2019

TRIAL

HIGHER SCHOOL CERTIFICATE

EXAMINATION, MATHEMATICS EXTENSION 1

GIRRAWEEN HIGH SCHOOL

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen.
- Board – approved calculators may be used.
- Laminated reference sheets are provided.
- Show all necessary working in Questions 11 - 15

Total marks - 70

Section 1 pages 2-3

10 marks

- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section 2 pages 4 – 9

60 marks

- Attempt Questions 11 - 15
- Allow about 1 hour 45 minutes for this section

SECTION 1(Multiple choice - 10 marks)

Colour in the appropriate circle on your multiple choice answer sheet.

1. What is the remainder when the polynomial $P(x) = 8x^3 + 4x^2 + 12x - 5$ is divided by $2x + 1$?

(A) -14 (B) -11 (C) 11 (D) 15

2. The point P divides the interval from $A(-1, 3)$ to $B(4, 8)$ externally in the ratio 3 : 2.

What are the coordinates of P?

(A) (14, 18) (B) (-14, 18) (C) (14, -18) (D) (-14, -18)

3. The coefficient of x^5 in the expansion of $\left(4x - \frac{1}{x^2}\right)^8$ is

(A) 4096 (B) -131072 (C) -4096 (D) -16384

4. Simplify: $\cos(x+y)\cos x + \sin(x+y)\sin x$

(A) $\sin y$ (B) $\sin x$ (C) $\cos y$ (D) $\cos x$

5. The function $f(x) = \log_e x^4$ has an inverse function $g(x)$. What is the gradient of the normal to $y = g(x)$ at the point (0, 1)?

(A) $\frac{1}{2}$ (B) -2 (C) -4 (D) $-\frac{1}{4}$

6. Which of the following is an expression for $\frac{d}{dx}\left(\tan^{-1}\frac{2}{x}\right)$?

(A) $-\frac{2x}{x^2+4}$ (B) $-\frac{x}{x^2+4}$ (C) $\frac{x^2}{x^2+4}$ (D) $-\frac{2}{x^2+4}$

7. What is the value of $\lim_{x \to 0} \left(\frac{\sin \frac{1}{4}x}{3x} \right)$?

(A) $\frac{1}{8}$ **(B)** $\frac{1}{6}$ **(C)** $\frac{1}{12}$ **(D)** $\frac{1}{24}$

8. From seven girls and five boys, a committee of 4 girls and 3 boys is to be chosen. How many different committees can be formed

(A) 350 **(B)** 792 **(C)** 495 **(D)** 220

9. What is the exact value of $\cos\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$?

(A) $-\frac{1}{2}$ **(B)** $\frac{1}{2}$ **(C)** $-\frac{1}{\sqrt{3}}$ **(D)** $\frac{1}{\sqrt{3}}$

10. What is the general solution of the equation $\tan 4\theta = -\frac{1}{\sqrt{3}}$?

(A) $\theta = \frac{n\pi}{4} - \frac{\pi}{24}$

(B) $\theta = \frac{n\pi}{4} + \frac{\pi}{24}$

(C) $\theta = \frac{n\pi}{4} - \frac{\pi}{12}$

(D) $\theta = \frac{n\pi}{4} + \frac{\pi}{12}$

Question 11 (12 marks) **Marks**

(a) Solve $\frac{4}{2x-1} \le \frac{2}{3}$ 3

(b) (i) Use the substitution $t = \tan\frac{x}{2}$ to show that $\frac{\sin x}{1-\cos x} = \cot\frac{x}{2}$ 2

(ii) Hence find the exact value of $\cot 22\frac{1}{2}°$ with a rational denominator. 2

(c) Use the substitution $u = \sqrt{x}$ to find $\int \frac{1}{(1+x)\sqrt{x}} dx$ 2

(d) Use the substitution $u = \sin x$ to find $\int_0^{\frac{\pi}{2}} \cos x e^{\sin x} dx$ 3

Question 12 (12 marks)

(a)) Find the acute angle between the lines $3x - y + 7 = 0$ and $x - 3y + 8 = 0$. Give your answer correct to the nearest degree. 3

(b) Consider the letters which form the word DESCARTES.

