GOSFORD HIGH SCHOOL

2011
TRIAL HSC EXAMINATION

EXTENSION 1 MATHEMATICS

General Instructions:

- Reading time: 5minutes.
- Working time: 2 hours
- Write using black or blue pen.
- Board-approved calculators may be used.
- Each question should be started on a separate writing booklet.
- All necessary working should be shown in every question.

Total marks: - 84

Attempt all Questions 1- 7.

Question 1 **Start a SEPARATE BOOKLET** **Marks**

a) Evaluate $\lim_{x\to 0} \frac{5\sin 2x}{4x}$ 1

b) Solve $\frac{x^2-3}{2x} > 0$ 3

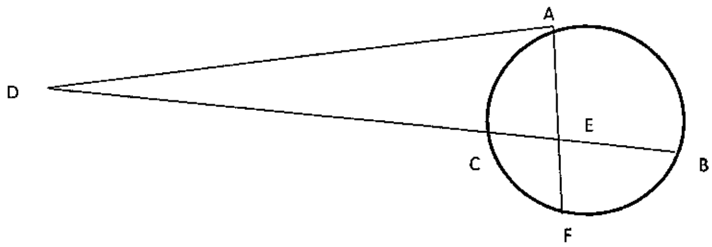
c) Evaluate $\int_0^1 \frac{dx}{\sqrt{4-x^2}}$ 2

d) Find $\int x\sqrt{x^2+1}\,dx$, using the substitution $u = x^2 + 1$ 3

e) Differentiate $\log_e\left(\frac{e^x+1}{e^x-1}\right)$ with respect to x 3

Question 2 **Start a SEPARATE BOOKLET** **Marks**

a) In the diagram, DA is a tangent to the circle and DCB is a straight line, cutting the circle in C and B. The point E is taken on CB so that DA = DE and AE produced, meets the circle at F.



(i) Copy the diagram into your answer booklet

(ii) Prove that AE bisects $\angle BAC$ 3
(hint: let $\angle DAC = \alpha$ & $\angle CAE = \beta$)

b) (i) Show that $\frac{d}{dx}\left(\frac{v^2}{2}\right) = \frac{dv}{dt}$ 2

(ii) When x metres from the origin, the velocity, v ms^{-1}, of a particle which moves along a straight line is given by

$$v^2 = 6(16 - x^4).$$

Find its acceleration when it is at $x = \frac{1}{2}$ 2

Question 2 continued

c) P, Q and R are the points (-5,12), (4, 9) and (0,2) respectively. X divides the interval PQ externally in the ratio 5:2
Prove that *angle* PRX is a right angle. **2**

d) Evaluate $\int_0^{\pi} cos^2 2x\, dx$ **3**

Question 3 **Start a SEPARATE BOOKLET** **Marks**

a) The area under the curve $y = \sin x + \cos x$, above the x axis and between $x = 0$ and $x = \frac{\pi}{2}$, is rotated about the x-axis. Find the volume of the solid of revolution formed. **4**

b) A disintegrating comet called Z, which is always spherical in shape, is decreasing in volume at a constant rate of $8m^3$/min. Find the rate at which the surface area is changing when the radius is 4m. **4**

c) α, β, γ are roots of the polynomial equation $2x^3 - 4x^2 + 5x - 3 = 0$. Find:

(i) $\alpha + \beta + \gamma$ **1**

(ii) $\alpha\beta\gamma$ **1**

(iii) $\frac{1}{\alpha\beta} + \frac{1}{\alpha\gamma} + \frac{1}{\beta\gamma}$ **2**

Question 4 **Start a SEPARATE BOOKLET** **Marks**

a) The line $y = mx$ makes an angle of 45^0 with the line $y = 2x - 3$.
Find the two possible values of *m*. **3**

b) The polynomial $P(x) = (x - a)^3 + b$ has a value zero at $x = 1$, and, when divided by x, the remainder is -7.
Find all possible values of a and b. **5**

c) Use the method of mathematical induction to prove that, for all positive integral values of n for $n \geq 1$,

$$1 \times 1! + 2 \times 2! + 3 \times 3! + \ldots + n \times n! = (n + 1)! - 1$$ **4**

Question 5 **Start a SEPARATE BOOKLET** **Marks**

a) Sketch the graph of the function $y = 3sin^{-1}(\frac{x}{2})$, stating clearly the domain and range. **3**

b) (i) Given that $x^2 + 4x + 5 \equiv (x+a)^2 + b^2$ find the values for a and b. **2**

