
FINAL MARK

GIRRAWEEN HIGH SCHOOL

Mathematics Extension 1

HSC ASSESSMENT

Trial Examination 2014

ANSWERS COVER SHEET

Name:______________________________Teacher:__________.

QUESTION	MARK	HE2	HE3	HE4	HE5	HE6	HE7
1 -6	/6						√
7-9	/3			√			√
10	/1					√	√
	/10						
11a-c	/6						√
d	/3			√			√
e	/3			√		√	√
f	/3						√
	/15						
12a	/4	√					√
b	/5				√		√
c	/2			√			√
d	/4						√
	/15						
13a	/4		√		√		√
b	/7				√		√
c	/4						√
	/15						
14ab	/9						√
c	/6		√				√
	/15						
TOTAL	**/70**	/4	/10	/11	/16	/4	**/70**

HSC Outcomes **Mathematics Extension 1**

HE1 appreciates interrelationships between ideas drawn from different areas of mathematics.

HE2 uses inductive reasoning in the construction of proofs.

HE3 uses a variety of strategies to investigate mathematical models of situations involving binomial probability, projectiles, simple harmonic motion and exponential growth and decay.

HE4 uses the relationship between functions, inverse functions and their derivatives

HE5 applies the chain rule to problems including those involving velocity and acceleration as functions of displacement.

HE6 determines integrals by reduction to a standard form through a given substitution.

HE7 evaluates mathematical solutions to problems and communicates them in an appropriate form.

GIRRAWEEN HIGH SCHOOL

MATHEMATICS

Trial Examination

Year 12 Extension 1 **August 2014**

Time allowed: *120 minutes (Plus 5 minutes reading time)*

INSTRUCTIONS:

- Attempt ALL questions.
- For Questions 1 to 10, write the letter corresponding to the correct response on your working paper.
- For Questions 11 to 14:
    - All questions are worth equal marks.
    - All necessary working should be shown in every question.
    Marks will be deducted for careless or badly arranged work.
    - Board-approved calculators may be used.
    - Each question attempted is to be returned on a *separate* piece of paper labelled clearly Question 11, Question 12, etc. Each sheet of paper should clearly show your name.
    - Diagrams are NOT to scale.

Question 1

The point that divides the interval between A(-8,12) and B (13,-9) externally in the ratio 2:5 is

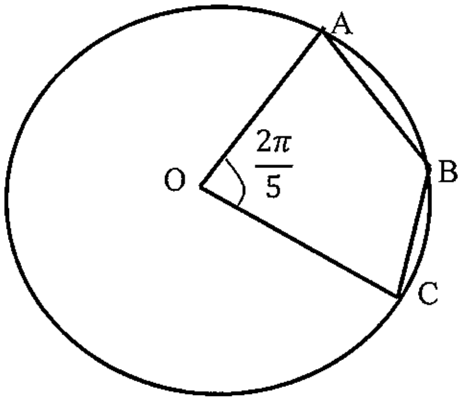
(A) (-22,26) (B) (-2,6) (C) (7,-3) (D) (22,-23)

Question 2

The remainder when a polynomial $P(x)$ is divided by $(x+1)(x-2)$ is $2x-3$. When $P(x)$ is divided by $(x+1)$ the remainder will be

(A) 0 (B) 1 (C) -5 (D) $(x-2)$

Question 3



In the diagram above O is the circle centre. If $\angle AOC = \frac{2\pi}{5}$, $\angle ABC =$

(A) $\frac{\pi}{5}$ (B) $\frac{2\pi}{5}$ (C) $\frac{3\pi}{5}$ (D) $\frac{4\pi}{5}$

Question 4

$$\int 6cos^2 3x.dx =$$

(A) $2sin^3 3x + C$ (B) $2cos^3 6x + C$ (C) $3x + \frac{1}{2}sin6x + C$

(D) $3x - \frac{1}{2}sin6x + C$

Question 5

The number of different committees of 2 men and 3 women which can be formed from a group of 7 men and 8 women is

(A) $\frac{15!}{5!10!}$ (B) $\frac{7!8!}{2!3!(5!)^2}$ (C) $\frac{7!8!}{2!3!}$ (D) $\frac{7!8!}{(5!)^2}$

Question 6

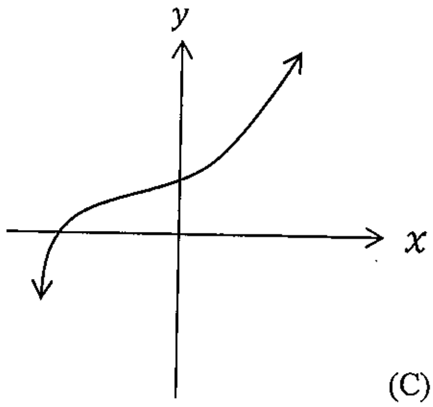
The solution to $3x + 2 = |2x - 1|$ is

(A) $x = -3$ (B) $x = -\frac{1}{5}$ (C) $x = -3 or\ x = -\frac{1}{5}$ (D)No solutions.