(i) How many distinct arrangements of the letters are possible? 1

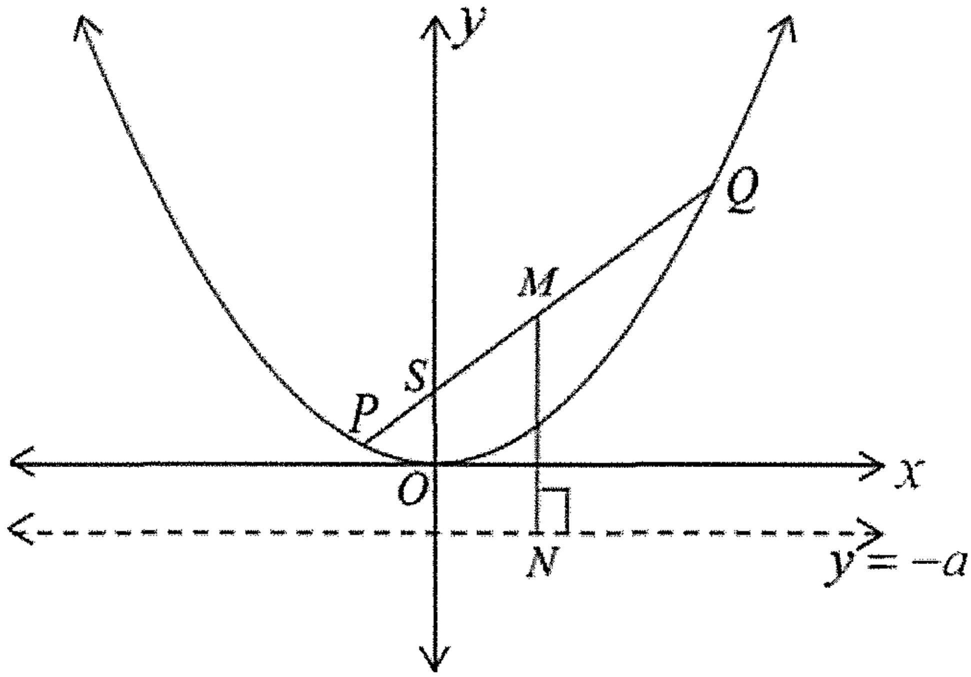
(ii) How many distinct arrangements are possible if the two E's are to be together. 1

(c) Roger and Mirka agree to play 6 sets of tennis. Based on past experience, Roger has a 0.8 probability of winning any one set played between them. What is the probability that Roger will win at least four of the six sets. 3

(d) Use mathematical induction to prove that $6^n - 5n - 1$ is divisible by 25 for all integers greater than 1. 4

Question 13 (12 marks)

(a) Let $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ be points on the parabola $x^2 = 4ay$ as shown in the diagram below.



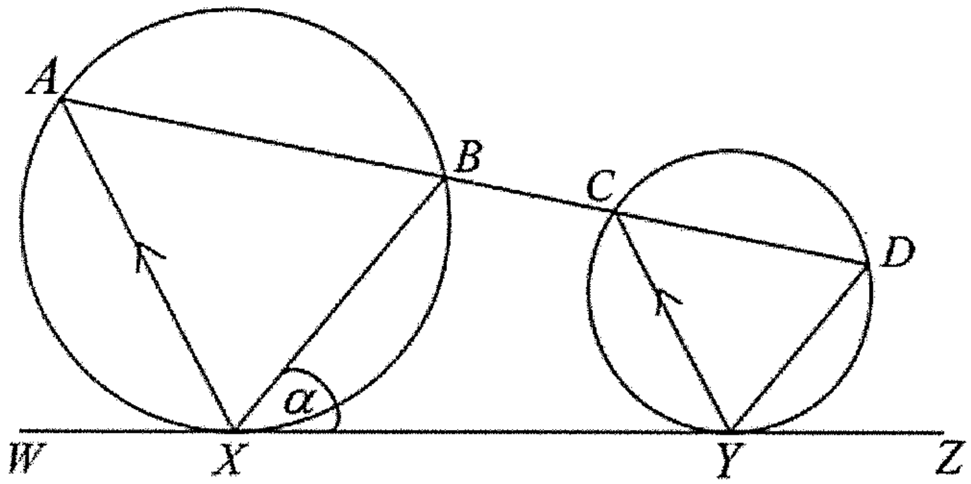
(i) Show that the equation of the chord PQ is $y = \frac{p+q}{2}x - apq$. 1

