GOSFORD HIGH SCHOOL

2020

Trial HSC Examination

Mathematics Extension 1

General Instructions

Total Marks – 70

All questions may be attempted

Section I (10 Marks)

Answer questions 1-10 on the Multiple Choice answer sheet provided.

Questions 1-10 are of equal values

Section II (60 Marks)

For Questions 11-14, start a new answer booklet for each question.

Questions 11-14 are of equal values

- Reading time -10 minutes
- Writing time - 2 Hours
- Writing using a black pen
- NESA approved calculators maybe used
- Leave your answers in the simplest exact form, unless otherwise stated
- Marks may be deducted for careless or badly arranged work
- All necessary working should be shown
- A Reference Sheet is provided

Collection

- Write your candidate number on each answer booklet and on your multiple choice answer sheet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question in section II, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Write your candidate number on this question paper and hand it in with your answers.

Candidate Number ------------------------------------

Section I – Multiple Choice (10 marks)

Attempt Questions 1 – 10

Read each question and choose an answer A, B, C or D.
Allow about 15 minutes for this section.
Use multiple- choice answer sheet for Question 1- 10

1. What is the value of $\sin 2x$ given that $\sin x = \frac{2\sqrt{3}}{4}$ and x is obtuse?

(A) $-\frac{\sqrt{3}}{4}$

(B) $-\frac{\sqrt{3}}{2}$

(C) $\frac{\sqrt{3}}{4}$

(D) $\frac{\sqrt{3}}{2}$

2. A ball is thrown from the origin O with a velocity V and an angle of elevation of θ, where $\theta \neq \frac{\pi}{2}$. What is the Cartesian equation of the flight path? Take $g = 10$ ms^{-1}.

(A) $y = x\tan\theta - \frac{5x^2}{V^2}(1 + \tan^2\theta)$

(B) $y = x\tan\theta - \frac{10x^2}{V^2}(1 + \tan^2\theta)$

(C) $x = V\cos\theta t$ and $y = -5t^2 + V\sin\theta t$

(D) $x = V\cos\theta t$ and $y = -10t^2 + V\sin\theta t$

3. An examination consists of 30 multiple-choice questions, each question having five possible answers. A student guesses the answer to every question. Let X be the number of correct answers. What is $E(X)$?

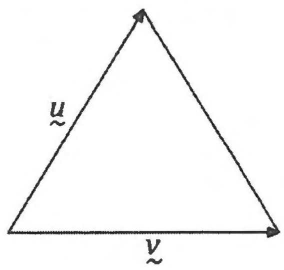
(A) 5

(B) 6

(C) 9

(D) 15

4. An equilateral triangle of side 3 units is shown below.
Vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are represented in the diagram.



What is the value of $\underset{\sim}{u} \cdot \underset{\sim}{v}$?

(A) 0

(B) $\frac{9}{\sqrt{2}}$

(C) $\frac{9}{2}$

(D) 9

5. If $y = \sin^{-1}\frac{a}{x}$ then $\frac{dy}{dx}$ equals:

(A) $\frac{-a}{x^2\sqrt{x^2 - a^2}}$

(B) $\frac{x}{\sqrt{x^2 - a^2}}$

(C) $\frac{-x}{\sqrt{x^2 - a^2}}$

(D) $\frac{-a}{x\sqrt{x^2 - a^2}}$

6. The equation $y = e^{ax}$ satisfies the differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$.

What are the possible values of a?

(A) $a = -2$ or $a = 3$

(B) $a = -1$ or $a = 6$

(C) $a = 2$ or $a = -3$

(D) $a = 1$ or $a = -6$

7. Which of the following is an expression for $\int \frac{x}{\sqrt{9 - x^2}} dx$? Use the substitution $u = 9 - x^2$.

(A) $-\sqrt{9 - x^2} + C$

(B) $-2\sqrt{9 - x^2} + C$

(C) $\sqrt{9 - x^2} + C$

(D) $2\sqrt{9 - x^2} + C$

8. Let $\underset{\sim}{u} = \underset{\sim}{i} + \underset{\sim}{j}$ and $\underset{\sim}{v} = \underset{\sim}{i} - \underset{\sim}{j}$. What is the angle between the two vectors.

(A) $\frac{\pi}{2}$

(B) $\frac{\pi}{4}$

(C) π

(D) 2π

9. Which of the following expressions represents the area of the region bounded by the curve $y = \sin^3 x$ and the x-axis from $x = -\pi$ to $x = 2\pi$? Use the substitution $u = \cos x$.

