11 continued.

f) $P(x) = x^3 + px^2 + qx + r$

(i) Let the roots be x_1, x_2 and x_3

$x_1 = \sqrt{k}, \quad x_2 = -\sqrt{k}, \quad x_3 = \alpha$

$x_1 + x_2 + x_3 = \frac{-b}{a} = -p$

$\therefore \sqrt{k} - \sqrt{k} + \alpha = -p \Rightarrow \alpha + p = 0$ (1)

(ii) $x_1 x_2 x_3 = -r$

$(\sqrt{k})(-\sqrt{k})\alpha = -r$

$-k\alpha = -r \Rightarrow k\alpha = r$ (1)

(iii) $x_1x_2 + x_2x_3 + x_1x_3 = q$

$(\sqrt{k} \cdot -\sqrt{k}) + (-\sqrt{k} \cdot \alpha) + (\sqrt{k} \cdot \alpha) = q$

$-k - \sqrt{k}\alpha + \sqrt{k}\alpha = q \Rightarrow -k = q$

Also $p = -\alpha$ from (i)

$\therefore pq = (-k)(-\alpha) = k\alpha = r$ (2)

Question 12

a) $\cos 6x = 1 - 2\sin^2 3x$

$\sin^2 3x = \frac{1}{2}(1 - \cos 6x)$ (1)

$\therefore \int \sin^2 3x \, dx = \frac{1}{2}\int (1 - \cos 6x)\,dx$

$= \frac{1}{2}\left(x - \frac{\sin 6x}{6}\right) + C$ (1)

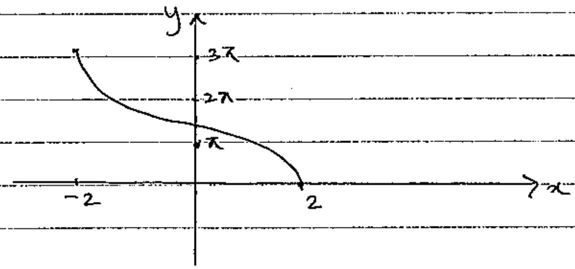
b) $\sin 2\theta(\tan\theta + \cot\theta) = 2\sin\theta\cos\theta\left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right)$

$= 2\sin^2\theta + 2\cos^2\theta = 2$ (2)

c) (i) $f(x) = 3\cos^{-1}\left(\frac{x}{2}\right)$

D: $-1 \leq \frac{x}{2} \leq 1 \Rightarrow -2 \leq x \leq 2$ (1)

R: $0 \leq f(x) \leq 3\pi$ (1)



(ii) $f(x) = 3\cos^{-1}\frac{x}{2} \quad \therefore f'(x) = 3 \cdot \frac{-1}{\sqrt{1 - \frac{x^2}{4}}} \cdot \frac{1}{2}$

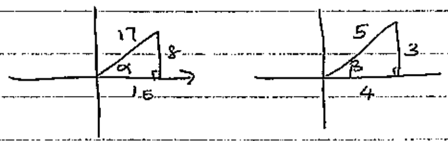
$= \frac{-3}{\sqrt{4 - x^2}}$

at $x = \sqrt{3}$, $f'(x) = \frac{-3}{\sqrt{4-3}} = -3$ (1)

Question 12 continued

d) $\alpha = \sin^{-1}\left(\frac{8}{17}\right) \Rightarrow \sin\alpha = \frac{8}{17}$

$\beta = \tan^{-1}\left(\frac{3}{4}\right) \Rightarrow \tan\alpha = \frac{3}{4}$



(1)

$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$= \frac{8}{17}\cdot\frac{4}{5} - \frac{15}{17}\cdot\frac{3}{5} = \frac{-13}{15}$ (2)

e) $\cos 2\theta = \cos\theta$ for $0 \le \theta \le 2\pi$.

$2\cos^2\theta - 1 = \cos\theta$

$2\cos^2\theta - \cos\theta - 1 = 0 \Rightarrow (2\cos\theta + 1)(\cos\theta - 1) = 0$

$\therefore \cos\theta = \frac{-1}{2}$ or $\cos\theta = 1$. (1)

$\therefore \theta = \frac{2\pi}{3}, \frac{4\pi}{3}, 0, 2\pi \therefore = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi$.