(ii) Show that if the chord PQ passes through the focus $S(0, a)$, then

$pq = -1$. 1

(iii) M is the midpoint of the focal chord PQ. N lies on the directrix such that MN is perpendicular to the directrix. T is the midpoint of MN. Find the Cartesian equation of the locus of T. 4

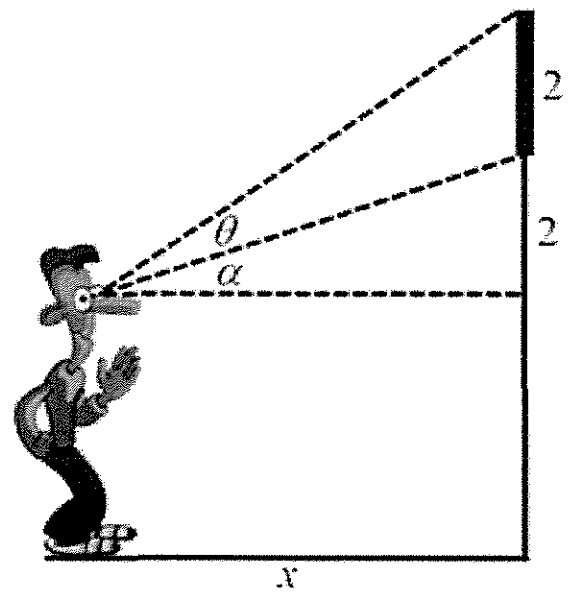
(b) In the diagram below, WZ is a common tangent to the two circles and AX is parallel to CY. AD is a straight line through B and C on the circles as shown. Let $\angle BXY = \alpha$.



(i)	Copy or trace this diagram into your writing booklet.	1
(ii)	Prove that $BX \parallel DY$.	3
(iii)	Prove that $BCYX$ is a cyclic quadrilateral.	2

Question 14 (12 marks)

(a) A person is viewing a painting hung on a wall. The vertical dimension of the painting is 2 metres and the bottom of the painting is 2 metre above the eye level of the viewer.



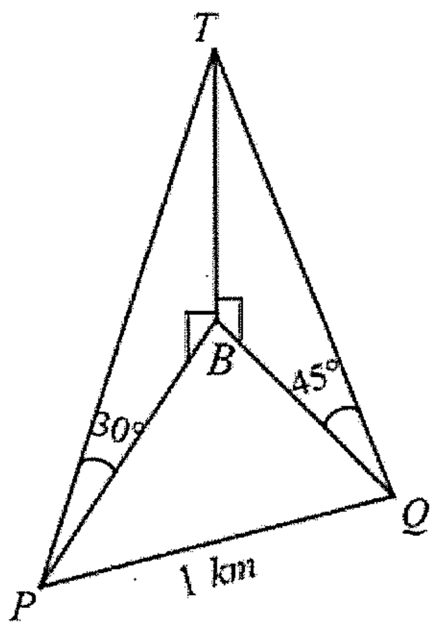
(i) Show that $\theta = \tan^{-1}\left(\frac{4}{x}\right) - \tan^{-1}\left(\frac{2}{x}\right)$ 2

(ii) Find the distance x that the viewer should stand from the wall in order to maximise the angle θ subtended by the painting. 3

(b) (i) Find $\frac{d}{dx}\sin^{-1}(e^x)$ 1

(ii) Hence evaluate $\int_{-\log_e 2}^{0} \frac{e^x}{\sqrt{1-e^{2x}}}dx$ 2

(c) The angle of elevation from a boat at P to a point T at the top of a vertical cliff is measured to be $30°$. The boat sails 1 km to a second point Q, from which the angle of elevation to T is measured to be $45°$. Let B be the point at the base of the cliff directly below T and let $h = BT$ be the height of the cliff in metres. The bearings of B from P and Q are $50°$ and $310°$ respectively.