(A) $-\int_{-\pi}^{2\pi} (1 - u^2)du$

(B) $-3\int_{0}^{\pi} (1 - u^2)du$

(C) $-\int_{-1}^{1} (1 - u^2)du$

(D) $3\int_{-1}^{1} (1 - u^2)du$

10. Emma made an error proving that $2^n + (-1)^{n+1}$ is divisible by 3 for all integers $n \geq 1$ using mathematical induction. The proof is shown below.

Step 1: To prove $2^n + (-1)^{n+1}$ is divisible by 3 (n is an integer)

To prove true for $n = 1$

$2^1 + (-1)^{1+1} = 2 + 1$

$= 3 \times 1$ Line 1

Result is true for $n = 1$

Step 2: Assume true for $n = k$

$2^k + (-1)^{k+1} = 3m$ (m is an integer) Line 2

Step 3: To prove true for $n = k + 1$

$2^{k+1} + (-1)^{k+1+1} = 2(2^k) + (-1)^{k+2}$ Line 3

$= 2[3m + (-1)^{k+1}] + (-1)^{k+2}$ Line 4

$= 2 \times 3m + 2 \times (-1)^{k+2} + (-1)^{k+2}$

$= 3[2m + (-1)^{k+2}]$

Which is a multiple of 3 since m and k are integers.

Step 4: True by induction

In which line did Emma make an error?

(A) Line 1

(B) Line 2

(C) Line 3

(D) Line 4

------------------------------ **End of Section I** ------------------------------

Section II (60 marks)

Attempt Questions 11 – 14. Allow about 1 hour and 45 minutes for this section

Questions 11 (15 marks) Use a SEPARATE writing booklet.

(a) Use the substitution $u = \cos x + 1$, find the exact value of the following integral:

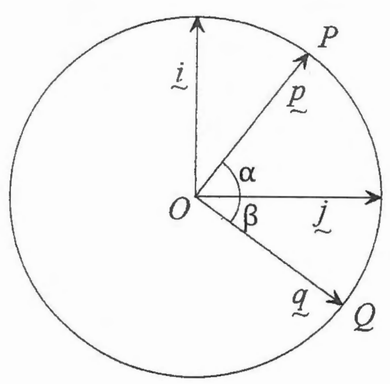
$$\int_0^{\frac{\pi}{2}} e^{\cos x+1} \sin x \, dx$$ **2**

(b) (i) Express $6\cos\theta + 8\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$, and $0 \le \alpha < 2\pi$.

Give α correct to 1 decimal place. **2**

(ii) Hence calculate the maximum value of $\dfrac{4}{12 + 6\cos\theta + 8\sin\theta}$ **2**

(c) Use the unit circle and the vectors in the diagram to derive the expansion of $\cos(\alpha + \beta)$



3

A man throws a ball from a point O on the horizontal ground so that it lands on the ground at a point P distant 80 m from him. The ball reaches the highest point 20 m above the ground. Neglect the sizes of the man and the ball. Take $g = 10\ m/s^2$

(d) (i) Find the vertical and horizontal components of velocity of projection of the ball. **2**

(ii) Find the velocity of projection of the ball. **2**

(iii) Find the further distance from the other boy B who starts running at the speed of $5\ m/s$

to catch the ball at the point P simultaneously. **2**

Questions 12 (15 marks) Use a SEPARATE writing booklet

(a) The roots of $x^3 - x^2 - 5x + 2 = 0$ are α, β, γ

(i) Show that $\beta + \gamma = 1 - \alpha$ **1**

(ii) Using part (i), and similar results, evaluate $\dfrac{\beta+\gamma}{\alpha} + \dfrac{\gamma+\alpha}{\beta} + \dfrac{\alpha+\beta}{\gamma}$ **2**

(b) Use mathematical induction to prove that $15^n + 2^{3n} - 2$ is a multiple of 7 for $n \geq 1$ **3**

(c) Points A, B have position vectors $\overrightarrow{OA} = 3i - 2j, \overrightarrow{OB} = -i + j$

(i) Find the unit vector along $\overrightarrow{AB}$ **1**

(ii) Suppose P is a point on AB such that $\overrightarrow{OP} \perp \overrightarrow{AB}$, Find $\overrightarrow{OP}$ **2**

(d) A particle is projected across horizontal ground from the origin O. Its initial velocity vector is $12i + 5j$ and its acceleration vector is $0i - 10j$.

