Girraween High School

2015 Year 12 Trial Higher School Certificate

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 11–14, show relevant mathematical reasoning and/or calculations

Total marks – 70

Section I

10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II

60 marks

- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

- For **Section II: Questions 11 – 14** MUST be returned in clearly marked ***separate* sections.**
- On each page of your answers, clearly write:
  - the **QUESTION** being answered
  - **YOUR NAME**
  - your **Mathematics TEACHER'S NAME.**
- Start each new question on a **NEW PAGE.**
- You may ask for extra pieces of paper if you need them.

For questions 1-10, fill in the response oval corresponding to the correct answer on your Multiple choice answer sheet.

1. What is the acute angle between the lines $y = 2x - 3$ and $3x + 5y - 1 = 0$, to the nearest degree?

A) 32^0 B) 50^0 C) 82^0 D) 86^0

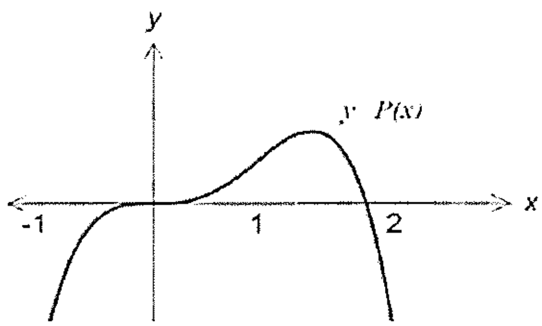
2. The number of different arrangements of the letters of the word REGISTER which begin and end with letter R is:

A) $\frac{6!}{(2!)^2}$ B) $\frac{8!}{2!}$ C) $\frac{6!}{2!}$ D) $\frac{8!}{2!2!}$

3. The middle term in the expansion $(2x-4)^4$ is

A) 81 B) $216x^2$ C) $384x^2$ D) $-96x^3$

4.



Which of the following could be the polynomial $y = P(x)$?

A) $P(x) = x^3(2-x)$ B) $P(x) = x^2(2-x)^2$

C) $P(x) = x^3(x-2)$ D) $P(x) = -x^3(x+2)$

5. The coordinates of the points that divides the interval joining (-7,5) and (-1,-7) externally in the ratio 1:3 are

A) (-10,8) B)(-10,11) C) (2,8) D)(2,11)

6. Which of the following represents the exact value of $\int_0^{\frac{\pi}{8}} \cos^2 x dx$?

A) $\frac{\pi - 2\sqrt{2}}{16}$ B) $\frac{\pi - 2\sqrt{2}}{8}$ C) $\frac{\pi + 2\sqrt{2}}{16}$ D) $\frac{\pi + 2\sqrt{2}}{8}$

7. Which of the following represents the derivative of $y = \cos^{-1}\left(\frac{1}{x}\right)$?

A) $-\frac{1}{x\sqrt{x^2 - 1}}$ B) $\frac{-1}{\sqrt{x^2 - 1}}$ C) $\frac{1}{\sqrt{x^2 - 1}}$ D) $\frac{1}{x\sqrt{x^2 - 1}}$

8. Let α, β, γ be the roots of $2x^3 + x^2 - 4x + 9 = 0$. What is the value of $\frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\alpha\gamma}$?

A) $\frac{-1}{9}$ B) $\frac{-1}{2}$ C) $\frac{1}{9}$ D) $\frac{1}{2}$

9. If $\cos\theta = -\frac{3}{5}$ and $0 < \theta < \pi$, then $\tan\frac{\theta}{2}$ is equal to:

A) $\frac{-1}{3}$ or 3 B) $\frac{1}{3}$ or 3 C) -2 D) 2

10. A particle is moving in Simple Harmonic Motion and its displacement, x units, at time t seconds is given by the equation $x = A\cos(nt) + 2$. The period of the motion is 4π seconds and the particle is initially at rest, 12 units to the right of the origin. Find the values of A and n.