(i) Show that $\angle PBQ = 100°$. 1

(ii) Find expressions for PB and QB in terms of h. 1

(iii) Hence show that $h^2 = \dfrac{1000^2}{\cot^2 30° + \cot^2 45° - 2\cot 30° \cot 45° \cos 100°}$ 1

(iv) Calculate the value of h, correct to the nearest metre. 1

Question 15 (12 marks)

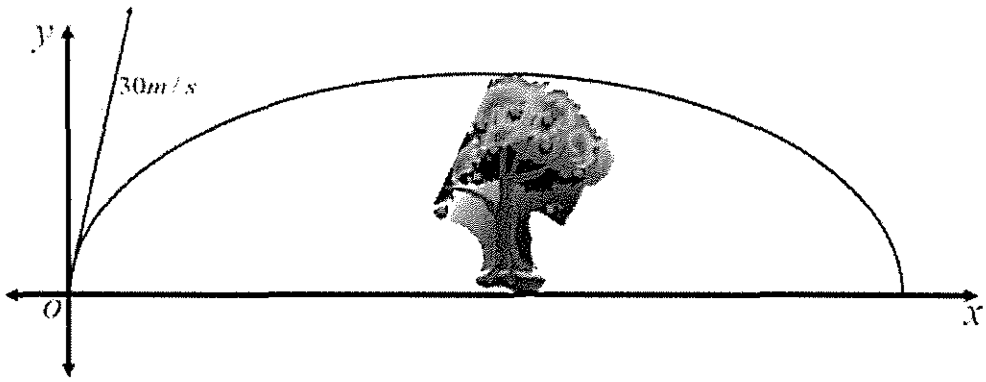
(a) A particle is moving in a straight line according to the equation $x = 10 + 8\sin 2t + 6\cos 2t$ where x is the displacement in metres and t is the time in seconds.

(i) Prove that the particle is moving in simple harmonic motion. 2

(ii) Express $8\sin 2t + 6\cos 2t$ in the form $A\sin(2t+\alpha)$ where $A > 0$ and α is an acute angle. 2

(iii) When is the displacement of the particle zero for the first time? 2

(b) A golf ball is thrown towards a tree which is 60 metres away and standing in the same horizontal plane as the ball. The tree is 20 metres high. The ball is thrown with an initial velocity of $30m/s$.



(i) Derive expressions for the horizontal displacement x metres and the vertical displacement y metres of the golf ball from O after time t seconds. Take α as the angle of projection and $g = 10m/s^2$. 3

(ii) Find the set of possible values of α, the angle of projection so that the ball clears the tree (Hint: $y > 20$) 3

END OF TEST

12 Extension 1 Trial HSC 2019 Solutions

1. $P(x) = 8x^3 + 4x^2 + 12x - 5$

$2x+1 = 2\left(x+\frac{1}{2}\right)$

remainder $= P\left(-\frac{1}{2}\right)$

$= -11$ (B)

2. $A(-1,3)$ $\quad B(4,8)$

$3 \quad -2$

$P\left(\dfrac{3\times 4 + -2\times -1}{3-2}, \dfrac{3\times 8 + -2\times 3}{3-2}\right)$

$= (14, 18)$ (A)

3. $T_{r+1} = (-1)^r\, {}^8C_r\, 4^{8-r}\, x^{8-3r}$

$8-3r = 5$

$r = 1$

$(-1)^1\, {}^8C_1\, 4^{8-1}$

$= -131072$ (B)

4. $\cos(x+y)\cos x + \sin(x+y)\sin x$

$= \cos(x+y-x) = \cos y$ (C)

5. $y = \log_e x^4$

$x = \log_e y^4$

$y^4 = e^x; \; y = e^{\frac{x}{4}}$

$y' = \frac{1}{4}e^{\frac{x}{4}}$

$y'\big|_{(0,1)} = \frac{1}{4}$

Gradient of normal $= -4$ (C)

6. $y' = \dfrac{1}{1+\frac{4}{x^2}} \times 2 \times \dfrac{-1}{x^2}$

$= \dfrac{-2}{x^2+4}$ (D)

7. $\lim\limits_{x\to 0} \dfrac{\sin\frac{1}{4}x}{3x} = \lim\limits_{x\to 0} \dfrac{\sin\frac{1}{4}x}{\frac{1}{4}x} \times \dfrac{\frac{1}{4}x}{3x}$

$= \frac{1}{12}$ (C)

8. ${}^7C_4 \times {}^5C_3 = 350$ (A)

9. $\cos\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$

$= \cos 300^\circ$

$= \cos 60^\circ = \frac{1}{2}$ (B)

10. $4\theta = n\pi + \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

$= n\pi - \frac{\pi}{6}$

$\theta = \frac{n\pi}{4} - \frac{\pi}{24}$ (A)

Question 11 (12 marks)