Find:

(i) the initial speed of the particle **1**

(ii) the angle of projection, correct to the nearest minute **1**

(iii) Beginning with its acceleration vector, derive the velocity at any time t in the form

$\underset{\sim}{v} = \dot{x}\underset{\sim}{i} + \dot{y}\underset{\sim}{j}$ **2**

(iv) Show that its cisplacement at time t is $\underset{\sim}{r} = (12t)\underset{\sim}{i} + (5t - 5t^2)\underset{\sim}{j}$ **1**

(v) Find the maximum height of the particle. **1**

Questions 13 (15 marks) Use a SEPARATE writing booklet

(a) $f(x)=\sqrt{4-\sqrt{x}}$

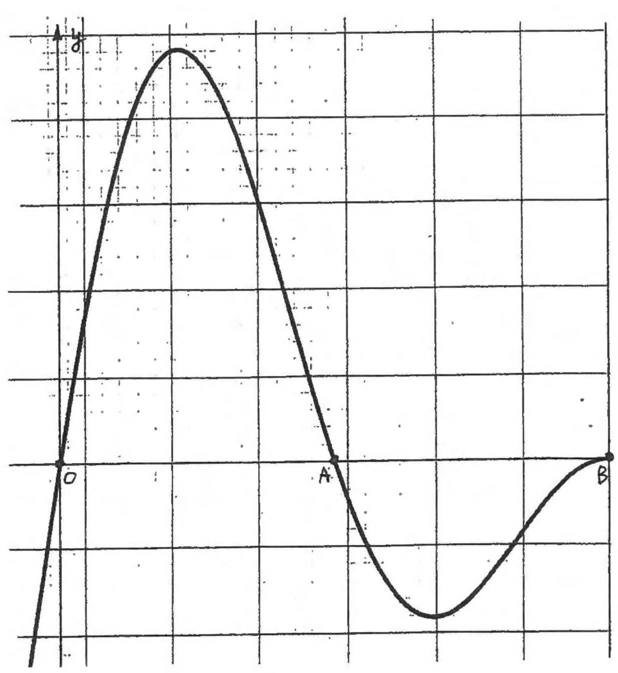
(i) Explain why the domain of $f(x)$ is $0\le x\le 16$ **1**

(ii) Prove $f(x)$ is a decreasing function and find its range. **2**

(iii) Since $f(x)$ is monotonic, an inverse function exists. Find the domain and range of $f^{-1}(x)$. Hence find $f^{-1}(x)$. **2**

(b) (i) Given that $\frac{d}{dx}(\sin 2x - 2x\cos 2x) = 4x\sin 2x$. The curve shown below is part of the function $y = x\sin 2x$. Write down the coordinates of the points A and B. **1**

(ii) If the area bounded by the curve and the line AB is k times that of the area of the region bounded by the curve and the line OA. Determine the value of k. **2**



(iii) Show that $\int_0^{\frac{\pi}{8}} x\sin 2x\,dx = \frac{\sqrt{2}}{32}(4-\pi)$ **2**

Questions 13 (Continued)

(c) On a roulette wheel, there are 18 red numbers, 18 black numbers, and 1 green number.

A ball is dropped onto the spinning wheel and lands on one of the numbers randomly.

Each result is independent. A gambler bets that the ball will land on any of the black numbers.

(i) Define the gambler's bet as a Bernoulli random variable X and give its mean and variance. **1**

(ii) If the gambler makes the same bet five times, let the random variable Y be the number of times the gambler wins. Describe the distribution of Y, and give its mean and variance. **2**

(iii) If the gambler makes the same bet five times, what is the probability he will win more times than he loses? Give your answer correct to three decimal places. **2**

Questions 14 (15 marks) Use a SEPARATE writing booklet

(a) Given that $t=\tan\frac{\theta}{2}$, then $\sin\theta=\frac{2t}{1+t^2}$ and $\cos\theta=\frac{1-t^2}{1+t^2}$ for any angle θ.

(DO NOT PROVE THIS)

(i) Show that the equation $7\cos\theta+4\sin\theta+5=0$ can be written as $t^2-4t-6=0$ **2**

(ii) θ_1,θ_2 are the solutions of the equation $7\cos\theta+4\sin\theta+5=0$ for $\left(\theta_1>0, 0<\theta_2<2\pi\right)$

Then without solving for θ, show that $\cot\frac{\theta_1}{2}+\cot\frac{\theta_2}{2}=-\frac{2}{3}$ **2**

(b) Show that, among any four integers, there are two integers whose difference is divisible by 3. **2**

(c) A tank contains 2500 litres of water and 25 Kg of dissolved salt. Fresh water enters the tank at a rate of 20 litres per minute. The solution is thoroughly mixed at all times and is drained from the tank at a rate of 15 litres per minute.