A) $A = 10, n = \frac{1}{2}$ B) $A = 10, n = 2$ C) $A = 12, n = \frac{1}{2}$ D) $A = 12, n = 2$

Question11.(15 marks)-show all necessary working) **Marks**

(a) Solve for x: $\dfrac{5}{x-1} > 2$ **2**

(b) Find the value of θ, such that $\sqrt{3}\cos\theta - \sin\theta = 1$, where $0 \le \theta \le 2\pi$. **3**

(c) Use the substitution $u = \sin^2 x$ to evaluate $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin 2x}{1+\sin^2 x}\,dx$

Give your answer in simplest form. **3**

(d) Use the mathematical induction to show that for all positive integers $n \ge 2$,

$$2\times1 + 3\times2 + 4\times3 + \dots\dots\dots\dots + n(n-1) = \frac{n(n^2-1)}{3}$$ **4**

(e) The coefficients of x^2 and x^{-1} in the expansion of $\left(ax - \dfrac{b}{x^2}\right)^5$ are the same, where a and b are non-zero. Show that $a + 2b = 0$. **3**

Question 12.(15 marks)

a) i) Find $\frac{d}{dx}\cos^{-1}\left(\frac{x-10}{10}\right)$. **2**

ii) Hence, evaluate $\int_5^{10} \frac{1}{\sqrt{20x-x^2}}dx$ **2**

b) Two points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$.

The general tangent at any point on the parabola with parameter t is given by

$y = tx - at^2$ (DO NOT prove this).

i) Find the coordinates of the point of intersection T of the tangents to the parabola at P and Q. **2**

ii) You are given that the tangents at P and Q intersect at an angle of 45^0. Show that $p - q = 1 + pq$ **2**

iii) By evaluating the expression $x^2 - 4ay$, or otherwise, find the locus of the point T when the tangents at P and Q meet as described in part (ii) above. **3**

c) The velocity v m/s of a particle moving in simple harmonic motion along the $x-$axis is given by $v^2 = 8 + 2x - x^2$.

i) Between what two points is the particle oscillating? **1**

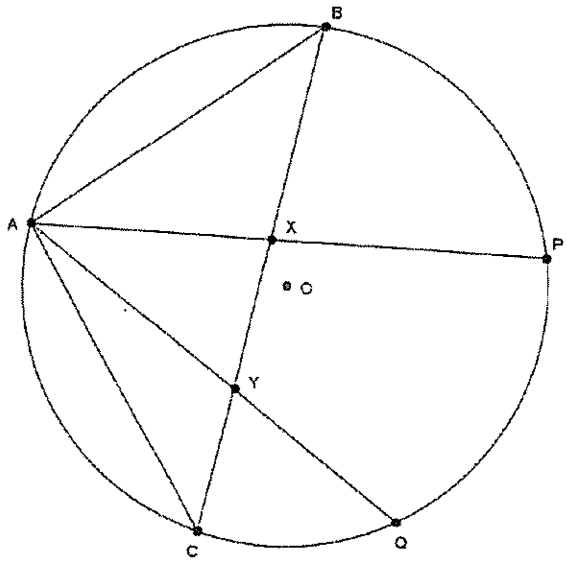
ii) What is the amplitude of the motion? **1**

iii) Find the acceleration of the particle in terms of x. **1**

iv) Find the period of oscillation. **1**

Question 13.(15 marks)

a) Let $ABPQC$ be a circle such that $AB = AC$, AP meets BC at X and AQ meets BC at Y as shown below. Let $\angle BAP = \alpha$ and $\angle ABC = \beta$.



i) Copy the diagram in your writing booklet, marking the information given above. **1**

