THE HILLS GRAMMAR SCHOOL

TRIAL HSC
1999

MATHEMATICS

3 UNIT

TIME ALLOWED: 2 Hours (plus 5 minutes reading time)

Teacher Responsible: Mrs B Spencer
Mrs S Maxton

INSTRUCTIONS:

- Attempt all questions.
- In every question show all necessary working.
- Silent calculators and approved templates may be used.
- Start each question on a NEW page and hand up your paper in ONE bundle with your name marked on EVERY page.
- All questions are of equal value.

Question 1

Marks

a) Express $\frac{1+\sqrt{2}}{\sqrt{5}+\sqrt{3}} + \frac{1-\sqrt{2}}{\sqrt{5}-\sqrt{3}}$ in the form of $a\sqrt{5}+b\sqrt{6}$ — 2

b) Show that $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$ — 2

c) Find the acute angle between the lines — 2

$y = 4x - 2$ and $2x + 3y - 9 = 0$

to the nearest minute.

d) Find the co-ordinates of the point P which divides the interval AB externally in the ratio 3:1. A is (–4,2) and B is (6,5). — 2

e) (i) What is the maximum value of $5\sin\theta + 12\cos\theta$; and — 2

(ii) What is the first positive value of θ for which this maximum occurs. — 2

Question 2

a) Find $\int x^2\sqrt{(x^3-9)}dx$ using $u = x^3 - 9$ — 3

b) Evaluate $\int_0^{\frac{\pi}{12}} \cos^2 x \, dx$. Answer in exact value terms. — 3

c) Find the term independent of x in the expansion of $(x^2 - \frac{1}{x})^{12}$. — 3

d) The remainder when $x^3 + ax + b$ is divided by $(x-2)(x+3)$ is $2x+1$. Find a and b. — 3

Question 3

Marks

a) The points P $(2ap, ap^2)$ and Q $(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$. The angle POQ is 90°. O is (0,0). Show that $pq = -4$ and hence find the equation of the locus of M, the midpoint of PQ. 2 3

b) A projectile is fired from O with a velocity of 20m/s at an angle of 60°. The projectile just clears a wall 25m from the point of projection. How high is the wall? Take $g = 10\text{m/sec}^2$. 3

c) $f(x) = x^3 - x^2 - x - 1$.

(i) Show that the equation $f(x) = 0$ has a root in the interval $1 < x < 2$. 1

(ii) Use Newton's method once to find a better approximation for the root, taking x to be 1.5. 3

Question 4

a) A stone is thrown into a pond and creates a circular ripple which expands so that $\frac{dr}{dt} = 1.5m/s$. Find the rate at which the area of the circle is increasing when the radius is 2m. Answer to 4 significant figures. 2

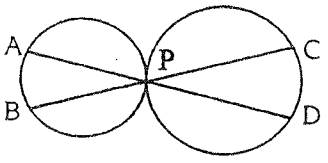
b) A particle moving along the x-axis has a velocity given by $v^2 = 15 - 2x - x^2$.

(i) Show that the centre of the motion is at $x = -1$. 2

(ii) Show that $a = -n^2 x$ and thus the motion is simple harmonic. 2

(iii) Find the amplitude and period of the motion. 2

c)



Two unequal circles touch at P. APD and BPC are straight lines.

(i) Copy the diagram in a larger scale. 4

(ii) Draw the common tangent XPY.

(iii) Prove that AB is parallel to CD.

Question 5

Marks

a) Use the expansion $(1+x)^n = \sum_{r=0}^{n} {}^nC_r x^r$ to prove that

$1 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \ldots\ldots\ldots + (-1)^n C_n = 0$. 2

b) In how many ways can a committee of 4 people be selected from a group of 10 people if;

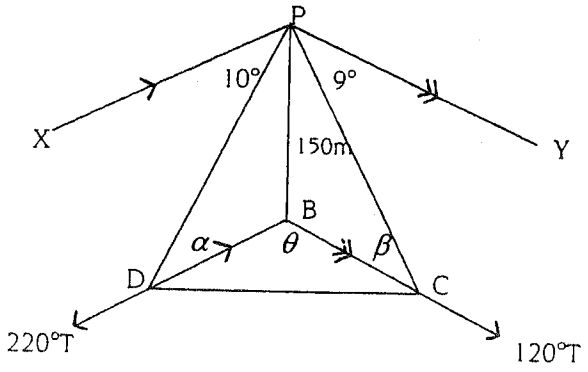
(i) 2 particular members P and Q are included; 3

(ii) 2 particular members P and Q are excluded. 3

c) I participate in a raffle where my chances of winning a prize are one in ten. If I buy a book of 12 tickets, what is the probability of my winning exactly two prizes. Leave answer in index form. 4

Question 6

a)



A ship is observed from the top of a 150m cliff BP with an angle of depression 9° when the ship is at the point C. Ten minutes later it is seen at D with an angle of depression of 10°. $\angle PDB = \alpha$, $\angle PCB = \beta$ and $\angle DBC = \theta$. BC bears 120°T. BD bears 220°T.