(i) Using y for the amount of salt in the tank in kilograms (as a function of time), and t for time in minutes, show that the concentrations of salt in the tank at time t can be given by $C=\frac{y}{2500+5t}$ **1**

(ii) Explain why the rate of change of salt in the tank can be given by $y'=-15C$ **1**

(iii) Find y, the amount of salt in the tank as a function of t. **4**

(d) Find the six solutions of the equation:

$$\sin\left(2\cos^{-1}\left(\cot\left(2\tan^{-1}x\right)\right)\right)=0$$ **3**

Give your answers as simplified surds

Ext 1 Trial 2020 (GHS)

1. B 2. A 3. B 4. C 5. D 6. C 7. A 8. A 9. D 10. D

Q11

a) $\int_0^{\pi/2} e^{\cos x + 1} \sin x \, dx$

$u = \cos x + 1$

$\frac{du}{dx} = -\sin x$

$dx = \frac{du}{-\sin x}$

$x = \frac{\pi}{2}$ $u = 1$

$x = 0$ $u = 2$

$I = \int_2^1 e^u \cdot - du$

$= \int_1^2 e^u \, du$

$= \left[e^u\right]_1^2$

$= e^2 - e$

(2)

b) i) $6\cos\theta + 8\sin\theta$

$\equiv R\cos(\theta - \alpha)$

$\therefore R\cos\alpha = 6$

$R\sin\alpha = 8$

$\therefore \tan\alpha = \frac{4}{3} \doteqdot 0.9$

$R^2(\sin^2\alpha + \cos^2\alpha) = 36 + 64$

$R = 10$

$\therefore 10\cos(\theta - 0.9)$

(2)

ii) $\frac{4}{12 + 6\cos\theta + 8\sin\theta}$

$= \frac{4}{12 + 10\cos(\theta - 0.9)}$

max at $10\cos(\theta - 0.9) = -1$

$\therefore \max = \frac{4}{12 - 10} = \frac{4}{2} = 2$

(2)

c) $\cos(\alpha + \beta) = \frac{\underset{\sim}{p} \cdot \underset{\sim}{q}}{|\underset{\sim}{p}||\underset{\sim}{q}|}$

& $|\underset{\sim}{p}| = |\underset{\sim}{q}| = 1$

$\therefore \cos(\alpha + \beta) = \underset{\sim}{p} \cdot \underset{\sim}{q}$

$\underset{\sim}{p} = \cos\alpha \, \underset{\sim}{i} + \sin\alpha \, \underset{\sim}{j}$

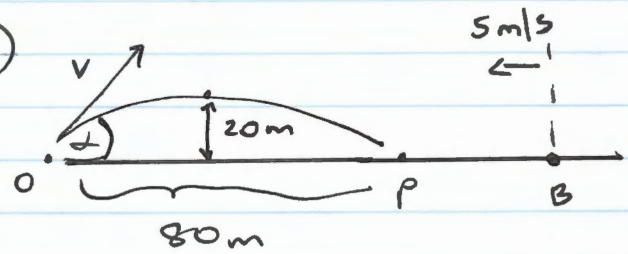
$\underset{\sim}{q} = \cos\beta \, \underset{\sim}{i} - \sin\beta \, \underset{\sim}{j}$

$\therefore \underset{\sim}{p} \cdot \underset{\sim}{q} = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\therefore \cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

(3)

d)



d i)

horiz

$\ddot{x} = 0$

$\dot{x} = 0t + c$

at $t=0$ $\dot{x} = V\cos\alpha = c$

$\dot{x} = V\cos\alpha$

$x = Vt\cos\alpha + c_1$

at $t=0$ $x=0$

$\therefore x = Vt\cos\alpha$

vert

$\ddot{y} = -g$

$\dot{y} = -gt + c_2$

at $t=0$ $\dot{y} = V\sin\alpha = c_2$

$\therefore \dot{y} = -10t + V\sin\alpha$

$y = -5t^2 + Vt\sin\alpha + c_3$

at $t=0$ $y = 0 = c_3$

$\therefore y = -5t^2 + Vt\sin\alpha$

max height at $\frac{dy}{dt} = 0$

$\therefore -10t + V\sin\alpha = 0$

$\therefore t = \frac{V\sin\alpha}{10}$

$$y = -\cancel{5}\left(\frac{V^2\sin^2\alpha}{\cancel{100}\,20}\right) + V\sin\alpha\left(\frac{V\sin\alpha}{10}\right)$$

$$20 = -\frac{V^2\sin^2\alpha}{20} + \frac{V^2\sin^2\alpha}{10}$$

$\therefore V^2\sin^2\alpha = 400$ as $V, \sin\alpha > 0$

$V\sin\alpha = 20$

& at $y=0$

$t(-5t + V\sin\alpha) = 0$ $t \neq 0$

$\therefore t = \frac{V\sin\alpha}{5}$

$\therefore x = V\left(\frac{V\sin\alpha}{5}\right)\cos\alpha$

$\therefore 80 = \frac{V^2\sin\alpha\cos\alpha}{5}$

& $V\sin\alpha = 20$

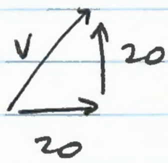
$\therefore 80 = \frac{V\cos\alpha \cdot 20}{5}$