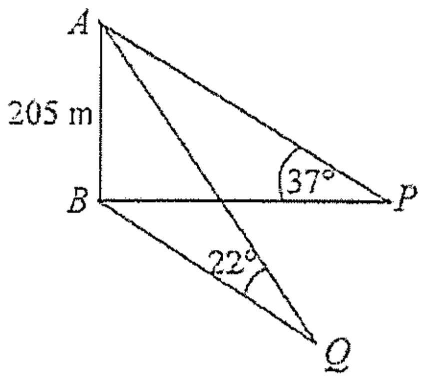
ii) State why $\angle AXC = \alpha + \beta$. **1**

iii) Prove that $\angle BQP = \alpha$. **1**

iv) Prove that $\angle BQA = \beta$. **1**

v) Prove that the quadrilateral $PQYX$ is cyclic. **2**

b) When the polynomial $P(x)$ is divided by $x^2 - 1$, the remainder is $3x + 1$. What is the remainder when $P(x)$ is divided by $x + 1$? **2** c)



NOT TO SCALE

A is 205 metres above the horizontal plane BPQ. AB is vertical. The angle of elevation of A from P is 37^0 and the angle of elevation of A from Q is 22^0. P is due East of B and Q is south 47^0 east from B. Calculate the distance from P to Q, to the nearest metre. **3**

d) Four people visit a town with four restaurants A, B, C and D.

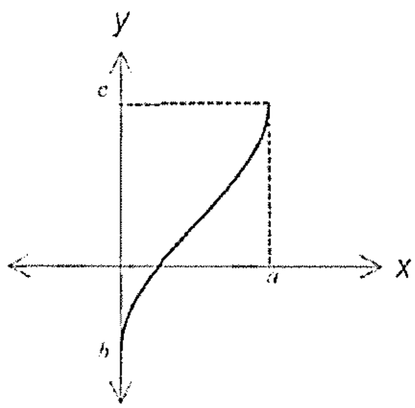
Each person chooses a restaurant at random.

i) Find the probability that they all choose different restaurants. **2**

ii) Find the probability that exactly two of them choose restaurant A. **2**

Question 14.(15 marks).

a) The graph of $y = 1 + 2\sin^{-1}(2x - 1)$ is shown in the diagram.



Determine the values of a, b and c. **2**

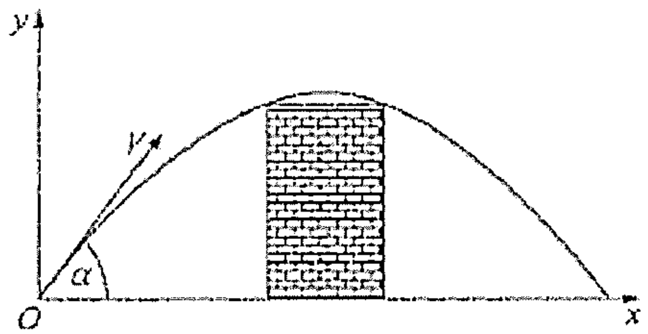
Question 14 continues on the next page

b) i) State the range of $y = \tan^{-1}\dfrac{\sqrt{x^2-4}}{2}$. **1**

ii) Find $\dfrac{dy}{dx}$ for the function $y = \tan^{-1}\dfrac{\sqrt{x^2-4}}{2}$ **2**

c) Find the volume of the solid when the region enclosed entirely by the curves $y = \sin x$ and $y = \sin 2x$ over the domain $0 \le x \le \dfrac{\pi}{2}$ is rotated about the x axis. **3**

d) A projectile is fired from the origin towards the wall of a fort with initial velocity Vms^{-1} at an angle α to the horizontal.



On its ascent, the projectile just clears one edge of the wall and on its decent it clears the other edge of the wall, as shown in the diagram.

The equations of motion of the projectile are

$x = Vt\cos\alpha$ and $y = Vt\sin\alpha - \dfrac{g}{2}t^2$ (Do not prove this.)

i) Show that the horizontal range R of the projectile is $\dfrac{V^2\sin 2\alpha}{g}$. **1**

Question 14 continues on the next page

ii) Hence, show that the equation of the path of the projectile is

$$y = x\left(1 - \frac{x}{R}\right)\tan\alpha.$$ **2**

iii) The projectile is fired at 45^0 and the wall of the fort is 10metres high. Show that the x coordinates of the edges of the wall are the roots of the equation

$x^2 - Rx + 10R = 0$. **1**

iv) If the wall of the fort is 4.5 metres thick, find the value of R. **3**

End of examination!!!