(1)

f) $y = e^x \cos^{-1}x$

$u = e^x$, $u' = e^x$ | $v = \cos^{-1}x$, $v' = \frac{-1}{\sqrt{1-x^2}}$

$\therefore y' = vu' + uv'$

$= \cos^{-1}x \cdot e^x + \frac{e^x(-1)}{\sqrt{1-x^2}}$

$= e^x\left[\cos^{-1}x - \frac{1}{\sqrt{1-x^2}}\right]$ (2)

Question 13

a) $1\times2^2 + 2\times3^2 + \dots + n(n+1)^2 = \frac{1}{12}n(n+1)(n+2)(3n+5)$

- prove true for $n=1$

LHS $= 1\times2^2 = 4$

RHS $= \frac{1}{12}(2)(3)(8) = 4$

$\therefore$ true for $n=1$

- assume true for $n=k$.

i.e. $1\times2^2 + 2\times3^2 + \dots + k(k+1)^2 = \frac{1}{12}k(k+1)(k+2)(3k+5)$

- prove true for $n=k+1$

i.e. $1\times2^2 + 2\times3^2 + \dots + k(k+1)^2 + (k+1)(k+2)^2$

$= \frac{(k+1)}{12}(k+3)(3k+8)$

LHS $1\times2^2 + 2\times3^2 + \dots + k(k+1)^2 + (k+1)(k+2)^2$

$= \frac{k}{12}(k+1)(k+2)(3k+5) + (k+1)(k+2)^2$

$= \frac{1}{12}(k+1)(k+2)\left[k(3k+5) + 12(k+2)\right]$

$= \frac{1}{12}(k+1)(k+2)\left[3k^2 + 17k + 24\right]$ (4)

$= \frac{1}{12}(k+1)(k+2)(k+3)(3k+8) = $ RHS

If true for $n=k$, then also true for $n=k+1$. But it is true for $n=1$

$\therefore$ Also true for $n=2,3,\dots$ (2)

$\therefore$ By induction, true for all positive integers

Question 13 continued

b) (i) $g(x) = 2x^3 + x + 4$

$g(-1) = 2(-1)^3 + (-1) + 4 = 1 > 0$

$f(-2) = 2(-2)^3 + (-2) + 4 = -14 < 0$

Since the sign changes and the curve is continuous, there is a root between $x=-1$ and $x=-2$.

1

(ii) $g(x) = 2x^3 + x + 4$

$g'(x) = 6x^2 + 1$

$x_1 = -1.5$, $g(x_1) = 2(-1.5)^3 + (-1.5) + 4$
$= -4.25$

$g'(x_1) = 6(-1.5)^2 + 1 = 14.5$

$\therefore x_2 = x_1 - \frac{g(x_1)}{g'(x_1)} = -1.5 - \frac{-4.25}{14.5}$
$= -1.2$ 2

c) (i) $v^2 = 8 - 2x - x^2$

At the end points $v=0$.

$\therefore 8 - 2x - x^2 = 0 \Rightarrow (4+x)(2-x) = 0$

$\therefore x = -4$ or $x = 2$ (1)

1

$\therefore$ the particle is oscillating between $x=-4$ and $x=2$

(ii) -4 ———— -1 (C.O.M) ———— 2

Centre of motion: $x = (-4+2) \div 2$
$x = -1$ (1)

Question 13 continued

c (iii) The maximum speed occurs at the centre of motion.

$\therefore v^2 = 8 - 2(-1) - (-1)^2 = 8 + 2 - 1 = 9$

$\therefore$ max. speed $= 3$ m/s 1 (1)

(iv) $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$= \frac{d}{dx}\left(4 - x - \frac{x^2}{2}\right) = -1 - x$ (2)

2

d) (i) $\cos x - \sin x = A\cos(x + \alpha)$ $0 < \alpha < \frac{\pi}{2}$

$\cos x - \sin x = A\cos x\cos\alpha - A\sin x\sin\alpha$

$A\cos\alpha = 1$; $A\sin\alpha = 1$

$\therefore A = \sqrt{2}$; $\alpha = \frac{\pi}{4}$ 1

$\therefore \cos x - \sin x = \sqrt{2}\cos\left(x + \frac{\pi}{4}\right)$ (1)

(ii) $\sqrt{2}\cos\left(x + \frac{\pi}{4}\right) = 1$ $0 \le x \le 2\pi$

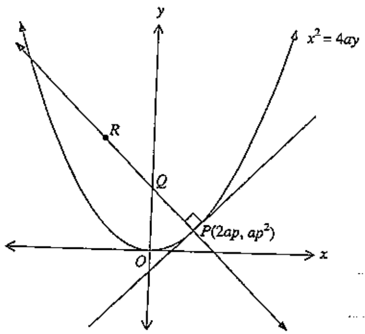
$\frac{\pi}{4} \le x + \frac{\pi}{4} \le \frac{9\pi}{4}$

$\therefore x + \frac{\pi}{4} = \frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}$

$\therefore x = 0, \frac{3\pi}{2}, 2\pi$ 2 (2)

Question 14

a)



(i) The eqⁿ of normal: $x + py - 2ap - ap^3 = 0$

intersects the y axis at Q. ∴ sub in $x = 0$.