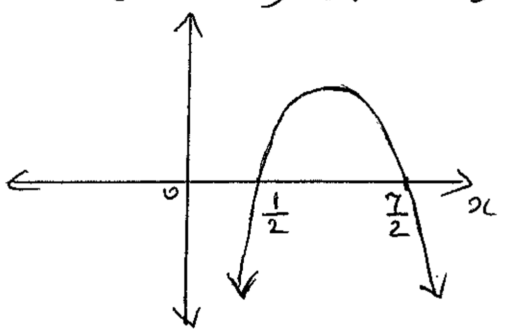
$$\frac{4}{2x-1} \le \frac{2}{3}$$

Multiply by $3(2x-1)^2$, $x \neq \frac{1}{2}$

$$12(2x-1) \le 2(2x-1)^2$$

$$12(2x-1) - 2(2x-1)^2 \le 0$$

$$2(2x-1)(7-2x) \le 0$$



(3)

$$x < \frac{1}{2} \text{ or } x \ge \frac{7}{2}$$

(b)(i) $t = \tan\frac{x}{2}$

$$\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}$$

$$\text{LHS} = \frac{\sin x}{1-\cos x}$$

$$= \frac{\frac{2t}{1+t^2}}{1 - \frac{1-t^2}{1+t^2}}$$

(2)

$$= \frac{\frac{2t}{1+t^2}}{\frac{2t^2}{1+t^2}}$$

$$= \frac{2t}{1+t^2} \times \frac{1+t^2}{2t^2}$$

$$= \frac{1}{t} = \frac{1}{\tan\frac{x}{2}} = \cot\frac{x}{2} = \text{RHS}$$

(ii) $\cot 22\frac{1}{2}^\circ = \frac{\sin 45}{1-\cos 45}$

$$= \frac{\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}} = \frac{\frac{1}{\sqrt{2}}}{\frac{\sqrt{2}-1}{\sqrt{2}}}$$

(2)

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}-1} = \frac{1}{\sqrt{2}-1} \times \frac{\sqrt{2}+1}{\sqrt{2}+1}$$

$$= \frac{\sqrt{2}+1}{2-1} = 1$$

(c) $u = \sqrt{x}$

$$\frac{du}{dx} = \frac{1}{2\sqrt{x}}$$

$$2\,du = \frac{dx}{\sqrt{x}}$$

$$I = \int \frac{2\,du}{1+u^2}$$

(2)

$$= 2\int \frac{du}{1+u^2}$$

$$= 2\tan^{-1}(u) + C$$

$$= 2\tan^{-1}(\sqrt{x}) + C$$

(d) $u = \sin x$

$\frac{du}{dx} = \cos x$

$\cos x \, dx = du$

when $x=0$, $u = \sin 0 = 0$

when $x = \frac{\pi}{2}$, $u = \sin\frac{\pi}{2} = 1$

$\underline{I} = \int_0^1 e^u \, du$ (3)

$= \left[e^u\right]_0^1$

$= e - 1$

Question 12 (12 marks)

(a) $3x - y + 7 = 0$, $x - 3y + 8 = 0$

$y = 3x + 7$ $y = \frac{1}{3}x + \frac{8}{3}$

$m_1 = 3$ $m_2 = \frac{1}{3}$

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$ (3)

$= \left|\frac{3 - \frac{1}{3}}{1 + 3 \times \frac{1}{3}}\right| = \frac{4}{3}$

$\theta = 53^\circ$

(b) $p = 0.8$, $q = 0.2$

$P(x=r) = {}^nC_r p^r q^{n-r}$

$P(x \geq 4) = P(x=4) + P(x=5) + P(x=6)$

$= {}^6C_4 (0.8)^4 (0.2)^2 + {}^6C_5 (0.8)^5 (0.2) + {}^6C_6 (0.8)^6 (0.2)^0$

$= 0.9011$ (3)

(c) (i) $\frac{9!}{2! \times 2!} = 90720$ (1)

(ii) $\frac{8!}{2!} = 20160$ (1)

(d) $n = 2$

$6^2 - 5 \times 2 - 1 = 36 - 10 - 1$
$= 25$ which is divisible by 25.