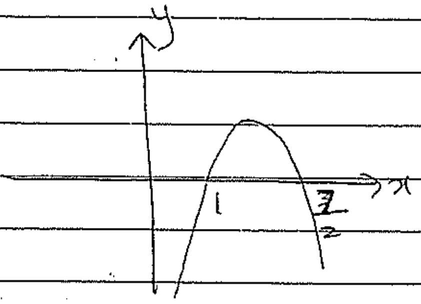
Section 1

1) D 2) C 3) C 4) A 5) B 6) C 7) D

8) C 9) D 10) A

Question 11

a) $\dfrac{5}{x-1} > 2$



$5(x-1) > 2(x-1)^2$

$5(x-1) - 2(x-1)^2 > 0$

D

C $\quad (x-1)[5-2(x-1)] > 0$

C $\quad (x-1)(7-2x) > 0$

A

B $\quad \therefore 1 < x < \dfrac{7}{2}$

C

D

C

D

A

b) $\sqrt{3}\cos\theta - \sin\theta = 1$

Let $\sqrt{3}\cos\theta - \sin\theta = R\cos(\theta+\alpha)$, $R>0$

$\therefore \sqrt{3}\cos\theta - \sin\theta = R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

equating coefficients, $\quad R\cos\alpha = \sqrt{3}$ — ①

$R\sin\alpha = 1$ — ②

$①^2 + ②^2 \Rightarrow \quad R^2 = 4 \quad \therefore R = 2 \quad (R>0)$

$\therefore \cos\alpha = \dfrac{\sqrt{3}}{2}$ and $\sin\alpha = \dfrac{1}{2}$

$\therefore \alpha$ is in 1st quadrant

$\alpha = \dfrac{\pi}{6}$

$\therefore \sqrt{3}\cos\theta - \sin\theta = 2\cos\left(\theta + \dfrac{\pi}{6}\right)$

$\cos\left(\theta + \frac{\pi}{6}\right) = \frac{1}{2}$

$\therefore \theta + \frac{\pi}{6} = \frac{\pi}{3}, \frac{5\pi}{3}$

$\therefore \theta = \frac{\pi}{6}, \frac{3\pi}{2}$

(3)

c) $u = \sin^2 x$

$\therefore du = 2\sin x \cos x \, dx = \sin 2x \, dx$

Limits: $x = \frac{\pi}{4}, \; u = \sin^2 \frac{\pi}{4} = \frac{1}{2}$

$x = \frac{\pi}{3}, \; u = \frac{3}{4}$

$\therefore \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin 2x}{1+\sin^2 x} \, dx = \int_{\frac{1}{2}}^{\frac{3}{4}} \frac{du}{1+u}$

(3)

$= \left[\ln(1+u)\right]_{\frac{1}{2}}^{\frac{3}{4}} = \ln\left(\frac{7}{4}\right) - \ln\left(\frac{3}{2}\right)$

$= \ln \frac{\frac{7}{4}}{\frac{3}{2}} = \ln \frac{7}{6}$

d) $2\times1 + 3\times2 + 4\times3 + \dots + n(n-1) = \frac{n(n^2-1)}{3}$ for $n \geq 2$

step 1 Prove true for $n = 2$

LHS: $2\times1 = 2$ | RHS: $\frac{2(4-1)}{3} = 2 = \text{LHS}$

$\therefore$ true for $n = 2$

step 2 assume true for $n = k$

$\therefore 2\times1 + 3\times2 + \dots + k(k-1) = \frac{k(k^2-1)}{3}$

step 3 Prove true for $n = k+1$

Question 11 Continued

i.e. $2\times1+3\times2+\cdots+k(k-1)+(k+1)k=\frac{(k+1)[(k+1)^2-1]}{3}$

LHS ~~$2\times1+$~~

$\underbrace{2\times1+3\times2+\cdots+k(k-1)}+k(k+1)$

$=\frac{k(k^2-1)}{3}+k(k+1)=\frac{k(k-1)(k+1)+3k(k+1)}{3}$

$=\frac{k(k+1)}{3}\left[k-1+3\right]$

$=\frac{k(k+1)(k+2)}{3}=\frac{k+1}{3}(k^2+2k)$

$=\frac{k+1}{3}\left[(k+1)^2-1\right]=$ RHS

$\therefore$ By the principle of mathematical induction, true for $n\geq2$.