Examination continues on the next page

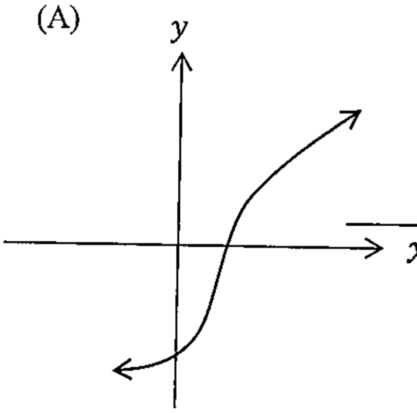
Question 7

The graph of $y = f(x)$ is below:

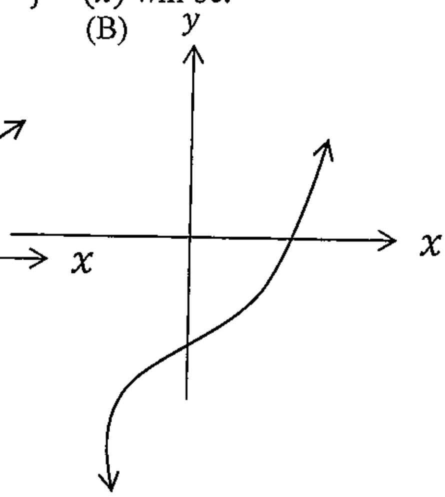


The graph of $y = f^{-1}(x)$ will be:

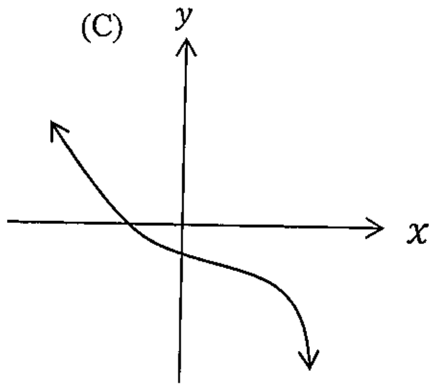
(A)



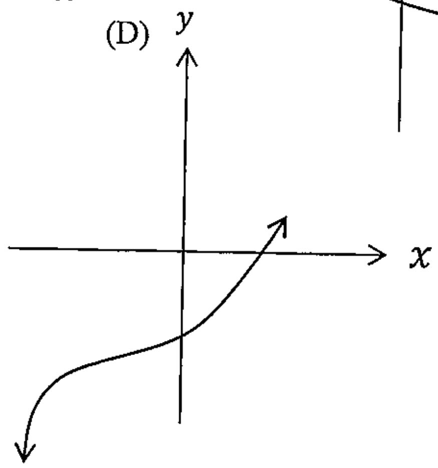
(B)



(C)



(D)



Question 8

The derivative of $y = x^2 sin^{-1}x$ is

(A) $y' = \frac{2x}{\sqrt{1-x^2}}$ (B) $y' = \frac{2x}{\sqrt{1-x^2}} + x^2 sin^{-1}x$

(C) $y' = 2x sin^{-1}x + \frac{x^2}{\sqrt{1-x^2}}$ (D) $y' = 2x cos^{-1}x + x^2 sin^{-1}x$

Question 9

The general solution to the equation $2sin2\boldsymbol{\theta} = -\sqrt{3}$ is

(A) $\boldsymbol{\theta} = (-1)^{n+1} \times \frac{\pi}{6} + \frac{n\pi}{2}$ (B) $\boldsymbol{\theta} = (-1)^{n+1} \times \frac{\pi}{3} + n\pi$

(C) $\boldsymbol{\theta} = (-1)^{n} \times \frac{\pi}{6} + \frac{n\pi}{2}$ (D) $\boldsymbol{\theta} = (-1)^{n} \times \frac{\pi}{3} + n\pi$

Question 10

$$\int_{-\frac{ln2}{2}}^{0} \frac{e^x}{\sqrt{1-e^{2x}}}.dx$$ (Note: use $u = e^x$)

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{2}$

Examination continues on the next page

Question 11 ***(15 Marks – show all necessary working on a separate piece of paper)*** **Marks**

(a) Solve for x: $\frac{3}{2x-1} \geq \frac{1}{5}$ **3**

(b) Find the acute angle between the lines $y = 2x - 1$ and $x + y = 3$. Answer to the nearest minute. **2**

(c) How many different ways are there of arranging the letters of the word HALLOWEEN in a circle? **1**

(d) Find **3**

$$\int_{\frac{-3}{2}}^{\frac{-1}{2}} \frac{e^x}{\sqrt{1-e^{2x}}} . dx$$

(Use the substitution $u = e^x$)

(e) Use the substitution $x = u^2$ to find **3**

$$\int \frac{1}{\sqrt{x(1-x)}} . dx$$

(f) The polynomial equation $2x^3 + px^2 + qx + r = 0$ has roots α, β and γ. If $\alpha + \beta + \gamma = \frac{3}{2}$, $\alpha^2 + \beta^2 + \gamma^2 = \frac{49}{4}$ and $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{2}{5}$ find the values of p, q and r. **3**

Question 12 ***(15 Marks – show all necessary working on a separate piece of paper)***

(a) Prove using mathematical induction for all integers $n \geq 1$: $1^3 + 2^3 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2$ **4**

Question 12 continues on the next page

Question 12 (continued) **Marks**

(b) A piece of metal with an initial temperature of 30^0C is placed in a furnace with temperature 900^0C. The temperature of the surface of the metal obeys Newton's law of cooling – that is, $\frac{dT}{dt} = k(900 - T)$ where T is the current temperature of the surface of the metal and t is the time in minutes since the metal was first put into the furnace.