$0 + py - 2ap - ap^3 = 0$

$\therefore y = \dfrac{2ap + ap^3}{p} = 2a + ap^2$

$\therefore Q = (0, 2a + ap^2)$ ①

(ii) Let $R = (x, y)$; Q is the mid-point of PR

$\therefore \dfrac{x + 2ap}{2} = 0 \Rightarrow x = -2ap$ — ①

$\dfrac{y + ap^2}{2} = 2a + ap^2 \Rightarrow y = 4a + ap^2$ — ②

eliminate p from (i) and (ii)

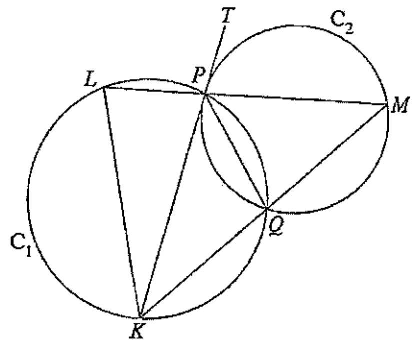
$y = 4a + a\cdot\left(\dfrac{-x}{2a}\right)^2 = 4a + \dfrac{x^2}{4a}$

$\therefore x^2 = 4a(y - 4a)$ ②

∴ Vertex $= (0, 4a)$

Question 14 continued.

b)



$\angle TPM = \angle LPK$ (vertically opposite ∠s are equal)

$\angle TPM = \angle PQM$ (∠ in the alternate segment in C_2 with tangent TP).

$\therefore \angle LPK = \angle PQM$ ①

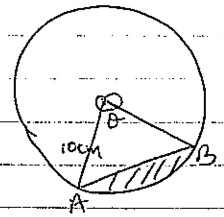
$\angle PQM = \angle PLK$ (exterior ∠ to a cyclic quadrilateral in C_1 = opposite interior ∠). ①

$\therefore \angle LPK = \angle PLK \Rightarrow PK = LK$

∴ Δ PKL is isosceles (sides opposite equal side ∠s).

Question 14 continued

c)



$$S = \frac{1}{2} \times 10^2 (\theta - \sin\theta) = 50(\theta - \sin\theta)$$

$$\therefore \frac{dS}{d\theta} = 50(1 - \cos\theta)$$

$$\frac{d\theta}{dt} = 0.01 \text{ rad/sec.}$$

$$\frac{dS}{dt} = \frac{dS}{d\theta} \times \frac{d\theta}{dt}$$

$$\therefore \frac{dS}{dt} = 50(1-\cos\theta) \times 0.01$$

when $\theta = \frac{\pi}{3}$, $\frac{dS}{dt} = 50(1-\frac{1}{2}) \times 0.01$
$= 0.25$

∴ Area increases at $0.25 \text{ cm}^2/\text{s}$.

d) (i) $T = 35 + Ae^{-kt}$

$$\frac{dT}{dt} = -k\,Ae^{-kt} = -k(T-35) \quad (1)$$

(ii) when $t = 0$, $T = 1400$

$\therefore 1400 = 35 + A \Rightarrow A = 1365$

when $t = 15$, $T = 995$

$\therefore 995 = 35 + 1365e^{-15k} \Rightarrow e^{-15k} = \frac{960}{1365}$

$\therefore -15k = \log \frac{960}{1365}$

$\therefore k = -\frac{1}{15} \log \frac{960}{1365} = 0.02346\ldots$
$= 0.023$

Question 14 continued

d) (iii) when $T = 200$,

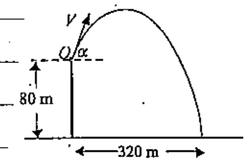
$$200 = 35 + 1365e^{-kt}$$

$$e^{-kt} = \frac{165}{1365}$$

$$\therefore -kt = \log\frac{165}{1365} = \log\frac{11}{91}$$

$$\therefore t = -\frac{\log\frac{11}{91}}{k} = 90.047\ldots$$
$= 90$ minutes

e) (i) $y = -5t^2 + Vt\sin\alpha$



$\dot{y} = -10t + V\sin\alpha$

when $t = 3$, $\dot{y} = 0$ $\therefore 0 = -30 + V\sin\alpha$

$\therefore V\sin\alpha = 30 \quad (1)$

(ii) when it hits the ground, $y = -80$.

$\therefore -80 = -5t^2 + 30t$

$\therefore t^2 - 6t - 16 = 0$

$(t-8)(t+2) = 0 \quad \therefore t = -2$ or 8

as $t \geq 0$, $t = 8$. (2)