e) $\left(ax-\frac{b}{x^2}\right)^5=\left(ax-bx^{-2}\right)^5$

$T_{k+1}={}^5C_k\,(ax)^{5-k}(-1)^k(bx^{-2})^k$

$={}^5C_k\,a^{5-k}\,(-1)^k\,b^k\,x^{5-k-2k}$

To find k,

For x^2: $2=5-3k\Rightarrow k=1$

For x^{-1}: $-1=5-3k\Rightarrow k=2$

$\therefore$ Coefficient of $T_2={}^5C_1\,a^4(-b)^1=-5a^4b$

Coefficient of $T_3={}^5C_2\,a^3b^2=10a^3b^2$

Since $T_2=T_3$, $-5a^4b=10a^3b^2$

$10a^3b^2+5a^4b=0\Rightarrow5a^3b(2b+a)=0$

$\therefore 2b+a=0$ as required

Question 12

a) $\frac{d}{dx}\cos^{-1}\left(\frac{x-10}{10}\right) = \frac{-1}{\sqrt{1-\left(\frac{x-10}{10}\right)^2}} \times \frac{1}{10}$

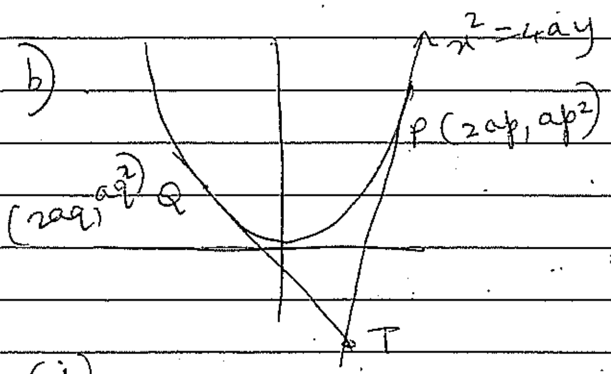
$= \frac{-1}{\sqrt{100-(x-10)^2}} = \frac{-1}{\sqrt{20x-x^2}}$ (2)

(ii) $\int_5^{10} \frac{dx}{\sqrt{20x-x^2}} = -\left[\cos^{-1}\left(\frac{x-10}{10}\right)\right]_5^{10} = -\cos^{-1}0 + \cos^{-1}\left(-\frac{1}{2}\right)$

$= -\frac{\pi}{2} + \frac{2\pi}{3}$

$= \frac{\pi}{6}$ (2)

b)



(i) Tangent at P: $y = px - ap^2$ — ①

Tangent at Q: $y = qx - aq^2$ — ②

① − ② $0 = x(p-q) - a(p^2-q^2)$

$\therefore x(p-q) - a(p-q)(p+q) = 0$

$(p-q)[x - a(p+q)] = 0$ (2)