(i) Prove that $T = 900 - Ae^{-kt}$ satisfies the differential equation $\frac{dT}{dt} = k(900 - T)$. **1**

(i) If the surface of the metal heats up to 540^0C after 30 minutes, find the values of A and k. **2**

(ii) The surface of the metal starts to melt at 700^0C. To the nearest minute, when will the surface have started to melt? **2**

(c) For the graph of $y = 2cos^{-1}\left\{\frac{x}{3}\right\}$

(i) State the domain and range. **1**

(ii) Sketch the graph over its full domain. **1**

(d) (i) Express $\sqrt{5}sin\boldsymbol{\theta} - \sqrt{3}\boldsymbol{cos\theta}$ in the form $Asin(\boldsymbol{\theta} - \boldsymbol{\alpha})$. **2**

(ii) Hence or otherwise solve the equation $\sqrt{5}sin\boldsymbol{\theta} - \sqrt{3}\boldsymbol{cos\theta} = \sqrt{3}$ for $0 \leq \boldsymbol{\theta} \leq \mathbf{360^0}$. **2**

Examination continues on the next page

Question 13 ***(15 Marks – show all necessary working on a separate piece of paper)*** **Marks**

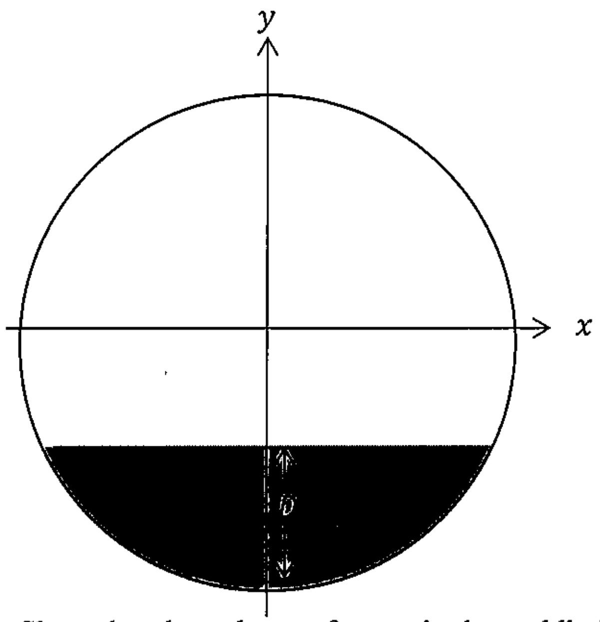
(a) A particle moves along a straight line so that its velocity obeys the rule $v^2 = -9x^2 + 90x - 189$.

(i) Show that the particle is moving with simple harmonic motion. **1**

(ii) By finding where the velocity is zero, find the endpoints and the amplitude of the motion. **2**

(iii) Find the period of the motion. **1**

(b) A puddle of water with depth D lies in the base of a sphere which has been formed by rotating the bottom part of the circle $x^2 + y^2 = 9$ about the y axis *(see diagram.)*



(i) Show that the volume of water in the puddle is given by $V = \frac{\pi}{3}\{27(D-3) - (D-3)^3 + 54\}$ **2**

(ii) Show that the surface area of the top of the puddle is given by $S = \pi(6D - D^2)$ **2**

Question 13 continues on the next page

(iii) If the volume of water in the puddle is decreasing at a rate proportional to its surface area (i.e. $\frac{dV}{dt} = -kS, k$ a constant), show that the rate at which the *depth* of the water is changing is $-kcm/min$. **2**

(iv) If the depth of the water is initially $1cm$, time is in minutes and $k = 0.02$ find out how long it will take for the puddle to evaporate entirely. **1**

(c) (i) Show that the equation $e^x + x = 0$ has a root between $x = 0$ and $x = -1$. **1**

(iii) Using $x_0 = -0.5$, use one application of Newton's method to find a better approximation for the root of $e^x + x = 0$. **3**

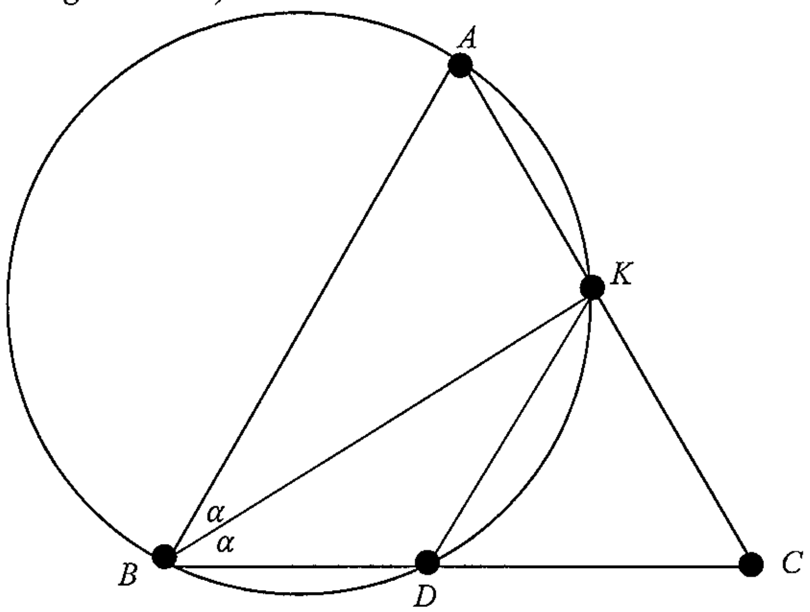
Examination continues on the next page

Question 14 ***(15 Marks – show all necessary working on a separate piece of paper)*** **Marks**

(a) In $\Delta ABC, AB = AC$ and BK bisects $\angle ABC$.

The circle through A, B and K cuts BC at D.