(ii) Using the result from part (i) above, find; $\int \frac{dx}{x^2+4x+5}$ **2**

c) A particle moves in a straight line and at time *t* seconds, its distance is x cms from a fixed origin point, O. The equation of the line is given by :

$$x = 1 + \frac{1}{2}cos2t$$

(i) Show the motion of the particle is in Simple Harmonic Motion **1**

(ii) State the period of motion for the particle. **1**

(iii) Sketch the graph of x as a function of *t* in the domain $0 \le t \le 2\pi$ **1**

(iv) Find the displacement of the particle when it is at rest and thus determine the length of its path. **2**

Question 6 **Start a SEPARATE BOOKLET** **Marks**

a) (i) Assuming that $\cos x \neq 0$,make *tan* x the subject of $\sin(x + \theta) = a\cos x$ **3**

(ii) Find the exact value of *tan* x when $\sin\left(x + \frac{\pi}{3}\right) = 2\cos x$, and the values of *x*, for $0 \le x \le 2\pi$, correct to 4 decimal places. **2**

b) (i) Derive the equation of the tangent to the parabola $x^2 = 4ay$ at the point P(2ap, ap^2). **2**

(ii) The tangent meets the line x = *a* at Q. Find the co-ordinates of Q. **2**

(ii) M is the mid-point of PQ. Prove that, as P moves on the parabola, M moves on a straight line. **3**

Question 7 **Start a SEPARATE BOOKLET** **Marks**

READ IT AS $f(x) = \sin(\frac{x}{10}) - e^{-x}$

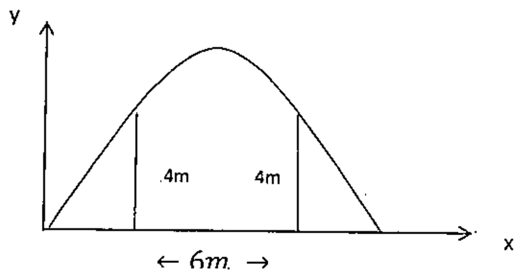
a) Find, correct to 2 decimal places, the root of $e^{-x} = \sin(\frac{x}{10})$, using one application of Newton's Method, and taking $x = 2$ as the first approximation. **3**

b) A ball is projected from a point O with speed V m / sec and at an angle α to the horizontal. Air resistance is ignored and g m / sec^2 is the acceleration due to gravity.

(i) Derive the expressions for the horizontal component x (t) and the vertical component y (t) of the ball's displacement after t seconds (neglect air resistance). **2**

(ii) If R is the range on the horizontal plane of this projectile, show that the cartesian equation of the path can be given by: $y = x\left(1 - \frac{x}{R}\right)\tan\alpha$ **4**

(iii)



If $\alpha = 45^0$ and the ball just clears two vertical posts, which are both 4 metres above the level of projection and 6 metres apart, calculate the range, R. **3**

END OF EXAM ☺

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

ANSWERS

2011 Ext one Trial

Q1 a) $\lim\limits_{x \to 0} \dfrac{5\sin 2x}{4x} = 5 \times \dfrac{2}{4}$

$= 2\frac{1}{2}$ (1)

b) $\dfrac{x^2-3}{2x} > 0$

Consider

$x > 0$	$x < 0$ (1)
$x^2-3>0$	$x^2-3<0$
$\therefore x < -\sqrt{3}$ or $x > \sqrt{3}$	$-\sqrt{3} < x < \sqrt{3}$ (1)
since $x>0$	since $x<0$
$x > \sqrt{3}$	$-\sqrt{3} < x < 0$ (1)

c) $\displaystyle\int_0^1 \frac{dx}{\sqrt{4-x^2}} = \left[\sin^{-1}\left(\frac{x}{2}\right)\right]_0^1$ (1)

$= \sin^{-1}\frac{1}{2} - \sin^{-1} 0$

$= \frac{\pi}{6} - 0$

$= \pi/6$ (1)

d) let $u = x^2+1$

$du = 2x$ (1)

$\therefore \displaystyle\int x\sqrt{x^2+1}\,dx = \frac{1}{2}\int \sqrt{u}\,du$ (1)