$\therefore V\cos\alpha = 20$

2

$\therefore$ vert & horiz. components of velocity $= 20$ m/s

ii)



$V^2 = 20^2 + 20^2$

$V = 20\sqrt{2}$ m/s

2

iii) using $x = Vt\cos\alpha$

$V\cos\alpha = 20$, $x = 80$

$\therefore$ time of flight $= \frac{80}{20}$

$t = 4$ s

$\therefore$

$\therefore$ Dist $= 5 \times 4$

$= 20$ m

2

or 100 m from man

Q12

a) $x^3 - x^2 - 5x + 2 = 0$

$\alpha + \beta + \gamma = \frac{-b}{a}$

$= \frac{-(-1)}{1}$

$= 1$

$\therefore \beta + \gamma = 1 - \alpha$ ①

b) $\frac{\beta+\gamma}{\alpha} + \frac{\gamma+\alpha}{\beta} + \frac{\alpha+\beta}{\gamma}$

from ① $\beta + \gamma = 1 - \alpha$
$\alpha + \beta = 1 - \gamma$
$\alpha + \gamma = 1 - \beta$

$\therefore \frac{1-\alpha}{\alpha} + \frac{1-\beta}{\beta} + \frac{1-\gamma}{\gamma}$

$= \frac{1}{\alpha} - 1 + \frac{1}{\beta} - 1 + \frac{1}{\gamma} - 1$

$= \left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right) - 3$

$= \frac{\Sigma \alpha\beta}{\alpha\beta\gamma} - 3$

$= \frac{-5}{-2} - 3$

$= -\frac{1}{2}$ ②

b) RTP $15^n + 2^{3n} - 2 = 7p$
p is +ve int
$n \geq 1$

① Prove $n=1$

$LHS = 15^1 + 2^{3(1)} - 2$

$= 15 + 8 - 2$

$= 21$

$\therefore$ divisible by 7

$\therefore$ true $n=1$

② Assume true $n=k$, $k \geq 1$

ie

$15^k + 2^{3k} - 2 = 7q$ q +ve int

$15^k = 7q - 2^{3k} + 2$

③ RTP $n = k+1$

ie $15^{k+1} + 2^{3(k+1)} - 2 = 7m$

m is +ve int

LHS

$= 15.15^k + 2^{3k}.2^3 - 2$

$= 15(7q - 2^{3k} + 2) + 8.2^{3k} - 2$

$= 7.15q - 15.2^{3k} + 30 + 8.2^{3k} - 2$

$= 7.15q - 7.2^{3k} + 28$

$= 7(15q - 2^{3k} + 4)$

$= 7m$

③

10

c) $\underset{\sim}{a} = \begin{pmatrix}3\\-2\end{pmatrix}$ $\quad \underset{\sim}{b} = \begin{pmatrix}-1\\1\end{pmatrix}$

i) $\underset{\sim}{c} = \vec{AB} = -\underset{\sim}{a} + \underset{\sim}{b}$

$$= \begin{pmatrix}-3\\2\end{pmatrix} + \begin{pmatrix}-1\\1\end{pmatrix}$$

$$= \begin{pmatrix}-4\\3\end{pmatrix}$$

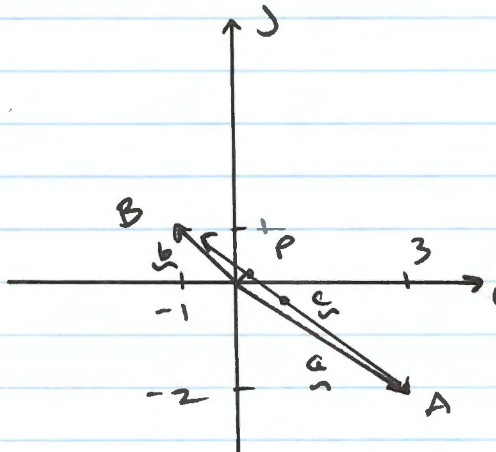
$|\underset{\sim}{c}| = \sqrt{16+9}$

$= 5$

$$\hat{\underset{\sim}{c}} = \frac{1}{5}\begin{pmatrix}-4\\3\end{pmatrix}$$

$$= \begin{pmatrix}-\frac{4}{5}\\ \frac{3}{5}\end{pmatrix}$$ (1)

ii)



let $P(x,y)$

$\vec{PO}\cdot\vec{AB} = 0 \qquad \vec{OP} \perp \vec{AB}$

$\therefore \begin{pmatrix}x\\y\end{pmatrix}\cdot\begin{pmatrix}-4\\3\end{pmatrix} = 0$

$\therefore -4x + 3y = 0$ (1)