(see diagram below)



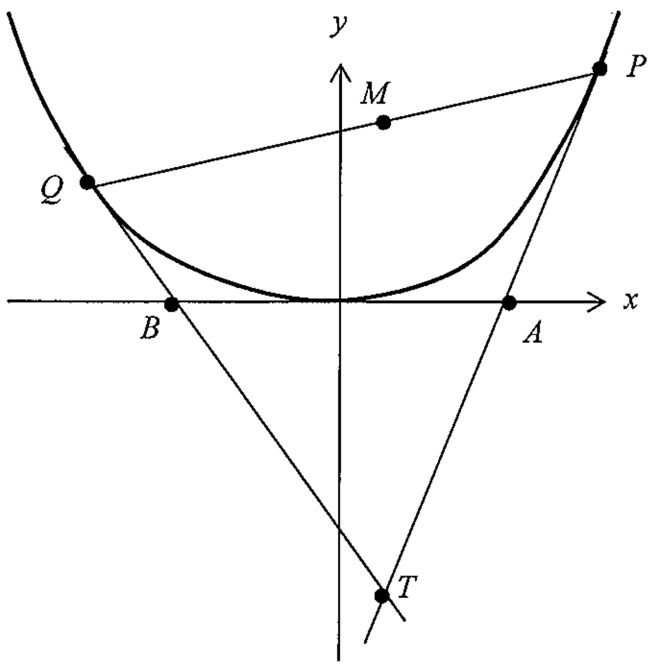
(i) Copy or trace the diagram on to your answer paper and state why $\angle DKC = 2\alpha$. **1**

(ii) Prove that $AK = DC$. **2**

Question 14 continues on the next page

Question 14 (continued) **Marks**

(b) $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are on the parabola $x^2 = 4ay$. The tangents at P and Q meet the x axis at A and B respectively and meet each other at T. M is the midpoint of PQ. Note that q is negative and $|p| > |q|$. *(See diagram.)*



(i) Find the gradient of the chord PQ and the gradient of the tangent at P. **1**

(ii) Find the coordinates of M. **1**

(iii) If $\Delta PQT \parallel\!| \Delta BAT$ show that $pq = -1$. **2**

(Hint: If $\Delta PQT \parallel\!| \Delta BAT$ then $\tan\angle QPT = \tan\angle ABT$.)

(iv) If $pq = -1$ find the locus of M. **2**

Question 14 continues on the next page

Question 14 (continued) **Marks**

(c) (i) A howitzer is located on a hill 100 metres high. It has a muzzle velocity of $500m/s$. For a shell fired from the howitzer at an angle α up from the horizontal, the equations of motion are

$x = 500t\cos\alpha$ and $y = -\frac{1}{2}gt^2 + 500t\sin\alpha + 100$.

(***Do NOT prove these***!)

*(A)*Show that $y = \frac{-gx^2}{500000}\sec^2\alpha + x\tan\alpha + 100$ **1**

*(B)*If the howitzer is aimed so that a shell fired from it will hit a target $5000m$ away on the plain below, find the two angles at which it could be fired to hit this target. **2**

(Use $g = 9.8m/s^2$)

(ii)Another howitzer with a muzzle velocity of Vm/s is on level ground (so the initial height is $0m$.)If it fires a shell at an angle α to the horizontal the equation for y in terms of x is

$y = \frac{-gx^2}{2V^2}\sec^2\alpha + x\tan\alpha$. ***(Do NOT prove this!)***

*(A)*If the shell is fired to hit a target R metres away show that **1**

$gR^2\tan^2\alpha - 2V^2R\tan\alpha + gR^2 = 0$.

*(B)*Show that $\tan\alpha = \frac{V^2 \pm \sqrt{V^4 - g^2R^2}}{gR}$ **1**

*(C)*Show that the two possible values of α which result in the shell hitting the target add up to 90^0 . **1**

Here endeth the examination!!!

Solutions / Marking Scheme p.1

Y12 HSC Extension 1 Trial

Girraween HS 2014

Multiple Choice:

(1) A (2) C (3) D (4) C (5) B (6) B (7) A (8) C
(9) A (10) B

(15)

(11)(a) By critical points:

Equality:

$\frac{3}{2x-1} = \frac{1}{5}$

$\times 5(2x-1)$

$15 = 2x-1$

$8 = x$.

Discontinuity: $x \neq \frac{1}{2}$. 1

$\times \times \frac{1}{2} \checkmark\checkmark\checkmark\checkmark 8 \times\times\times$ 1

Test: $x=0$	Test: $x=2$	Test: $x=11$
$\frac{3}{2\times 0-1}$	$\frac{3}{2\times 2-1}$	$\frac{3}{2\times 11-1}$
$=-3 \ngeq \frac{1}{5}$	$=1 \geq \frac{1}{5}$	$=\frac{1}{7} \ngeq \frac{1}{5}$

3

Solution: $\frac{1}{2} < x \leq 8$. 1

(OR) By "New York"

$\frac{3}{2x-1} \geq \frac{1}{5}$

$\times$ BS by $5(2x-1)^2$

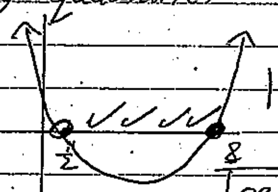
$15(2x-1) \geq (2x-1)^2$ 1

$0 \geq (2x-1)^2 - 15(2x-1)$

$0 \geq (2x-1)[2x-1-15]$

$0 \geq (2x-1)(2x-16)$

Sketching quadratic:



$\frac{1}{2} < x \leq 8$. 1

OR 3

(b) m of $x+y=3$

$y=-x+3$

$m=-1$.

$\tan\theta = \left|\frac{m_1 - m_2}{1+m_1 m_2}\right|$

$= \left|\frac{2+1}{1+2\times -1}\right|$ 1

$= |-3|$

$\tan\theta = 3$. $\rightarrow$ $\therefore \theta = 71^\circ 34'$ 1

2

p. 2

(11)(c) HALLOWEEN

$\rightarrow$ 9 letters, 2 L's, 2 E's.

$= \frac{8!}{2!2!}$

$= 10\,080$ ways.