Assume that the result is true for $n = k$

ie $6^k - 5k - 1 = 25P$ —— (1)

where P is an integer

To prove that the result is true for $n = k+1$

ie $6^{k+1} - 5(k+1) - 1 = 25Q$ —— (2)

where Q is an integer

LHS of (2) $= 6^{k+1} - 5(k+1) - 1$

$= 6^k \times 6 - 5k - 5 - 1$

$= (25P + 5k + 1) \times 6 - 5k - 5 - 1$

(by assumption (1))

$= 25P \times 6 + 30k + 6 - 5k - 5 - 1$

$= 25P \times 6 + 25k$

$= 25(6P + k)$ (4)

$= 25Q$ = RHS of (2)

$\therefore$ the result is true for $n = k+1$ if it is true for $n = k$.

Hence by the principle of mathematical induction the result is true for $n > 1$.

Question 13 (12 marks)

(a) $P(2ap, ap^2)$ $Q(2aq, aq^2)$

$$m_{PQ} = \frac{ap^2 - aq^2}{2ap - 2aq}$$

$$= \frac{a(p+q)(p-q)}{2a(p-q)}$$

$$= \frac{p+q}{2}$$

Equation of PQ

$$y - ap^2 = \frac{p+q}{2}(x - 2ap)$$

$$2y - 2ap^2 = (p+q)(x - 2ap)$$

$$2y = px + qx - 2apq$$

$$y = \frac{(p+q)x}{2} - apq \quad (1)$$

(ii) Substitute $S(0, a)$

in $y = \frac{(p+q)x}{2} - apq$ (1)

$$a = 0 - apq$$

$$1 = -pq \quad \therefore pq = -1$$

(iii) $M = \left(\frac{2ap + 2aq}{2}, \frac{ap^2 + aq^2}{2}\right)$

$$= \left(ap + aq, \frac{ap^2 + aq^2}{2}\right)$$

$$N = (ap + aq, -a)$$

$$T = \left(\frac{2ap + 2aq}{2}, \frac{\frac{ap^2 + aq^2}{2} - a}{2}\right)$$

$$= \left(ap + aq, \frac{ap^2 + aq^2 - 2a}{4}\right)$$

$x = a(p+q)$

$$y = \frac{a(p^2+q^2-2)}{4}$$

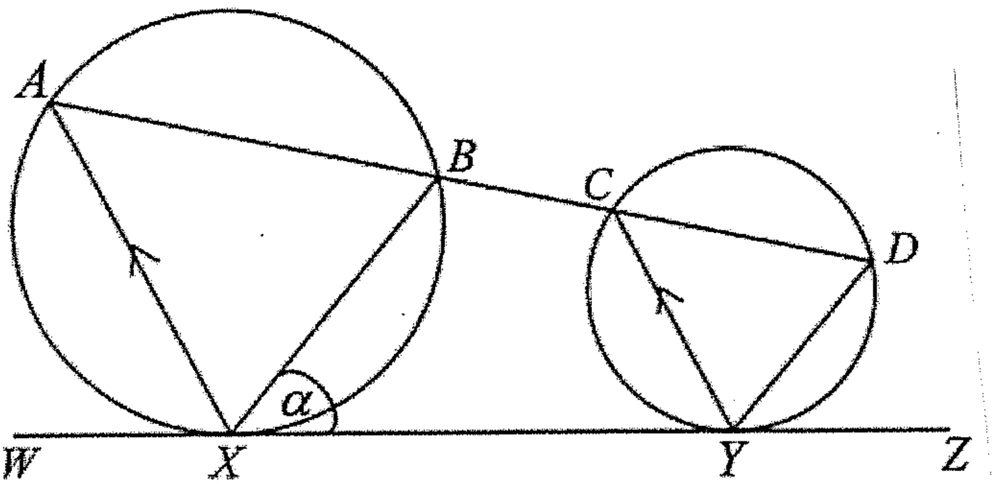
$$= \frac{a(p^2+q^2+2pq)}{4}$$

$$= \frac{a(p+q)^2}{4} \qquad 4$$

$$= \frac{a\left(\frac{x}{a}\right)^2}{4}$$

$$y = \frac{x^2}{4a}; \quad \underline{\underline{x^2 = 4ay}}$$

(b) (i)



1

(ii) Given $\angle BXY = \alpha$

$\angle BAX = \alpha$ (angle between tangent and chord at the point of contact is equal to the angle in the alternate segment)

$\angle DCY = \alpha$ (corresponding angles, $AX \parallel CY$)

$\angle DYZ = \alpha$ (angle between tangent and chord at the point of contact is equal to the angle in the alternate segment) (3)

$\angle DYZ = \angle BXY$

$\therefore$ $BX \parallel DY$ (corresponding angles are equal.