(i) Show that $\alpha = 10°$, $\beta = 9°$ and $\theta = 100°$. 3

(ii) Show that BD = 150 $\cot\alpha$ and BC = 150 $\cot\beta$ and hence that $CD^2 = 150^2 (\cot^2\alpha + \cot^2\beta - 2\cot\alpha\cot\beta\cos\theta)$. 4

(iii) Find the speed of the ship in km/h to three significant figures. 2

c) Solve for all x, $2\sin^2 x = \sin 2x$. Answer in radian measure. 3

Question 7

Marks

a) My loungeroom is kept at a constant temperature of 25°c. A cup of tea left standing in the room cools at a rate proportional to the difference in temperature between the tea and its surroundings so that $\frac{dT}{dt} = k(T - 25)$. After 20 minutes the temperature of the tea has dropped from 95°c to 65°c.

(i) Show that $T = 25 + Ae^{kt}$ is a solution of $\frac{dT}{dt} = k(T - 25)$. 1

(ii) Find the values of A and k. 3

(iii) Find the temperature of the tea after a further 10 minutes. 2

b) Consider the function $f(x) = x \sin^{-1} x$.

(i) Show that $f(x)$ is an even function. 1

(ii) Find $f'(x)$ and hence the co-ordinates of the only turning point. 2

(iii) Determine the domain and range of $f(x)$. 2

(iv) Sketch $f(x)$. 1

Solution. Hills Grammar

3U 1999 Trial HSC.

Question 1

a) $\dfrac{1+\sqrt{2}}{\sqrt{5}+\sqrt{3}} + \dfrac{1-\sqrt{2}}{\sqrt{5}-\sqrt{3}}$

$= \dfrac{\sqrt{5}-\sqrt{3}+\sqrt{10}-\sqrt{6}+\sqrt{5}+\sqrt{3}-\sqrt{10}-\sqrt{6}}{(\sqrt{5}+\sqrt{3})(\sqrt{5}-\sqrt{3})}$

$= \dfrac{2\sqrt{5}-2\sqrt{6}}{2}$

$= \sqrt{5}-\sqrt{6}$ $\quad a = 1 \quad b = -1.$ (2)

b) $\dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{n!}{(n-r)!\,r!} \div \dfrac{n!}{(n-r+1)!\,(r-1)!}$

$= \dfrac{n!}{(n-r)!\,r!} \times \dfrac{(n-r+1)!\,(r-1)!}{n!}$

$= \dfrac{n-r+1}{r}$ as reqd. (2)

c) $y = 4x-2$ $\quad m_1 = 4$

$2x+3y=9$ $\quad m_2 = \dfrac{-2}{3}$

$\tan\theta = \left|\dfrac{m_1-m_2}{1+m_1m_2}\right|$

$= \left|\dfrac{4+\frac{2}{3}}{1-\frac{8}{3}}\right|$

$= \dfrac{14}{5}$

$\theta = 70°21'$ (2)

d) $A(-4,2)$ $\quad B(6,5)$ $\quad P(15, 6\tfrac{2}{5})$ (2)

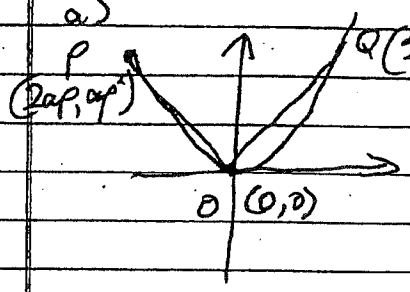
3 : 2

e) $5\sin\theta + 12\cos\theta = 13\left(\frac{5}{13}\sin\theta + \frac{12}{13}\cos\theta\right)$ $\quad \alpha = 67°23'$

$= 13\sin(\theta+\alpha)$ where [illegible]

Question 3

a)



$PO \perp QO$

$\therefore m_1 \times m_2 = -1$

$\frac{p}{2} \times \frac{q}{2} = -1$

$pq = -4$ (2)