(d) $\int_{-\frac{3}{2}}^{-\frac{1}{2}} \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$

$u = e^x$

$du = e^x\,dx$

$= \int_{-\frac{3}{2}}^{-\frac{1}{2}} \frac{1}{\sqrt{1-e^{2x}}} \cdot e^x\,dx$

$= \int_{e^{-\frac{3}{2}}}^{e^{-\frac{1}{2}}} \frac{1}{\sqrt{1-u^2}}\,du$

$= \left[\sin^{-1} u\right]_{e^{-\frac{3}{2}}}^{e^{-\frac{1}{2}}}$

$= \sin^{-1}(e^{-\frac{1}{2}}) - \sin^{-1}(e^{-\frac{3}{2}})$

$= 0.4267$ [4DP].

Final a... fo... s...

Solutions: p.3

Q. (11)(e) $\int \frac{1}{\sqrt{x(1-x)}}.dx$ $x = u^2$, $dx = 2u.du$

$= \int \frac{1}{\sqrt{u^2(1-u^2)}}.2u.du$ 1

$= \int \frac{1}{u\sqrt{1-u^2}}.2u.du$

$= \int \frac{2}{\sqrt{1-u^2}}.du$

$= 2\sin^{-1}(u) + C$ 1 **3**

$= 2\sin^{-1}(\sqrt{x}) + C$ 1

(f) $\alpha+\beta+y = -\frac{p}{2} = \frac{3}{2}$

$\therefore p = -3.$ 1

$\alpha^2+\beta^2+y^2 = (\alpha+\beta+y)^2 - 2(\alpha\beta+\alpha y+\beta y)$

$\frac{49}{4} = (\frac{3}{2})^2 - 2(\alpha\beta+\alpha y+\beta y)$

$\therefore 2(\alpha\beta+\alpha y+\beta y) = -10$

$\alpha\beta+\alpha y+\beta y = -5$ **3**

$\frac{q}{2} = -5$

$q = -10.$ 1

$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{y} = \frac{2}{5}$

$\frac{\alpha\beta+\alpha y+\beta y}{\alpha\beta y} = \frac{2}{5}$

As $\alpha\beta+\alpha y+\beta y = -5$,

$\frac{-5}{\alpha\beta y} = \frac{2}{5}.$

$-25 = 2\alpha\beta y$

$-\frac{25}{2} = \alpha\beta y$

$\therefore -\frac{r}{2} = -\frac{25}{2}$

$r = 25.$ 1

$\therefore p=-3, q=-10, r=25.$

Solutions: p.4 (15)

Q. (12)(a) Step 1: Show true for $n=1$:

LHS $= 1^3 = 1.$

RHS: $= \frac{1}{4}\times 1^2(1+1)^2 = 1$

LHS = RHS

True for $n=1$.

Step 2: Assume true for $n=k$ 1

i.e. $1^3+2^3+\ldots+k^3 = \frac{1}{4}k^2(k+1)^2.$

Step 3: Prove true for $n=k+1$

i.e. $1^3+2^3+\ldots+k^3+(k+1)^3 = \frac{1}{4}(k+1)^2(k+2)^2.$ 1

LHS:

$1^3+2^3+\ldots+k^3+(k+1)^3$

$= \frac{1}{4}k^2(k+1)^2 + (k+1)^3$ [using Step 2]. 1

$= \frac{1}{4}(k+1)^2[k^2+4(k+1)]$

$= \frac{1}{4}(k+1)^2[k^2+4k+4]$ **4**

$= \frac{1}{4}(k+1)^2(k+2)^2$ 1

= RHS.

$\therefore$ If it is true for $n=k$, it will be true for $n=k+1$

$\therefore$ As it is true for $n=1$ it will be true for $n=1+1=2$ and so on for all positive integers n.

(b)(i) $T = 900 - Ae^{-kt}$

LHS	RHS
$\frac{dT}{dt} = Ake^{-kt}$	$k(900-T)$
	$= k[900-(900-Ae^{-kt})]$
	$= k\times Ae^{-kt}$
	$= kAe^{-kt}$

LHS = RHS $\therefore T = 900 - Ae^{-kt}$ satisfies equation.

Solutions: p. 5

Q. (12)(b)(ii) When $t = 0$, $T = 30$

$\therefore T = 900 - Ae^{-kt}$

$30 = 900 - Ae^{-k\times 0}$

$30 = 900 - A$

$\therefore A = 870.$ 1

When $t = 30$, $T = 540$

$T = 900 - 870e^{-kt}$

$540 = 900 - 870e^{-30k}$

$e^{-30k} = \frac{360}{870}$

$e^{-30k} = \frac{12}{29}.$ 2

$e^{30k} = \frac{29}{12}$

$30k = \ln\left(\frac{29}{12}\right)$

$k = \frac{\ln\left(\frac{29}{12}\right)}{30}$ $[\doteqdot 0{\cdot}02941\ldots]$. 1

(iii) Starts to melt at 700°C

i.e. Find t when $T = 700$

$T = 900 - 870e^{-\frac{\ln(29/12)}{30}t}$

$700 = 900 - 870e^{-\frac{\ln(29/12)}{30}t}$ 1

$\frac{20}{87} = e^{-\frac{\ln(29/12)}{30}t}$

$\ln\left(\frac{20}{87}\right) = -\frac{\ln\left(\frac{29}{12}\right)}{30}t$ 2

$49{\cdot}98\ldots = t$

The surface will start to melt after 50 minutes.

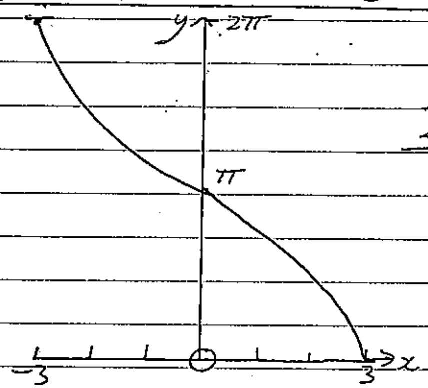
Solutions: p. 6

Q. (12)(c)(i) $y = 2\cos^{-1}\left(\frac{x}{3}\right)$

Domain: $-1 \le \frac{x}{3} \le 1$. | Range: $0 \le \frac{y}{2} \le \pi$.