& $\vec{PB} = \lambda\vec{AB}$

$\therefore$ slope $\vec{PB}$ = slope $\vec{AB}$

$$\frac{1-y}{-1-x} = \frac{-2-1}{3+1} = \frac{-3}{4}$$

$4 - 4y = 3 + 3x$

$3x + 4y - 1 = 0$ (2)

$-12x + 9y = 0$ (1) $\times 3$

$+12x + 16y - 4 = 0$ (2) $\times 4$

$25y - 4 = 0$

$y = +\frac{4}{25} \qquad x = +\frac{3}{25}$

$P\left(\frac{3}{25}, \frac{4}{25}\right)$ (2)

d) $v = 12\underset{\sim}{i} + 5\underset{\sim}{j}$ $\qquad \underset{\sim}{a} = 0\underset{\sim}{i} - 10\underset{\sim}{j}$

at $t = 0$

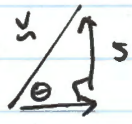
$\underset{\sim}{v} = 12\underset{\sim}{i} + 5\underset{\sim}{j} \qquad \underset{\sim}{a} = 0\underset{\sim}{i} - 10\underset{\sim}{j}$

i) at $t = 0$

$|\underset{\sim}{v}| = \sqrt{12^2 + 5^2}$

$= 13$ m/s (1)

ii)



$\tan\theta = \frac{5}{12}$

$\therefore \theta = 22°37'$ (1)

iii) $\underset{\sim}{a} = 0\underset{\sim}{i} - 10\underset{\sim}{j}$

$\ddot{x} = 0 \qquad \ddot{y} = -10$

$\dot{x} = 0t + c \qquad \dot{y} = -10t + c_1$

at $t = 0 \quad \dot{x} = 12, \dot{y} = 5$

$\therefore c = 12, \quad c_1 = 5$

$\dot{x} = 12 \qquad \dot{y} = -10t + 5$

$\therefore \underset{\sim}{v} = 12\underset{\sim}{i} + (-10t + 5)\underset{\sim}{j}$ (2)

iv)

$x = 12t + c_2 \qquad y = \frac{-10t^2}{2} + 5t + c_3$

at $t = 0 \quad x = 0 \quad y = 0$

$\therefore c_2 = 0 = c_3$

$x = 12t \qquad y = -5t^2 + 5t$

$\therefore \underset{\sim}{r} = 12t\,\underset{\sim}{i} + (5t - 5t^2)\underset{\sim}{j}$ ①

v) max height at $\dot{y} = 0$

$\therefore -10t + 5 = 0$

$10t = 5$

$t = 0.5$

$\therefore y = -5(0.5)^2 + 5(0.5)$

$= 1.25 \text{ m}$ ①

Q 13.

a) (i) $f(x) = \sqrt{4-\sqrt{x}}$

For $\sqrt{x}$: $x \geq 0$

For $\sqrt{4-\sqrt{x}}$: $4-\sqrt{x} \geq 0$

$4 \geq \sqrt{x}$

$16 \geq x \quad (x \geq 0)$

$\therefore \quad 0 \leq x \leq 16$ (1 mark)

(ii) $f(x) = (4 - x^{\frac{1}{2}})^{\frac{1}{2}}$

$f'(x) = \frac{1}{2}(4-x^{\frac{1}{2}})^{-\frac{1}{2}} \cdot -\frac{1}{2}x^{-\frac{1}{2}}$

$= -\dfrac{1}{4\sqrt{x}\sqrt{4-\sqrt{x}}}$

$\sqrt{x} \geq 0 \quad \forall x$

$\sqrt{4-\sqrt{x}} \geq 0 \quad \forall x$

$\therefore f'(x) = \dfrac{-1}{4\ (+)\ (+)}$

< 0

$\therefore$ f(x) is a decreasing function

$\Rightarrow$ Range can be found by evaluating f(x) at endpoints

$f(0) = 2$

$f(16) = 0$

$\therefore$ Range $y \in [0, 2]$

(2 marks)

(iii) $f(x)$ D: $x \in [0,16]$
R: $y \in [0,2]$

$f^{-1}(x)$ D: $x \in [0,2]$
R $y \in [0,16]$

$f: \ y = \sqrt{4-\sqrt{x}}$

$f^{-1}: \ x = \sqrt{4-\sqrt{y}}$

$x^2 = 4-\sqrt{y}$

$x^2 - 4 = -\sqrt{y}$

$\sqrt{y} = 4 - x^2$

$y = (4-x^2)^2$ for $(0 \le x \le 2)$

(2 marks)

b) (i) $x \sin 2x = 0$

$x = 0$ or $\sin 2x = 0$

$2x = 0, \pi, 2\pi, \ldots$

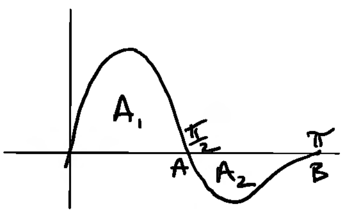
$x = 0, \frac{\pi}{2}, \pi, \ldots$

$\therefore A\left(\frac{\pi}{2}, 0\right)$

$B(\pi, 0)$

(1 mark)