(iii) $\angle BCY = 180 - \alpha$ (BCD is a straight line)

$\angle BCY + \angle BXY = 180^\circ$

$\therefore$ $BCYX$ is a cyclic quadrilateral as one pair of opposite angles are supplementary. (2)

Question 14 (12 marks)

(a) (i) $\tan(\theta + \alpha) = \dfrac{4}{x}$

$\therefore \theta + \alpha = \tan^{-1}\left(\dfrac{4}{x}\right)$

$\tan \alpha = \dfrac{2}{x}$

$\alpha = \tan^{-1}\left(\dfrac{2}{x}\right)$ (2)

$\theta = \theta + \alpha - \alpha$

$= \tan^{-1}\left(\dfrac{4}{x}\right) - \tan^{-1}\left(\dfrac{2}{x}\right)$

(ii) $\dfrac{d\theta}{dx} = \dfrac{x^2}{x^2+16} \times \dfrac{-4}{x^2} + \dfrac{x^2}{x^2+4} \times \dfrac{2}{x^2}$

$= \dfrac{2}{x^2+4} - \dfrac{4}{x^2+16}$

For stationary points $\dfrac{d\theta}{dx} = 0$

$\dfrac{2(x^2+16) - 4(x^2+4)}{(x^2+4)(x^2+16)} = 0$

$-2x^2 + 16 = 0$

$x^2 = 8$

$x = \pm\sqrt{8}$ (3)

$x = \sqrt{8}$ (since $x > 0$)

$= 2\sqrt{2}$ m.

x	2	$2\sqrt{2}$	3
$\dfrac{d\theta}{dx}$	$\dfrac{1}{20}$	0	-6.154×10^{-3}

θ is maximum when $x = 2\sqrt{2}$ m

(b) (i) $\frac{d}{dx}\sin^{-1}(e^x)$

$= \frac{1}{\sqrt{1-e^{2x}}} \times e^x = \frac{e^x}{\sqrt{1-e^{2x}}}$ (1)

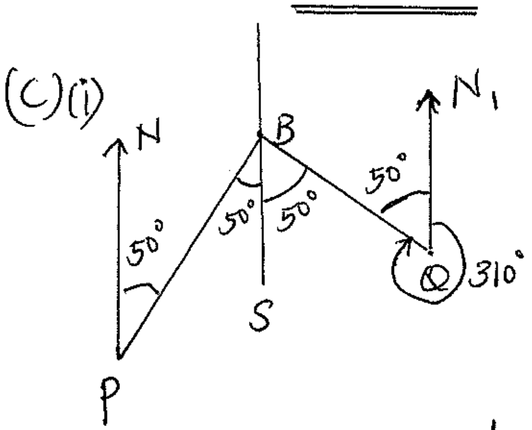
(ii) $\int_{-\log_e 2}^{0} \frac{e^x}{\sqrt{1-e^{2x}}}dx = \int_{-\log_e 2}^{0} \sin^{-1}(e^x)dx$

$= \sin^{-1}(e^0) - \sin^{-1}(e^{-\log_e 2})$

$= \sin^{-1}(1) - \sin^{-1}(\frac{1}{2})$ (2)

$= \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$

(c)(i)



$\angle PBS = 50°$ (alternate $\angle s$, PN ∥ BS)

$\angle BQN_1 = 360 - 310$
$= 50$ (angles at a point)

$\angle QBS = 50°$ (alternate $\angle s$, BS ∥ QN₁)

$\angle PBQ = \angle PBS + \angle QBS$ (1)
$= 50 + 50 = 100$
(adjacent $\angle s$)

(ii) $\cot 30 = \frac{PB}{TB}$

$PB = TB \times \cot 30$
$= h \cot 30$

$\cot 45 = \frac{QB}{TB}$ (1)