$-3 \le x \le 3$. $0 \le y \le 2\pi$. 1

(ii)



2

(d)(i) Let $\sqrt{5}\sin\theta - \sqrt{3}\cos\theta \equiv A\sin(\theta - \alpha)$

$\therefore \sqrt{5}\sin\theta - \sqrt{3}\cos\theta \equiv A\sin\theta\cos\alpha - A\cos\theta\sin\alpha$

Equating parts:

$\sqrt{5}\sin\theta = A\sin\theta\cos\alpha$	$-\sqrt{3}\cos\theta = -A\cos\theta\sin\alpha$.
$\sqrt{5} = A\cos\alpha$ (1)	$\sqrt{3} = A\sin\alpha$ (2)

Squaring & adding (1) & (2)

$A^2(\cos^2\alpha + \sin^2\alpha) = (\sqrt{5})^2 + (\sqrt{3})^2$

$A = 2\sqrt{2}.$ 1 2

Sub. $A = 2\sqrt{2}$ in (1): $\cos\alpha = \frac{\sqrt{5}}{2\sqrt{2}}$ Sub. $A = 2\sqrt{2}$ in (2): $\sin\alpha = \frac{\sqrt{3}}{2\sqrt{2}}$ → Q. (sin, cos +).

$\alpha = 37°46'$. 1

$\therefore \sqrt{5}\sin\theta - \sqrt{3}\cos\theta \equiv 2\sqrt{2}\sin(\theta - 37°46')$ [Rounded].

(ii) $\sqrt{5}\sin\theta - \sqrt{3}\cos\theta = \sqrt{3}$

$2\sqrt{2}\sin(\theta - 37°46') = \sqrt{3}.$

$\sin(\theta - 37°46') = \frac{\sqrt{3}}{2\sqrt{2}}.$

in domain $-37°46' \le \theta - 37°46' \le 322°14'$

$\theta - 37°46' = 37°46', 142°14'.$

$\theta = 75°32', 180°.$

[if not rounded, $\theta = 75°31', 180°$]. 1

Solutions: p.7

Q. (12)(d)(ii) By t formulae: ALTERNATIVE [if students want the hard way!].

$\sqrt{5}\sin\theta - \sqrt{3}\cos\theta = \sqrt{3}$

$\sqrt{5}\times\frac{2t}{1+t^2} - \sqrt{3}\frac{(1-t^2)}{1+t^2} = \sqrt{3}$ $\quad \times 1+t^2$

$\therefore 2t\sqrt{5} - \sqrt{3} + t^2\sqrt{3} = \sqrt{3} + t^2\sqrt{3}$

$2t\sqrt{5} = 2\sqrt{3}$

$t = \frac{\sqrt{3}}{\sqrt{5}}$

$\tan\left(\frac{\theta}{2}\right) = \frac{\sqrt{3}}{\sqrt{5}}$

$\frac{\theta}{2} = 37°46'$

$\theta = 75°32'$ ($75°31'$ with no rounding)

OR $[\pi]$

Test $\theta = 180°$: $\sqrt{5}\sin 180° - \sqrt{3}\cos 180° = \sqrt{3}$

$\rightarrow 180°$ IS a solution.

$\theta = 75°32', 180°$.

Q. (13)(a)(i) $v^2 = -9x^2 + 90x - 189$. (15)

$\frac{d}{dx}(v^2) = -18x + 90$

$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = -9x + 45$

$\therefore \ddot{x} = -9(x-5)$. 1

As $\ddot{x}$ obeys $\ddot{x} = -n^2(x-c)$, c the centre of motion, particle is moving with SHM.

(ii) $v = 0$: $-9x^2 + 90x - 189 = 0$

$x^2 - 10x + 21 = 0$

$(x-7)(x-3) = 0$ 2

$x = 7$ & $x = 3$ when $v = 0$] 1

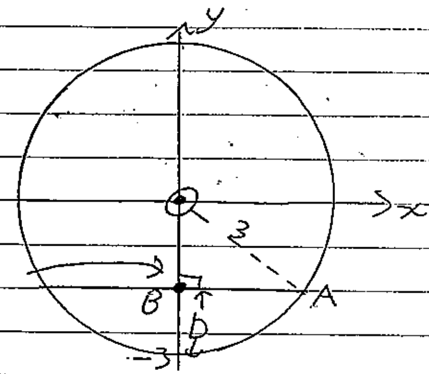
Endpoints of motion are $x=3, x=7$. Amplitude $= 2$.] 1

(iii) Period: From (1) $\ddot{x} = n^2(x-c) = -9(x-5)$

$\therefore n^2 = 9$, $n = 3$ | Period of motion $= \frac{2\pi}{3}$. 1

Solutions: p.8

Q. (13)(b)(i)



$y = -3 + D$ or $D - 3$

$V[\text{puddle}] = \pi\int_{-3}^{D-3} x^2\,dy$

$= \pi\int_{-3}^{D-3} 9 - y^2\,dy$ 1

2

$= \pi\left[9y - \frac{1}{3}y^3\right]_{-3}^{D-3}$

$= \pi\left[9(D-3) - \frac{1}{3}(D-3)^3 - \left(9\times -3 - \frac{1}{3}\times(-3)^3\right)\right]$

$= \pi\left[9(D-3) - \frac{1}{3}(D-3)^3 + 18\right]$

$V = \frac{\pi}{3}\left[27(D-3) - (D-3)^3 + 54\right]$ 1

$=$ RHS QED.