$= \frac{1}{2} \times \frac{2}{3} u^{\frac{3}{2}} + c$

$= \frac{1}{3} u^{\frac{3}{2}} + c$

$= \frac{1}{3}(x^2+1)^{\frac{3}{2}} + c$ (1)

e) ↗

e) $y = \log_e\left(\dfrac{e^x+1}{e^x-1}\right)$

$y = \log_e(e^x+1) - \log_e(e^x-1)$

$\dfrac{dy}{dx} = \dfrac{e^x}{e^x+1} - \dfrac{e^x}{e^x-1}$ (1)

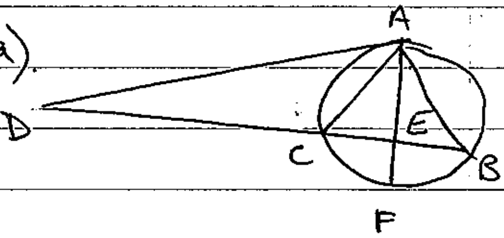
$= \dfrac{e^x(e^x-1) - e^x(e^x+1)}{(e^x+1)(e^x-1)}$ (1)

$= \dfrac{e^{2x} - e^x - e^{2x} - e^x}{e^{2x}-1}$

$= \dfrac{-2e^x}{e^{2x}-1}$ (1)

Question 2

a)



ii) let $\angle ABE = \alpha$,

$\angle BAE = \beta$

(1)

$\therefore \angle AEC = (\alpha+\beta)$ (ext $\angle$ at ΔAE

$\therefore \angle EAD = (\alpha+\beta)$ (isos ΔAED)

But $\angle DAC = \angle ABE$

$= \alpha$ (angle in alt seg) (1)

$\therefore \angle EAC = \beta$

$\therefore \angle BAE = \angle EAC$

$= \beta$

$\therefore AE$ Bisects $\angle BAC$

Q2 b

(i) $\frac{d}{dx}\left(\frac{v^2}{2}\right) = \frac{d\left(\frac{v^2}{2}\right)}{dv} \times \frac{dv}{dx}$ (1)

$= v \times \frac{dv}{dx}$

$= \frac{dx}{dt} \times \frac{dv}{dx}$ (1)

$= \frac{dv}{dt}$

$\therefore \frac{d}{dx}\left(\frac{v^2}{2}\right) = \frac{dv}{dt}$

(ii) $v^2 = 6(16 - x^4)$

$a = \frac{dv}{dt}$

$= \frac{d}{dx}\left(\frac{v^2}{2}\right)$

$= \frac{d}{dx}\left(3(16-x^4)\right)$ (1)

$= -12x^3$

Acceleration at $x = \frac{1}{2}$

$a = -12\left(\frac{1}{2}\right)^3$

$= -1\frac{1}{2}\,ms^{-2}$ (1)

c Co-ordinate of x $\left(\frac{-5\times2+4\times5}{5-2}, \frac{12(-2)+9\times5}{5-2}\right)$

$= (10, 7)$ (1)

Gradient of $PR = \frac{2-12}{0+5}$ grad of $RX = \frac{7-2}{10-0}$

$= -2$ $\qquad = \frac{1}{2}$

$\therefore M_{PR} \times M_{RX} = -2 \times \frac{1}{2}$

$= -1$ (1)

$\therefore \angle PRX$ is a right angle

d (1)

$\int_0^{\pi} \cos^2 2x\, dx$ $\qquad \cos 2x = \cos^2 x + \sin^2$

$= 2\cos^2 x - 1$

$= \frac{1}{2}\int_0^{\pi}(\cos 4x + 1)\,dx$ $\quad \therefore \cos^2 x = \frac{\cos 2x + 1}{2}$

hence $\cos^2 2x = \frac{\cos 4x + 1}{2}$

$= \frac{1}{2}\left[\frac{\sin 4x}{4} + x\right]_0^{\pi}$ (1)

$= \left[\frac{\sin 4x}{8} + \frac{x}{2}\right]_0^{\pi}$

$= \frac{\sin 4\pi}{8} + \frac{\pi}{2} - \frac{\sin 4 \times 0}{2} - 0$

$= 0 + \frac{\pi}{2} - 0$

$= \frac{\pi}{2}$ (1)

Question 3

a) $V = \int_0^{\frac{\pi}{2}} \pi y^2\, dx$

$= \pi\int_0^{\frac{\pi}{2}} (\sin x + \cos x)^2\, dx$ (1)