$x = a(p+q)$ $(p \neq q)$

$\therefore y = apq$

(ii) Angle between tangents at P and Q

$\tan 45° = \left|\frac{p-q}{1+pq}\right| \Rightarrow 1 = \frac{p-q}{1+pq}$

$\therefore p - q = 1 + pq$ (2)

b) (iii) $\cancel{T = a(p+q), apq}$

$X = a(p+q)$ and $Y = apq$

$$X^2 - 4aY = a^2(p+q)^2 - 4a(apq)$$
$$= a^2[p^2+2pq+q^2-4pq]$$
$$= a^2(p^2-2pq+q^2)$$
$$= a^2(p-q)^2 = a^2(1+pq)^2$$
$$= a^2\left(1+\frac{y}{a}\right)^2 = a^2\left(1+\frac{y^2}{a^2}+\frac{2y}{a}\right)$$
$$\therefore X^2 - 4aY = a^2 + y^2 + 2ay$$
$$\therefore X^2 = a^2 + 6aY + Y^2$$

c) (i) $v^2 = 8+2x-x^2$

at extreme points, $v=0$

$\therefore 8+2x-x^2 = 0.$

$(4-x)(2+x) = 0 \quad \therefore x = -2$ and $x = 4.$

1

(ii) amplitude $= \frac{4 - -2}{2} = 3.$

(iii) $v^2 = 8+2x-x^2$

$\frac{1}{2}v^2 = 4+x-\frac{1}{2}x^2$

$\therefore \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dx}\left(4+x-\frac{x^2}{2}\right)$

$\therefore a = 1-x = -(x-1)$ 1

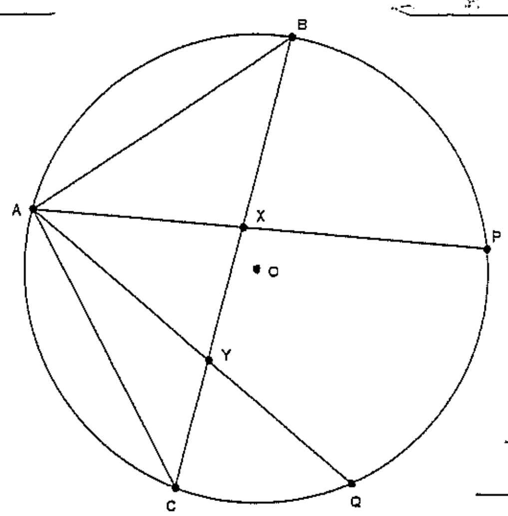
$\therefore \ddot{x} = -n^2(x-x_0)$

$n=1$

(iv) $T = \frac{2\pi}{n} = 2\pi$ seconds

Question 13

a (i)



(1)

(ii) $\angle AXC = \alpha + \beta$ (exterior $\angle$ of $\triangle ABX$ = opposite interior angles).

(1)

(iii) Construction: Join BQ, PQ

$\angle BQP = \angle BAP$ (Angles in the same segment on arc ~~PB~~)

(1)

(iv) $\angle BQA = \angle BCA$ (angles in same segment on ~~arc~~ AB)

$\angle BCA = \angle ABC$ (Angles opposite equal sides are equal)

$\therefore \angle BQA = \beta$

(1)

(v) $\angle PQA = \cancel{\alpha + \beta}\ \angle BQP + \angle BQA = \alpha + \beta$

$\angle AXC = \alpha + \beta$ (from ii)

$\angle PQA = \angle AXC$

$\therefore$ PQXY is a cyclic quadrilateral.

(external $\angle$ equals opposite interior angle)

(2)