(ii) From diagram above, distance $OA = 3$ [radius]

$OB = 3 - D$

$\therefore (BA)^2 = (OA)^2 - (OB)^2$

$= 3^2 - (3-D)^2$

$= 6D - D^2$ 1

$\therefore S$ [top of puddle $= \pi r^2$ where $r = BA \Rightarrow r^2 = (BA)^2 = 6D - D^2$

$= \pi(6D - D^2)$ 1

$=$ RHS QED. 2

PTO →

Q. (13)(b)(iii) $\frac{dD}{dt} = \frac{dD}{dV} \times \frac{dV}{dt}$ (1)

Note: $V = \frac{\pi}{3}\left[27(D-3) - (D-3)^3 + 54\right]$

$\frac{dV}{dD} = \frac{\pi}{3}\left[27 - 3(D-3)^2\right]$.

$= \pi\left[6D - D^2\right]$

$\therefore \frac{dD}{dV} = \frac{1}{\pi(6D-D^2)}$. (2) | 1

We are given $\frac{dV}{dt} = -kS$

$= -k\pi(6D - D^2)$ (3)

$\therefore$ Sub. (2) and (3) in (1): 2

$\frac{dD}{dt} = \frac{dD}{dV} \times \frac{dV}{dt}$

$= \frac{1}{\pi(6D-D^2)} \times -k\pi(6D-D^2)$ | 1

$\frac{dD}{dt} = -k$ as required.

(iv) $\frac{dD}{dt} = -0.02$ cm/min.

$D = \int -0.02\,dt$

$= -0.02t + C$

As $D = 1$ when $t = 0$,

$1 = C$

$\therefore D = 1 - 0.02t$.

Finding t when $D = 0$

$1 - 0.02t = 0$ 1

$t = 50$

Puddle will evaporate after 50 minutes.

Q. (13)(c)(i) At $x = -1$, $e^x + x$

$= e^{-1} - 1$

$\doteqdot -0.63$.

At $x = 0$, $e^x + x$

$= e^0 + 0$

$= 1$. | 1

As $e^x + x < 0$ at $x = -1$ & $e^x + x > 0$ at $x = 0$ and is continuous for $-1 \le x \le 0$, $e^x + x = 0$ must have a root between $x = 0$ & $x = -1$.

(ii) $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ | 1 $f'(x) = e^x + 1$.

$= -0.5 - \frac{e^{-0.5} - 0.5}{e^{-0.5} + 1}$ | 1   3

$= -0.5663\ldots$ | 1

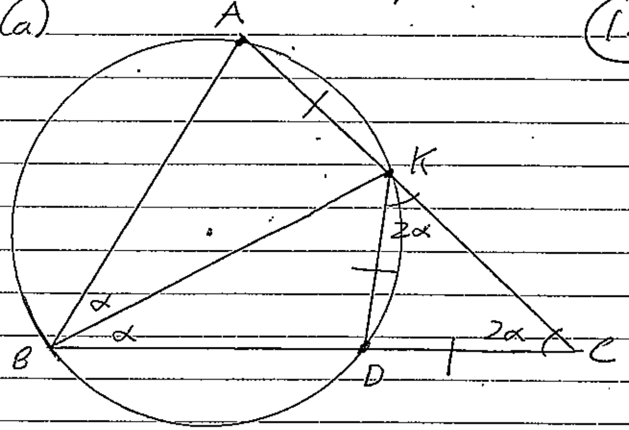
Better approximation for solution of $e^x + x = 0$ is $x = -0.5$

[$\doteqdot 0.5663\ldots$].

(15)

Q. (14)(a)

(i)



$\angle DKC = 2\alpha$ (exterior $\angle$ cyclic quadrilateral (DKAB)] 1

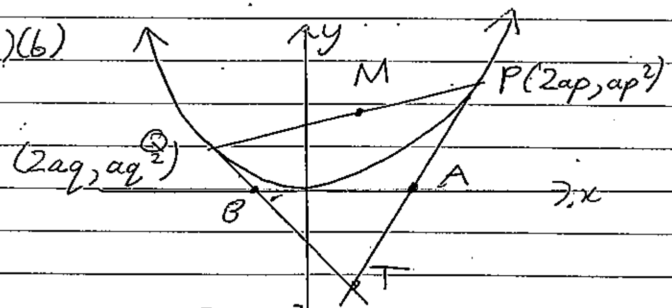
(ii) $\angle ACB = 2\alpha$ ($\angle$'s opposite = sides in isosceles $\triangle ABC$]. 1

DK = DC [sides opposite = $\angle$'s in isosceles $\triangle$ CKD].

DK = AK [= chords subtend = $\angle$'s $\angle ABK$ & $\angle DBK$ at circumference] 1/2

$\therefore$ AK = DC.

Q. (14)(b)



(i) $m\,PQ = \dfrac{ap^2 - aq^2}{2ap - 2aq}$

$= \dfrac{a(p-q)(p+q)}{2a(p-q)}$

$= \dfrac{p+q}{2}, p \neq q.$

m [tangent at P]

$x^2 = 4ay$

$\dfrac{1}{4a}x^2 = y$

$\dfrac{x}{2a} = \dfrac{dy}{dx}.$

At P (2ap, ap²)

$\dfrac{dy}{dx} = \dfrac{2a}{2a} = p.$ | m of tangent = p. 1

Q. (14)(b) (ii) $M = \left(\dfrac{2ap+2aq}{2}, \dfrac{ap^2+aq^2}{2}\right)$

$= \left[a(p+q), \dfrac{a}{2}(p^2+q^2)\right].$ 1

(iii) In $\triangle PQT$ and $\triangle BAT$

$\angle PTQ = \angle BTA$ [same angle].