$QB = TB \times \cot 45$
$= h \cot 45$

(iii) Apply cosine rule in ΔPBQ

$PQ^2 = PB^2 + QB^2 - 2 \times PB \times QB \times \cos \angle PBQ$

$1000^2 = h^2\cot^2 30 + h^2\cot^2 45 - 2h\cot 30 \times h\cot 45 \cos 100$

$= h^2(\cot^2 30 + \cot^2 45 - 2\cot 30 \cot 45 \cos 100)$

$h^2 = \frac{1000^2}{\cot^2 30 + \cot^2 45 - 2\cot 30 \times \cot 45 \cos 100}$ (1)

(iv) $h = 466\,m$ (1)

Question 15 (12 marks)

(a) $x = 10 + 8\sin 2t + 6\cos 2t$

$\dot{x} = 16\cos 2t - 12\sin 2t$

$\ddot{x} = -32\sin 2t - 24\cos 2t$

$= -4(8\sin 2t + 6\cos 2t)$

$= -4(x - 10)$ (2)

which is of the form

$\ddot{x} = -n^2(x - x_0)$

$\therefore$ the particle is moving in SHM.

(ii) Let $8\sin 2t + 6\cos 2t = A\sin(2t + \alpha)$

$= A\sin 2t\cos\alpha + A\cos 2t\sin\alpha$

Equating coefficients of $\sin 2t$ and $\cos 2t$ we get

$8 = A\cos\alpha$

$6 = A\sin\alpha$

$A^2 = 100$ (2)

$A = \pm 10$

$A = 10$ (since $A > 0$)

$\sin\alpha = \frac{3}{5}$; $\cos\alpha = \frac{8}{10}$

$\alpha = 0.6435^c$

$8\sin 2t + 6\cos 2t$

$= 10\sin(2t + 0.6435)$

(iii) $x = 0$

$8\sin 2t + 6\cos 2t + 10 = 0$

$10\sin(2t + 0.6435) = -10$

$\sin(2t + 0.6435) = -1$

$2t + 0.6435 = \frac{3\pi}{2}$ (2)

$t = 2.034$ seconds

(b) (i) $\ddot{x} = 0$ $\quad \ddot{y} = -10$

$\dot{x} = A$ $\quad \dot{y} = -10t + B$

where A and B are constants

when $t = 0$, $\dot{x} = V\cos\alpha = 30\cos\alpha$ $\quad \dot{y} = V\sin\alpha = 30\sin\alpha$

$30\cos\alpha = A$ $\quad 30\sin\alpha = 0 + B$

$\dot{x} = 30\cos\alpha$ $\quad \dot{y} = -10t + 30\sin\alpha$

$x = (30\cos\alpha)t + C$

$y = -5t^2 + (30\sin\alpha)t + D$

when $t = 0$, $x = 0$ and $y = 0$

$0 = 0 + C \quad \therefore C = 0$

$0 = 0 + 0 + D \quad \therefore D = 0$ (3)

$x = (30\cos\alpha)t$

$y = -5t^2 + (30\sin\alpha)t$

(ii) For the ball to clear the tree $x = 60$ and $y > 20$

$$60 = (30\cos\alpha)t \quad \text{— (1)}$$

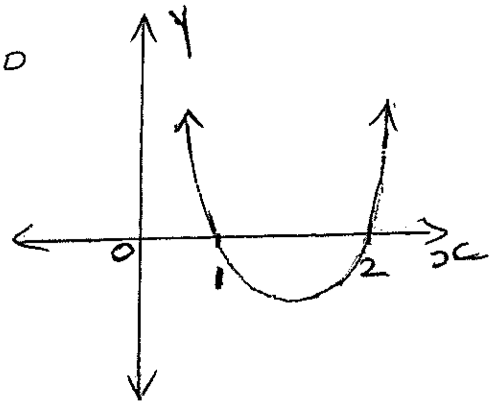
$$t = \frac{60}{30\cos\alpha} = \frac{2}{\cos\alpha}$$

Substitute in $y > 0$

$$-5 \times \frac{4}{\cos^2\alpha} + (30\sin\alpha) \times \frac{2}{\cos\alpha} > 20 \qquad (3)$$

$$-20\sec^2\alpha + 60\tan\alpha > 20$$

$$20\tan^2\alpha - 60\tan\alpha + 40 < 0$$



$$\tan^2\alpha - 3\tan\alpha + 2 < 0$$

$$(\tan\alpha - 2)(\tan\alpha - 1) < 0$$

$$1 < \tan\alpha < 2$$

$$\underline{\underline{45^\circ < \alpha < 63^\circ 26'}}$$