Question 13 continued

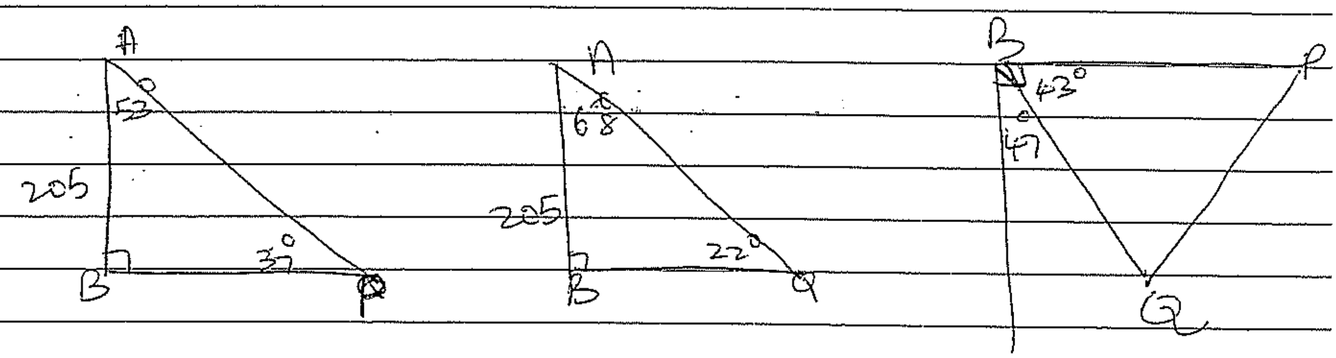
b) $P(x) = Q(x)(x^2-1) + 3x+1$

$P(x) = Q(x)(x-1)(x+1) + 3x+1$

$P(-1) = 0 - 3 + 1 = -2$

$\therefore$ remainder $= -2$ (2)

c)



$\tan 53^\circ = \frac{BP}{205} \Rightarrow BP = 205\tan 53^\circ$

$\tan 68^\circ = \frac{BQ}{205} \Rightarrow BQ = 205\tan 68^\circ$

In ΔPBQ, $PQ^2 = (205\tan 53)^2 + (205\tan 68)^2 - 2\times 205^2 \times \tan 53 \tan 68 \cos 43^\circ$

$\therefore PQ = \sqrt{\text{Answer}} = 359.9350\ldots$

$= 360$ m (nearest metre)

(3)

(d) (i) P(all choose different restaurants)

$$= \frac{4!}{4^4} = \frac{3}{32}$$

(ii) P(a person chooses restaurant A) $= \frac{1}{4}$

$\therefore$ P(exactly 2 choose A) $= {}^4C_2 \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)^2 = \frac{27}{128}$

OR

P(exactly 2 choose A) $= \frac{{}^4C_2 \times 3 \times 3}{4^4} = \frac{27}{128}$

Question 14 $\quad y = 1 + 2\sin^{-1}(2x-1)$

a) Domain: $-1 \leq 2x - 1 \leq 1$

$0 \leq 2x \leq 2 \Rightarrow 0 \leq x \leq 1$

Range: $-\frac{\pi}{2} \leq \frac{y-1}{2} \leq \frac{\pi}{2}$

$1 - \pi \leq y \leq 1 + \pi$.

$\therefore a = 1$; $b = 1 - \pi$; $c = 1 + \pi$ (2)

Question 14 continued

b) (i) Range $\{0 \le y < \frac{\pi}{2}\}$ ②

(ii) $y = \tan^{-1}\frac{\sqrt{x^2-4}}{2}$

Let $u = \frac{\sqrt{x^2-4}}{2}$ $\therefore y = \tan^{-1}u$

$\frac{dy}{du} = \frac{1}{1+u^2} = \frac{1}{1+\frac{x^2-4}{4}}$

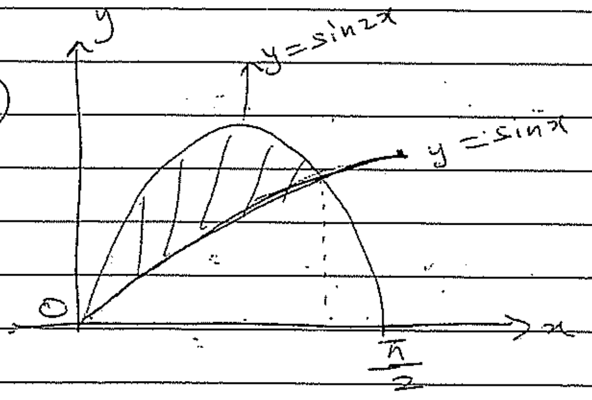
$= \frac{4}{x^2}$

$\frac{du}{dx} = \frac{1}{2}(x^2-4)^{-\frac{1}{2}} \times 2x \times \frac{1}{2}$

$= \frac{x}{2\sqrt{x^2-4}}$

$\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{4}{x^2} \times \frac{x}{2\sqrt{x^2-4}} = \frac{2}{x\sqrt{x^2-4}}$ ②

c)