$= \pi\int_0^{\frac{\pi}{2}} (\sin^2 x + 2\sin x\cos x + \cos^2 x)\,dx$

$= \pi\int_0^{\frac{\pi}{2}} (1 + 2\sin x\cos x)\, dx$

$= \pi\int_0^{\frac{\pi}{2}} (1 + \sin 2x)\, dx$

$= \pi\left[x - \frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{2}}$ (1)

$= \pi\left[\frac{\pi}{2} - \frac{1}{2}\cos\pi - 0 - \frac{1}{2}\cos 0\right]$

$= \pi\left(\frac{\pi}{2} + \frac{1}{2} + \frac{1}{2}\right)$

$= \pi\left(\frac{\pi}{2} + 1\right) u^3$ (1)

Q3 Contd.

b) $V=\frac{4}{3}\pi R^3$

$\frac{dV}{dt}=\frac{dV}{dr}\times\frac{dR}{dt}$ (1)

$8=4\pi R^2\frac{dR}{dt}$

$\frac{dR}{dt}=\frac{2}{\pi R^2}$ (1)

Now $SA=4\pi R^3$

$\frac{ds}{dt}=\frac{ds}{dR}\times\frac{dR}{dt}$

$=8\pi R\times\frac{2}{\pi R^2}$ (1)

$=16/R$

when $R=4$

$\frac{ds}{dt}=\frac{16}{4}$

$=4$ m/min (1)

c) $2x^3-4x^2+5x-3=0$

i) $\alpha+\beta+\gamma=\frac{-b}{a}$

$=4/2$

$=2$ (1)

ii) $\alpha\beta+\alpha\gamma+\beta\gamma=\frac{c}{a}$

$=\frac{5}{2}$ (1)

iii) $\frac{1}{\alpha\beta}+\frac{1}{\alpha\gamma}+\frac{1}{\beta\gamma}=\frac{\alpha+\beta+\gamma}{\alpha\beta\gamma}$ (1)

$=\frac{2}{3/2}$

$=\frac{4}{3}$ (1)

Question 4.

a) $y=mx$ has gradient m

$y=2x-3$ has gradient of 2

$\tan\theta=\left|\frac{m-2}{1+2m}\right|$

$=\tan 45°$

$=1$

$\therefore\left|\frac{m-2}{1+2m}\right|=1$ (1)

$\frac{m-2}{1+2m}=1$ $\quad$ $\frac{m-2}{1+2m}=-1$ (1)

$m-2=1+2m$ $\quad$ $m-2=-1-2m$

$m=-3$ $\quad$ $m=\frac{1}{3}$ (1)

b) $P(x)=(x-a)^3+b$

$P(1)=0$ $\therefore(1-a)^3+b=0$ $\boxed{1}$ (1)

$P(0)=-7$ $\therefore(0-a)^3+b=-7$ (1)

$-a^3+b=-7$ $\boxed{2}$

$\boxed{1}-\boxed{2}$

$(1-a)^3+a^3=7$

$1-3a+3a^2-a^3+a^3=7$ (1)

$3a^2-3a-6=0$

$a^2-a-2=0$

$(a-2)(a+1)=0$

$a=2,\ a=-1$ (1)

$\therefore b=1,\ b=-8$ (1)

Q4 contd

c) $S_n = 1\times 1! + 2\times 2! + 3\times 3! + \dots + n\times n!$

$= (n+1)! - 1$

1, when $n=1$ $\quad S_1 = 1\times 1!$

$= (1+1)! - 1$

$= 1$ (1)

∴ true for $n=1$

2, Assume true for $n=k$ and prove $n=k+1$

$S_{k+1} = S_k + T_{k+1}$

$= (k+1)! - 1 + (k+1)\times(k+1)!$ (1)

$= (k+1)! + (k+1)(k+1)! - 1$

$= (k+1)!\,\{1 + k + 1\} - 1$

$= (k+1)!\,(k+2) - 1$

$= (k+2)! - 1$

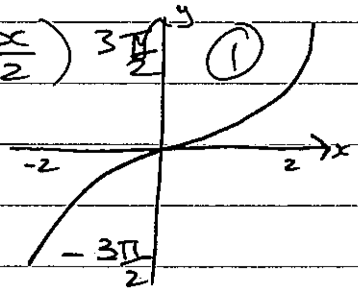
∴ true for $n = k+1$ (1)

3, since true for $n=1$, $n=k$ and $n=k+1$

∴ it is true for all positive integral values for $n \geq 1$ (1)

Question 5

a) $y = \sin^{-1}\left(\frac{x}{2}\right)$ (1)