(ii)



$A_2 = kA_1$

$-\int_{\frac{\pi}{2}}^{\pi} x \sin 2x \, dx = k \int_0^{\pi} x \sin 2x \, dx$

$-\frac{1}{4}\left[\sin 2x - 2x\cos x\right]_{\frac{\pi}{2}}^{\pi} = \frac{k}{4}\left[\sin 2x - 2x\cos x\right]_0^{\frac{\pi}{2}}$

$\vdots$

$\frac{3\pi}{4} = \frac{k\pi}{4}$

$\therefore k = 3$

(2 marks)

(iii) $\int_0^{\pi/8} x \sin 2x \, dx$

$= \frac{1}{4}\left[\sin 2x - 2x\cos 2x\right]_0^{\pi/8}$

$= \frac{1}{4}\left[\left(\frac{1}{\sqrt{2}} - \frac{\pi}{4}\times\frac{1}{\sqrt{2}}\right)\right]$

$= \frac{1}{4\sqrt{2}}\left(1 - \frac{\pi}{4}\right)$

$= \frac{\sqrt{2}}{8}\left(\frac{4-\pi}{4}\right)$

$= \frac{\sqrt{2}}{32}(4-\pi)$. (2 marks)

(c) (i) This is a Bernoulli random variable with $p = 18/37$.
This may be written

$X \sim Ber\left(\frac{18}{37}\right)$ or $X \sim Bin\left(\frac{18}{37}, 1\right)$.

$\mu = \frac{18}{37}$, $\sigma^2 = \frac{18}{37} \times \frac{19}{37} = \frac{342}{1369}$ (1 mark)

(ii) This is a binomial random variable with $p = 18/37$ and $n = 5$. This may be written

$Y \sim Bin\left(\frac{18}{37}, 5\right)$

$\mu = 5 \times \frac{18}{37} = \frac{90}{37} \approx 2.43$

$\sigma^2 = 5 \times \frac{18}{37} \times \frac{19}{37} = \frac{1710}{1369} \approx 1.25$ (2 marks)

(iii) This is a binomial probability and we are looking for $P(W \geq 3)$, where W is the number of wins in five bets

$P(W \geq 3) = P(W=3) + P(W=4) + P(W=5)$

$= \binom{5}{3}\left(\frac{18}{37}\right)^3\left(\frac{19}{37}\right)^2 + \binom{5}{4}\left(\frac{18}{37}\right)^4\left(\frac{19}{37}\right)^1$

$+ \binom{5}{5}\left(\frac{18}{37}\right)^5\left(\frac{19}{37}\right)^0$

$= 0.475$ (2 marks)

2020 Trial Extension 1 Q14 Solutions

a) i) $7 \times \left(\frac{1-t^2}{1+t^2}\right) + 4 \times \left(\frac{2t}{1+t^2}\right) + 5 = 0$

$7(1-t^2) + 8t + 5(1+t^2) = 0$

$7 - 7t^2 + 8t + 5 + 5t^2 = 0$

$12 + 8t - 2t^2 = 0$

$-2(t^2 - 4t - 6) = 0$

$\therefore t^2 - 4t - 6 = 0$ as Req.

ii) Solution 1.

$t = \frac{4 \pm \sqrt{16 - 4 \times 1 \times (-6)}}{2 \times 1}$

$= \frac{4 \pm \sqrt{40}}{2}$

$= 2 \pm \sqrt{10}$

Since $t = \tan\frac{\theta}{2}$, then $\frac{1}{t} = \cot\frac{\theta}{2}$

So $\cot\frac{\theta_1}{2} + \cot\frac{\theta_2}{2} = \frac{1}{2+\sqrt{10}} + \frac{1}{2-\sqrt{10}}$

$= \frac{2-\sqrt{10} + 2+\sqrt{10}}{4-10}$

$= \frac{4}{-6}$

$= \frac{-2}{3}$ as Req

Solution 2

Since roots are θ_1 and θ_2, let $t_1 = \tan\left(\frac{\theta_1}{2}\right)$ and $t_2 = \tan\left(\frac{\theta_2}{2}\right)$

hence $\cot\left(\frac{\theta_1}{2}\right) = \frac{1}{t_1}$ and $\cot\left(\frac{\theta_2}{2}\right) = \frac{1}{t_2}$

If t_1 and t_2 are the roots of $t^2 - 4t - 6 = 0$,

then $t_1 + t_2 = -(-4) = 4$

and $t_1 \times t_2 = -6$

So $\cot\left(\frac{\theta_1}{2}\right) + \cot\left(\frac{\theta_2}{2}\right) = \frac{1}{t_1} + \frac{1}{t_2}$

$= \frac{t_1 + t_2}{t_1 \times t_2}$

$= \frac{4}{-6} = \frac{-2}{3}$ as Req.

b) Integers can go into 3 pigeon holes. When the integer is divided by 3

Box 0	Box 1	Box 2
Remainder = 0	Remainder = 1	Remainder = 2

By Pigeonhole Principle at least 1 box contains 2 integers.