M is $a(p+q)$, $\frac{a(p^2+q^2)}{2}$

$x = a(p+q)$ $\quad 2y = a(p^2+q^2)$

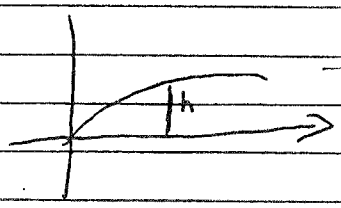
$\frac{x^2}{a^2} = p^2 + q^2 - 8$

$\frac{x^2}{a^2} + 8 = \frac{2y}{a}$

$x^2 = 2ay - 8a^2$

$x^2 = 2a(y - 4a)$ (3)

b)



$x = Vt\cos\theta$

$25 = 20t \times \cos 60°$

$t = 2.5$

$h = -5t^2 + 20 \times 2.5 \sin 60°$

$h = 12.05\,m$ (3)

c) (i) $f(x) = x^3 - x^2 - x - 1$

$f(1) = -2$ negative

$f(2) = 1$ positive

$\therefore$ there is a root between 1 and 2 (1)

(ii) $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= 1.5 + \frac{1.375}{2.75}$

$= 2$

$f(x) = x^3 - x^2 - x - 1$

$f(1.5) = -1.375$

$f'(x) = 3x^2 - 2x - 1$

$f'(1.5) = 2.75$ (3)

Question 5.

a) $(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \dots + {}^nC_n x^n$

Let $x = -1$ (2)

$0 = 1 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \dots + (-1)^n {}^nC_n$ as required.

6) (i) If A and B are included, I need only choose 2 from the other eight

${}^8C_2 = 28$ ways (3)

(ii) If A and B are excluded, I have 4 choose 4 from the remaining 8

${}^8C_4 = \frac{8 \times 7 \times 6 \times 5}{4 \times 3 \times 2 \times 1}$

$= 70$ ways (3)

a) $p = \text{win } \frac{1}{10}$

$q = \text{lose } \frac{9}{10}$

$(p+q)^{12}$ select the p^2 term.

${}^{12}C_{10} p^2 q^{10}$ (4)

$= \frac{12 \times 11}{2 \times 1} \times \left(\frac{1}{10}\right)^2 \times \left(\frac{9}{10}\right)^{10}$

$= \frac{66 \times 9^{10}}{10^{12}}$

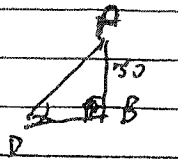
Question 6

(i) $X\hat{P}D = P\hat{D}B$ (alternate $XP \parallel PB$) (1)

$Y\hat{P}C = P\hat{C}B$ (" $PY \parallel BC$) (1)

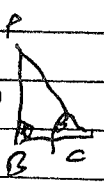
$T\hat{O}B = 220° - 120° = 100°$ (given bearings) (1)

(ii)



$\cot \alpha = \frac{DB}{150}$

$DB = 150 \cot \alpha$



$\cot \beta = \frac{BC}{150}$

$BC = 150 \cot \beta$ (2)

(iii) By cosine rule

$CD^2 = DB^2 + BC^2 - 2 DB BC \cos \theta$

Question 7

a) $\frac{dT}{dt} = k(T-25)$

(i) $T = 25 + Ae^{kt}$ $\quad Ae^{kt} = T-25$

$\frac{dT}{dt} = kAe^{kt}$

$= k(T-25)$ as required. (1)

(ii) When $t=0$ $\quad T=95$

$95 = 25 + A$ $\quad A = 70$ ... (1)

when $t=20$ $\quad T=65$

$65 = 25 + 70e^{20k}$

$\frac{40}{70} = e^{20k}$

$k = \frac{\ln \frac{4}{7}}{20}$ (1)

$= -0.02798$

(iii) when $t=30$

$T = 25 + 70e^{30k}$

$= 55°$ (2)

b) $f(x) = x\sin^{-1}x$

(i) $f(x) = x\sin^{-1}x$

$f(-x) = -x\sin^{-1}(-x)$

$= -x \times -\sin^{-1}(x)$

$= x\sin^{-1}x$

$\therefore$ function is even. (2)

(iii) $f'(x) = \frac{x}{\sqrt{1-x^2}} + \sin^{-1}x$

$f'(x) = 0$ when $x=0$

$\therefore$ T.P. @ $(0,0)$

x	$-\frac{1}{2}$	0	$\frac{1}{2}$
y	-0.1	0	0.1
slope	\	$-$	/

Min T.P. (2)

(iii) domain $-1 \le x \le 1$

range $0 \le y \le \frac{\pi}{2}$ (2)

(iv) (1)