Dom: $-1 \leq \frac{x}{2} \leq 1$

$-2 \leq x \leq 2$ (1)

RANGE $-\frac{\pi}{2} \leq \frac{y}{3} \leq \frac{\pi}{2}$ (1)

$-\frac{3\pi}{2} \leq y \leq \frac{3\pi}{2}$

b) i) $x^2 + 4x + 4 + 1$

$= (x+2)^2 + 1$

∴ If $x^2+4x+5 \equiv (x+a)^2 + b^2$

$a = 2 \quad b = \pm 1$ (2)

ii) $\displaystyle\int \frac{dx}{x^2+4x+5} = \int \frac{dx}{(x+2)^2+1}$ (1)

$= \tan^{-1}(x+2) + c$ (1)

c) i, Replace $x-1$ by X

∴ $X = \frac{1}{2}\cos 2t$

$\dot{X} = -\sin 2t$

$\ddot{X} = -2\cos 2t$

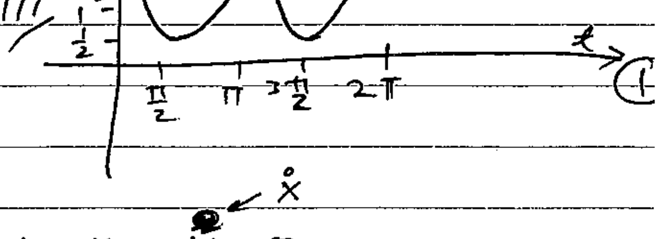
$= -4\left(\frac{1}{2}\cos 2t\right)$

$= -(2^2)X$ (1)

∴ in SHM

ii, $T = \frac{2\pi}{n} = \frac{2\pi}{2} = \pi$ sec (1)

iii,



(1)

iv, When $\dot{X} = 0$

$-\sin 2t = 0$

$t = \frac{n\pi}{2} \quad n = 0, 1, 2, 3, \dots$

when $t = \frac{n\pi}{2}$ $\quad X = x - 1 = \frac{1}{2}\cos 2\left(\frac{n\pi}{2}\right)$

$= \frac{1}{2}\cos n\pi$ (1)

$= \frac{1}{2}$ or $-\frac{1}{2}$

∴ The displacement are $\frac{3}{2}$ or $\frac{1}{2}$ the origin.

length $= \frac{3}{2} - \frac{1}{2} = 1$ (1)

Question 6

a) $\sin(x+\theta) = \sin x\cos\theta + \cos x\sin\theta$

$\therefore \sin x\cos\theta + \cos x\sin\theta = a\cos x$ (1)

Divide by $\cos x$

$$\frac{\sin x\cos\theta}{\cos x} + \frac{\cos x\sin\theta}{\cos x} = \frac{a\cos x}{\cos x} \quad (1)$$

$\tan x\cos\theta + \sin\theta = a$

$$\therefore \tan x = \frac{a-\sin\theta}{\cos\theta} \quad (1)$$

ii) If $\sin\left(x+\frac{\pi}{3}\right) = 2\cos x$
then $\theta = \frac{\pi}{3}$ $a=2$ (1)

$$\tan x = \frac{2-\sin\frac{\pi}{3}}{\cos\frac{\pi}{3}} = \frac{2-\frac{\sqrt{3}}{2}}{\frac{1}{2}} \quad (1)$$

$= 4-\sqrt{3}$

$x = 1.1555, 4.2971$ (1)

b) i) $x^2 = 4ay$ $\therefore y = \frac{x^2}{4a}$

$$\frac{dy}{dx} = \frac{2x}{4a} = \frac{x}{2a} = \frac{2ap}{2a} \text{ at } (2ap, ap^2)$$

$= p$ (1)

Eqn of Tangent is

$y - ap^2 = p(x-2ap)$

$y - ap^2 = p(x-2ap)$

$= px - 2ap^2$ (1)

b ii) $y - px + ap^2 = 0$

sub in $x = a$

$\therefore y - ap + ap^2 = 0$ (1)

$\therefore Q$ is $(a, ap-ap^2)$

iii) $P(2ap, ap^2)$ $Q(a, ap-ap^2)$

Find the midpoint of M

$x = \frac{2ap+a}{2}$ $y = \frac{ap^2+ap-ap^2}{2}$

$= \frac{ap}{2}$ (1)