If the 2 integers are in Box (x) where $x = 0, 1, 2$ then the integers can be written as $3M+x$ and $3N+x$, where M and N are integers.

$\therefore$ The difference between the integers is $(3M+x) - (3N+x) = 3(N-M)$

Hence the difference is divisible by 3.

c) i) Amount of liquid in the tank, at t minutes is

$$A = 2500 + 20t - 15t$$
$$= 2500 + 5t$$

Concentration of Salt $= \dfrac{\text{Amount of Salt}}{\text{Amount of Water}}$

$$\therefore C = \frac{y}{2500+5t}$$

ii) Rate of Change of Salt = Concentration of Salt per litre × Amount of liquid leaving the tank.

$$\therefore y' = C \times (-15)$$
$$= -15C$$

iii)
$$\frac{dy}{dt} = -15C = \frac{-15y}{2500+5t}$$

$$\therefore dy \times \frac{1}{y} = \frac{-15}{2500+5t} \times dt$$

Hence $\displaystyle\int \frac{1}{y} \cdot dy = -3 \int \frac{5}{2500+5t} \cdot dt$

So $\ln|y| = -3 \ln|2500+5t| + k$

$$\ln|y| + 3\ln|2500+5t| = k$$
$$\ln|y \times (2500+5t)^3| = k$$
$$\therefore y \times (2500+5t)^3 = e^k$$
$$y = e^k \times \frac{1}{(2500+5t)^3}, \quad \text{let } A = e^k$$
$$\therefore y = \frac{A}{(2500+5t)^3}$$

when $t=0$, $y=25$ So $25 = \dfrac{A}{(2500+5\times 0)^3}$ $\therefore A = 25 \times 2500^3$

Hence $y = \dfrac{25 \times 2500^3}{(2500+5t)^3}$

d) let $\alpha = \tan^{-1}x$ then $x = \tan\alpha$

Now $\cot(2\tan^{-1}x) = \dfrac{1}{\tan(2\tan^{-1}x)} = \dfrac{1}{\tan 2\alpha}$

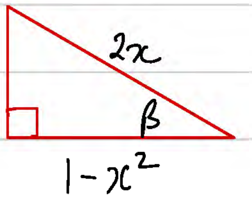
$$\cot 2\alpha = \frac{1}{\tan 2\alpha} = \frac{1-\tan^2\alpha}{2\tan\alpha}$$

$$\therefore \cot 2\alpha = \frac{1-x^2}{2x}$$

Hence $\sin\left(2\cos^{-1}\left(\dfrac{1-x^2}{2x}\right)\right) = 0$

let $\beta = \cos^{-1}\left(\dfrac{1-x^2}{2x}\right)$ then $\cos\beta = \dfrac{1-x^2}{2x}$ and $\sin 2\beta = 0$

Now $\sin 2\beta = 2\sin\beta\cos\beta$



$$\therefore \text{3rd Side} = \sqrt{(2x)^2 - (1-x^2)^2}$$
$$= \sqrt{4x^2 - (1-2x^2+x^4)}$$
$$= \sqrt{-x^4+6x^2-1}$$

Hence $\sin\beta = \dfrac{\sqrt{-x^4+6x^2-1}}{2x}$

So $\sin 2\beta = 2\sin\beta\cos\beta = 2\times\dfrac{\sqrt{-x^4+6x^2-1}}{2x}\times\dfrac{1-x^2}{2x} = 0$

Which makes $1-x^2 = 0$ or $-x^4+6x^2-1 = 0$

$\therefore x = \pm 1$

$$x^4 - 6x^2 + 1 = 0$$
$$x^4 - 6x^2 + 9 = 8$$
$$(x^2-3) = 8$$
$$x^2 = 3 \pm 2\sqrt{2}$$
$$= 1 \pm 2\sqrt{2} + 2$$
$$= (1 \pm \sqrt{2})^2$$
$$\therefore x = \pm(1 \pm \sqrt{2})$$

$\therefore$ The six solutions are ± 1, $\pm 1+\sqrt{2}$, $\pm 1-\sqrt{2}$