$\tan\angle ABT = -q$ [as q is negative & $\angle ABT < 90°$].

By $\tan\theta = \dfrac{m_1 - m_2}{1 + m_1 m_2}$

$\tan\angle TPQ = \dfrac{mPT - mPQ}{1 + mPT\, mPQ}$

$= \dfrac{p - \frac{p+q}{2}}{1 + p \times \frac{p+q}{2}}$ 1

If $\triangle PQT \parallel\!\mid \triangle BAT$ then $\angle ABT = \angle TPQ$

$\tan\angle ABT = \tan\angle TPQ$

$-q = \dfrac{p - \frac{p+q}{2}}{1 + \frac{p(p+q)}{2}}$

$-q = \dfrac{2p - (p+q)}{2 + p(p+q)}$ 2

$-q = \dfrac{p-q}{2+p(p+q)}$ 1

$-2q - p(p+q) = p - q$

$-pq(p+q) = p+q$

$pq = -1.$

(iv) M: $\dfrac{x}{a} = p+q.$

$y = \dfrac{a}{2}(p^2+q^2)$

$= \dfrac{a}{2}\left[(p+q)^2 - 2pq\right]$ 1

Using $p+q = \dfrac{x}{a}$, $pq = -1$.

$y = \dfrac{a}{2}\left[\left(\dfrac{x}{a}\right)^2 + 2\right]$ 1

$y = \dfrac{a}{2}\left[\dfrac{x^2}{a^2} + 2\right].$ 2

$y = \dfrac{x^2}{2a} + a$

Solutions: p. 13

Q. (14)(c)(i)(A) $x = 500t\cos\alpha$

$$\frac{x}{500\cos\alpha} = t$$

$$y = -\frac{1}{2}gt^2 + 500t\sin\alpha + 100$$

$$= -\frac{g}{2}\times\left(\frac{x}{500\cos\alpha}\right)^2 + 500\times\frac{x\sin\alpha}{500\cos\alpha} + 100$$

$$y = \frac{-gx^2}{500\,000}\sec^2\alpha + x\tan\alpha + 100 \quad 1$$

(B) Finding α when $y=0$; $x=5000$

$$y = \frac{-gx^2}{500\,000}\sec^2\alpha + x\tan\alpha + 100$$

$$0 = \frac{-g}{500\,000}\times 5000^2\sec^2\alpha + 5000\tan\alpha + 100$$

$$0 = -50g\sec^2\alpha + 5000\tan\alpha + 100$$

$\div$ BS by -50

$$0 = g\sec^2\alpha - 100\tan\alpha - 2$$

$$= g(1+\tan^2\alpha) - 100\tan\alpha - 2$$

$$= g + g\tan^2\alpha - 100\tan\alpha - 2$$

$$0 = g\tan^2\alpha - 100\tan\alpha + g - 2 \quad 1$$

Letting $g = 9.8$

$$9.8\tan^2\alpha - 100\tan\alpha + 7.8 = 0$$

$$49\tan^2\alpha - 500\tan\alpha + 39 = 0 \quad 2$$

$$\tan\alpha = \frac{500 \pm \sqrt{500^2 - 4\times 49\times 39}}{2\times 49}$$

$$\tan\alpha = \frac{500+\sqrt{242356}}{98} \text{ or } \frac{500-\sqrt{242356}}{98}$$

$$\alpha = 84°22' \text{ or } 4°30' \quad 1$$

Solutions p. 14

Q. (14)(c)(ii)(A)

$$y = \frac{-gx^2}{2V^2}\sec^2\alpha + x\tan\alpha$$

As $y = 0$ when $x = R$

$$-\frac{gR^2}{2V^2}\sec^2\alpha + R\tan\alpha = 0$$

$\times$ BS by $-2V^2$

$$gR^2\sec^2\alpha - 2V^2R\tan\alpha = 0$$

$$gR^2(1+\tan^2\alpha) - 2V^2R\tan\alpha = 0 \quad 1$$

$$gR^2\tan^2\alpha - 2V^2R\tan\alpha + gR^2 = 0 \text{ as required.}$$

(B) $$\tan\alpha = \frac{2V^2R \pm \sqrt{(2V^2R)^2 - 4\times gR^2\times gR^2}}{2gR^2}$$

$$= \frac{2V^2R \pm \sqrt{4V^4R^2 - 4g^2R^4}}{2gR^2}$$

$$= \frac{2V^2R \pm 2R\sqrt{V^4 - g^2R^2}}{2gR^2}$$

$$\tan\alpha = \frac{V^2 \pm \sqrt{V^4 - g^2R^2}}{gR} \text{ as required.} \quad 1$$

(C) If the two values of α add up to $90°$ then $\tan\alpha_2 = \frac{1}{\tan\alpha_1}$ i.e. $\tan\alpha_1 \times \tan\alpha_2 = 1$.

As $\tan\alpha_1 = \frac{V^2 + \sqrt{V^4 - g^2R^2}}{gR}$, $\tan\alpha_2 = \frac{V^2 - \sqrt{V^4 - g^2R^2}}{gR}$

$$\tan\alpha_1\times\tan\alpha_2 = \frac{(V^2 + \sqrt{V^4 - g^2R^2})}{gR}\times\frac{(V^2 - \sqrt{V^4 - g^2R^2})}{gR}$$

$$= \frac{V^4 - (V^4 - g^2R^2)}{g^2R^2}$$

$$= \frac{g^2R^2}{g^2R^2} \quad 1$$

$\alpha_1 + \alpha_2 = 90°$ as required.