From $y = \frac{ap}{2}$ $p = \frac{2y}{a}$

$$\therefore x = \frac{2a\left(\frac{2y}{a}\right)+a}{2} \quad (1)$$

$2x = 4y + a$

$2x - 4y - a = 0$

ie locus of M is a straight line (1)

Question 7

a) $f(x) = \sin\left(\frac{x}{10}\right) - e^{-x}$

$f'(x) = \frac{1}{10}\cos\left(\frac{x}{10}\right) + e^{-x}$ (1)

$x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

$x_2 = 2 - \dfrac{f(2)}{f'(2)}$

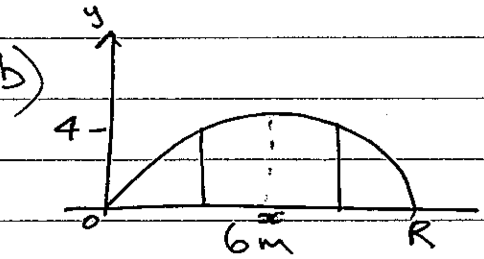
Now $f(2) = \sin\frac{1}{5} - e^{-2}$

$f'(2) = \frac{1}{10}\cos\frac{1}{5} + e^{-2}$

$x_2 = 2 - \dfrac{\sin\frac{1}{5} - e^{-2}}{\frac{1}{10}\cos\frac{1}{5} + e^{-2}}$ (1)

$\doteqdot 1.073$ (1)

b)



when $t = 0$ $\dot{x} = V\cos\alpha$

$\dot{y} = V\sin\alpha$

i) $\ddot{x} = 0$

$\therefore \dot{x} = V\cos\alpha$

$x = (V\cos\alpha)t + c_1$

when $t = 0$ $x = 0$

$\therefore c_1 = 0$

$\therefore x = Vt\cos\alpha$ $\boxed{1}$ (1)

$\ddot{y} = -g$

$\dot{y} = -gt + c_2$

when $t = 0$ $\dot{y} = V\sin\alpha$

$\therefore C_2 = V\sin\alpha$

$\therefore \dot{y} = -gt + V\sin\alpha$

$y = -\frac{1}{2}gt^2 + (V\sin\alpha)t + c_3$

(1)

when $t = 0$ $y = 0$ $\therefore c_3 = 0$

$\therefore y = -\frac{1}{2}gt^2 + Vt\sin\alpha$ $\boxed{2}$

ii) $t = \dfrac{x}{V\cos\alpha}$ from $\boxed{1}$

(1)

$\therefore y = -\frac{1}{2}g\left(\dfrac{x^2}{V^2\cos^2\alpha}\right) + \dfrac{V.x\sin\alpha}{V\cos\alpha}$

$y = \dfrac{-gx^2}{2V^2\cos^2\alpha} + x\tan\alpha$

$\boxed{3}$

To find R let $y = 0$

$\therefore 0 = x\left(\tan\alpha - \dfrac{gx}{2V^2\cos^2\alpha}\right)$

$x = \dfrac{\tan\alpha \times 2V^2\cos^2\alpha}{g}$

$= \dfrac{2V^2\sin\alpha\cos\alpha}{g}$

hence

$R = \dfrac{2V^2\sin\alpha\cos\alpha}{g}$ (1)

$\therefore 2V^2 = \dfrac{Rg}{\sin\alpha\cos\alpha}$

sub into $\boxed{3}$

$y = \dfrac{-gx^2}{\frac{Rg}{\sin\alpha\cos\alpha} \times \cos^2\alpha} + x\tan\alpha$ (1)

$= \dfrac{-x^2\tan\alpha}{R} + x\tan\alpha$

$y = x\left(1 - \frac{x}{R}\right)\tan\alpha$ (1)

iii) at

$\rightarrow$ over

Q7 contd

at $\alpha = 45^\circ$

$y = x\left(1 - \frac{x}{R}\right)$

when space halved:

when $x = \frac{R}{2} - 3$ $y = 4$ (1)

$\therefore 4 = \left(\frac{R}{2} - 3\right)\left(1 - \frac{\left(\frac{R}{2} - 3\right)}{R}\right)$

$4R = \left(\frac{R}{2} - 3\right)\left(R - \frac{R}{2} + 3\right)$

$4R = \frac{R^2}{4} - 9$

$16R = R^2 - 36$

$R^2 - 16R - 36 = 0$ (1)

$(R - 18)(R + 2) = 0$

$R = 18$ or -2

as $R > 0$

$\underline{R = 18m}$. (1)