GOSFORD HIGH SCHOOL

MATHEMATICS - 3 UNIT

1997 TRIAL EXAMINATION YEAR 12

TIME ALLOWED : TWO HOURS

(5 minutes reading time is allowed)

NAME..

DIRECTIONS TO CANDIDATES

- *ALL* questions may be attempted
- *ALL* questions are of equal value
- All necessary working should be shown in every question. Marks may not be awarded for careless or badly arranged work.
- Approved silent calculators may be used
- *Each* question attempted is to be returned on a *separate sheet of paper* clearly marked with your name and the question number

QUESTION	MARK
1	
2	
3	
4	
5	
6	
7	
TOTAL	

Question 1

a) Find x and y if $\frac{4^x}{16} = 8^{x+y}$ and $2^{2x+y} = 128$ (2)

b) Solve for x

$$\frac{x^2 - 9}{2x} \geq 0$$ (2)

c) $\lim_{x \to 0} \frac{4 \sin^2 x}{x^2}$ (2)

d) Use the table of standard integrals to show (2)

$$\int_0^{\frac{\pi}{6}} \sec 2x \tan 2x \, dx = \frac{1}{2}$$

e) Show that the condition for 2 of the roots of $x^3 + px^2 + qx + r = 0$ to differ only by sign is that $pq - r = 0$ (2)

f) Taking $x = 1.5$ as the first approximation use Newton's method to find a second approximation to the root of $1 - x + \sin x = 0$ to 2 decimal places (2)

Question 2 (Answer on a new page)

a) i) Evaluate $\int_{0}^{\frac{\pi}{6}} \sin^2 x \, dx$ (2)

ii) Use the substitution $x = \cos 2\theta$ (4)

$$\int_{\frac{1}{2}}^{1} \sqrt{\frac{1-x}{1+x}} \, dx$$

b) If $a > 0$ prove $e^{\log a} = a$ and hence determine (3)

$$\int_{-\log 2}^{\log 2} e^{x} \, dx$$

(Note all logs are to base e)

c) A sector of a circle with centre O and radius r is bounded by radii OP, OQ and the arc PQ. The angle POQ is θ radians

i) given r and θ vary such that the area of the sector POQ is a constant 100cm^2 show $\theta = \frac{200}{r^2}$ (1)

ii) Given that the radius is also increasing at a constant rate of 0.5cm/sec find the rate at which the angle POQ is decreasing when r = 10cm. (2)

Question 3 (Answer on a new page)

a) i) show that $\frac{d}{dt}(\sin^3 t) = 3\cos t - 3\cos^3 t$ (4)

ii) Hence $\int_0^{\frac{\pi}{4}} \cos^3 t \, dt$ (leave answer in surd form)

b) $S_n = \frac{1}{1 \times 3} + \frac{1}{3 \times 5} + \dots\dots\dots + \frac{1}{(2n-1)(2n+1)}$ (5)
for all positive integers n

i) Prove by induction that $S_n = \frac{1}{2}\left(1 - \frac{1}{2n+1}\right)$

ii) What is $\lim_{n \to \infty} S_n$?

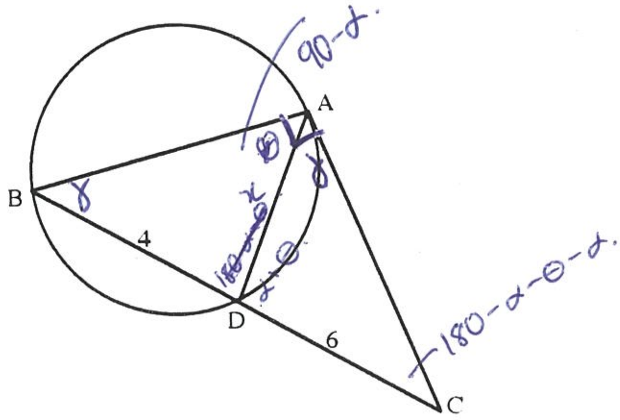
c) A is (2, 2) and B (-1, -2). The interval AB meets the line $2x + 3y = 4$ at C. Find the ratio AC:CB (3)

Question 4 (Answer on a new page)

a) In the diagram AC is a tangent to the circle and $\angle$ CAB is a right angle.

i) Show ΔABD is similar to ΔCAD

ii) Calculate AD and the radius of the circle. (3)



b) A particle moves in such a way that its displacement x metres from the origin O is given by $x = \sqrt{3}\cos 3t - \sin 3t$

i) Show that the equation can be expressed as (5)