$V = \pi\int_0^{\frac{\pi}{3}}(\sin^2 2x - \sin^2 x)\,dx$

$= \pi\int_0^{\pi/3} \frac{1}{2}(1-\cos 4x) - \frac{1}{2}(1-\cos 2x$

Point of intersection:

~~$\sin 2x = \sin x$~~

~~$2\sin x\cos x - \sin x = 0$~~

$\sin x(2\cos x - 1) = 0$

$\therefore x = 0$ or $x = \frac{\pi}{3}$

$= \frac{\pi}{2}\int_0^{\pi/3}(\cos 2x - \cos 4x)\,dx$

$= \frac{\pi}{2}\left[\frac{1}{2}\sin 2x - \frac{1}{4}\sin 4x\right]_0^{\pi/3}$

$= \frac{\pi}{8}\left[2\times\sin\frac{2\pi}{3} - \sin\frac{4\pi}{3}\right] - [0 - 0]$

$= \frac{\pi}{8}\left[2\times\frac{\sqrt{3}}{2} - (-\frac{\sqrt{3}}{2})\right] = \frac{3\sqrt{3}\pi}{16}$ u³

d) (i) For the horizontal range R, sub. in $y = 0$ to find the time of flight.

$$0 = t\left(v\sin\alpha - \frac{gt}{2}\right)$$

$$\therefore t = \frac{2v\sin\alpha}{g}$$

$$\therefore x = v\left(\frac{2v\sin\alpha}{g}\right)\cos\alpha = \frac{v^2 2\sin\alpha\cos\alpha}{g}$$

$$= \frac{v^2\sin 2\alpha}{g} \quad (1)$$

(ii) $x = vt\cos\alpha \Rightarrow t = \dfrac{x}{v\cos\alpha}$

sub. in $t = \dfrac{x}{v\cos\alpha}$ into $y = vt\sin\alpha - \dfrac{gt^2}{2}$

$$y = v\sin\alpha\left(\frac{x}{v\cos\alpha}\right) - \frac{g}{2}\left(\frac{x^2}{v^2\cos^2\alpha}\right)$$

$$= x\tan\alpha - \frac{x^2 g}{2v^2\cos^2\alpha} \quad \text{and} \quad R = \frac{v^2\sin 2\alpha}{g}$$

$$\therefore y = x\tan\alpha - \frac{x^2 g}{2\cos^2\alpha}\cdot\frac{\sin 2\alpha}{Rg}$$

$$= x\tan\alpha - \frac{x^2}{2\cos^2\alpha}\cdot\frac{2\sin\alpha\cos\alpha}{R}$$

$$= x\tan\alpha - \frac{x^2\tan\alpha}{R}$$

$$= x\tan\alpha\left(1 - \frac{x}{R}\right) \quad (3)$$

(iii) $\alpha = 45^\circ$; $y = 10$

$$\therefore 10 = x\left(1 - \frac{x}{R}\right)\tan 45^\circ$$

$$10R = x(R - x) \Rightarrow x^2 - Rx + 10R = 0 \quad (1)$$

Question 14 continued

d) (iv) Let x_1 and x_2 be the coordinates of the edge of the wall, with $x_1 > x_2$

$$x^2 - Rx + 10R = 0$$

$$\therefore x_1 = \frac{R + \sqrt{R^2 - 40R}}{2} \text{ and } x_2 = \frac{R - \sqrt{R^2 - 40R}}{2}$$

$x_1 - x_2 = 4.5$ (given).

$$\therefore x_1 - x_2 = \sqrt{R^2 - 40R} = 4.5$$

$$\therefore R^2 - 40R = \left(\frac{9}{2}\right)^2$$

$$4R^2 - 160R - 81 = 0$$

$$(2R + 1)(2R - 81) = 0$$

$$\therefore R = 40.5\text{m} \quad (\because R > 0)$$