$x = 2\cos\left(3t + \frac{\pi}{6}\right)$

ii) Show that the motion of the particle is simple harmonic. Hence find the period and amplitude.

iii) Find the time when the particle first passes through the origin.

c) i) Show that the function $f(x) = \frac{x-4}{x-2}$ $x \neq 2$ (4)
is an increasing function for all values of x in its domain.

ii) Sketch the function showing any points of intersection with the co-ordinate axes and also the equations of any asymptotes.

iii) Find the inverse function $f^{-1}(x)$ and state its range.

Question 5 (Answer on a new page)

a) i) Prove that the expression for acceleration is (8)

$$\frac{dv}{dt} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

ii) the acceleration of a particle which moves back and forth in a straight line is given by

$$\ddot{x} = -9x$$

where x is its displacement from 0.
Initially it is at rest 2cm to the positive side of 0.

1) Show that its speed is given by $3\sqrt{4 - x^2}$

2) Determine the equation for its displacement in terms of t

3) Show that the particle returns to any particular position and travelling with the same velocity in the same direction after every $\frac{2\pi}{3}$ seconds.

b) A machine produces a computer component. 95% of components are found satisfactory.For a sample of 12 components find the probability that (4)

i) exactly 2 are unsatisfactory

ii) at least 2 are unsatisfactory

Question 6 (Answer on a new page)

a) The straight line $y = mx + b$ meets the parabola (8)
$x = 2At$ $Y = At^2$ in real distinct points P, Q, which correspond respectively to the values $t = p$, $t = q$

i) prove $pq = \frac{-b}{A}$

ii) prove that $p^2 + q^2 = 4m^2 + \frac{2b}{A}$

iii) show the equation of the normal to the parabola at P is

$$x + py = 2\,Ap + Ap^3$$

iv) The point N is the point of intersection of the normals to the parabola at P and Q. Prove that the co-ordinates of N are $(-Apq\,(p + q),\ A\,(2 + p^2 + pq + q^2)$ and express these co-ordinates in terms of A, m and b.

b) Given $x\,(1 + x)^n = \sum_{r=0}^{n} \binom{n}{r} x^{r+1}$

show by differentiation that

$$\binom{n}{0} + 2\binom{n}{1} + 3\binom{n}{2} \ldots + (n+1)\binom{n}{n} = (n+2)\,2^{n-1}$$ (4)

Question 7 (Answer on a new page)

a) (6)

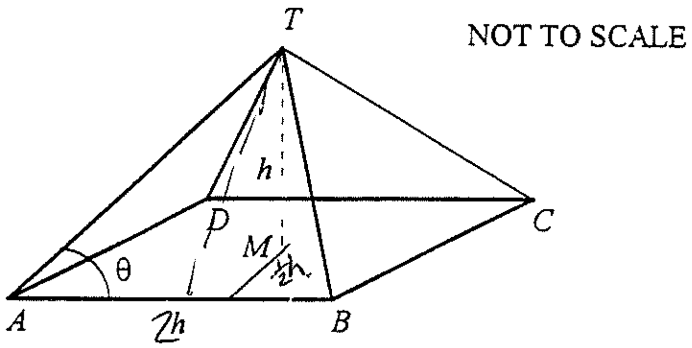
A person stands on the roof of a tall building 45m above the ground and shoots an arrow in a horizontal direction with an initial speed of 30m/sec. Using $g = 10\text{m/sec}^2$

i) Find the equations of motion of the arrow and calculate how long it takes for the arrow to strike the ground?

ii) How far from the base of the building does the arrow strike the ground?

iii) Find the size of the acute angle between the path of the arrow and the horizontal at the moment the arrow strikes the ground.

b) (3)



The diagram shows a right square pyramid with base ABCD, vertex T and altitude TM. TM = AB = h

Show that if $\angle TAB = \theta$ then $\cos\theta = \frac{1}{\sqrt{6}}$

c) In a colony of 2000 birds the number N infected with a disease at time t is given by $N = \frac{2000}{1 + Ke^{-2000t}}$ where K is a constant and t is in years. (3)

i) Show that eventually all the birds will be infected.

ii) If at $t = 0$ one bird is infected, after how many hours will 1000 birds be infected. (2 dp).(Assume 365.25 days in a year).