THE HILLS GRAMMAR SCHOOL

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

2003

MATHEMATICS

EXTENSION 1

Time Allowed: Two hours (plus 5 minutes reading time)

Teacher Responsible: Mr D Price

SPECIAL INSTRUCTIONS:

- This paper contains 7 questions. ALL questions to be attempted.
- ALL questions are of equal value.
- ALL necessary working should be shown in every question in the booklets provided.
- Start each question in a new booklet.
- A table of standard integrals is supplied at the back of this paper.
- Board approved calculators may be used.
- Hand up your paper in ONE bundle, together with this question paper.
- ALL HSC course outcomes are being assessed in this task. The Course Outcomes are listed on the back of this sheet.

Students are advised that this is a Trial Examination only and cannot in any way guarantee the content or the format of the Higher School Certificate Examination.

Question One **Marks**

(a) Evaluate $\int_0^{2\sqrt{3}} \frac{dx}{4+x^2}$ 2

(b) Differentiate $\cos^3 x$ 2

(c) Find the point which divides the line joining (4, 6) to (13, 5) externally in the ratio 4:1 2

(d) Write down the equation of the vertical asymptote of $y = \frac{2x}{3x-1}$ 1

(e) Solve for x: $\frac{3}{x+5} \le 1$ 2

(f) Evaluate $\int_0^{\frac{1}{\sqrt{2}}} \frac{2x^3}{\sqrt{1-x^4}}\,dx$ using the substitution $u = x^4$ 3

Question Two (Start a NEW booklet) Marks

(a) Evaluate $\lim_{x \to 0} \frac{\tan 2x}{4x}$ 1

(b) Solve the equation

$\sin\theta + \sqrt{3}\cos\theta = 1$ for $0 \le \theta \le 2\pi$ 4

(c) Air is being pumped into a spherical balloon at the rate of $450\text{cm}^3\,\text{s}^{-1}$. Calculate the rate at which the radius of the balloon is increasing at the instant when the radius reaches 15cm. $\left[V = \frac{4}{3}\pi r^3\right]$ 3

(d) Let $f(x) = \cos x - \ln x$ 4

(i) Show that a root to $f(x) = 0$ lies between 0·5 and 1·5.

(ii) Starting with a value of $x = 1$, use one application of Newton's method to find a better approximation to this root of $f(x) = 0$.

Question Three (Start a NEW booklet) Mar

(a) The region R is bounded by the curve $y = \cos x$, $x = 0$, $x = \frac{\pi}{2}$ and the x-axis. :

(i) Sketch R.

(ii) Find the exact volume of the solid generated when the region R is rotated about the x-axis.

(b) If α, β, γ, are the roots of the cubic polynomial equation $x^3 + 4x^2 - 6x - 8 = 0$ find the value of $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ 3

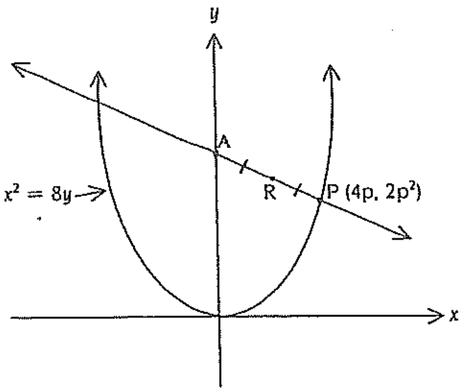
(c) Find the term which is independent of x in the expansion of $\left(2x^3 + \frac{1}{3x^2}\right)^5$ 3

(d) The remainder when $x^3 + ax + b$ is divided by $(x-2)(x+3)$ is $2x+1$. Find the values of a and b. 3

Question Four (Start a NEW booklet)

Marks

(a) 7



$P\,(4p, 2p^2)$ is a variable point on the parabola $x^2 = 8y$ as shown in the diagram above.

The normal at P cuts the y-axis at A and R is the midpoint of AP.

(i) Show that the normal at P has equation $x + py = 4p + 2p^3$

(ii) Show that R has coordinates $(2p,\ 2p^2 + 2)$

(iii) Show that the locus of R is a parabola and show that the vertex of this parabola is the focus of the parabola $x^2 = 8y$.

(b) (i) Evaluate $\int_1^3 \frac{dx}{x}$ 5

(ii) Use Simpson's rule with 3 function values to approximate $\int_1^3 \frac{dx}{x}$

(iii) Use your results to parts (i) and (ii) to obtain an approximation for e. Give your answer correct to 3 decimal places.

Question Five (Start a NEW booklet)

Marks

(a) Evaluate $\cos\left[\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right]$ 1

(b) When the temperature T of a certain body is 65°C it is cooling at the rate of 1°C per minute. 8

Assuming Newton's law of cooling: $\frac{dT}{dt} = -k(T - S)$ where

T is the temperature of the body at time t minutes
S is the temperature of the surrounding medium, assumed constant
k is a constant

(i) Show that $T = S + Ae^{-kt}$ is a solution of the given differential equation, where A is also a constant.

(ii) Show that the value of k is 0·02 given that S is 15°C.

(iii) Find T when $t = 20$ minutes, giving your answer to the nearest degree. (You may assume that initially T = 65)

(iv) How long will it take for the temperature of the body to fall to 35°C?

(b) The acceleration of a particle P, moving along a straight line has an acceleration given by 3

$$\frac{d^2x}{dt^2} = -4\left(x + \frac{16}{x^3}\right)$$

Given that P is initially at rest at the point $x = 2$, show that the velocity v at any time is given by

$$v^2 = 4\left(\frac{16 - x^4}{x^2}\right)$$

Question Six (Start a NEW booklet) Marks

(a) Prove by induction that, for all integers $n \geq 1$, 3

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$$

(b) Let $f(x)=x^2+6x$ for $x \geq -3$ 6

(i) Write down the range of $f(x)$.

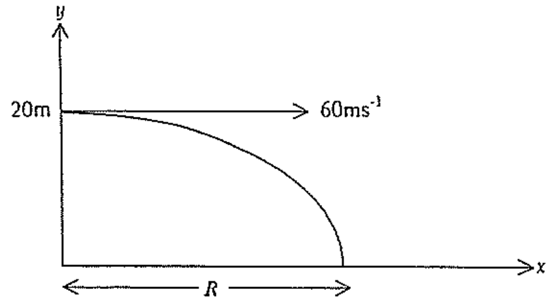
(ii) Briefly explain why the inverse function $f^{-1}(x)$ exists. Write down the domain and range of $f^{-1}(x)$.

(iii) Find $f^{-1}(x)$. Sketch the graph of $y=f^{-1}(x)$.

(c) Sketch the graph of $y=3\cos^{-1}(\frac{x}{2}-1)$. 3

Question Seven (Start a NEW booklet) Ma

(a) 8



An arrow is fired horizontally with a speed of 60ms^{-1} from the top of a 20m high wall on level ground as represented in the diagram above.

It is given that $\ddot{x}=0$ and $\ddot{y}=-10$ where (x, y) is the position of the arrow at time t seconds after firing.

(i) Using calculus, show that $x=60t$ and $y=20-5t^2$.

(ii) Find the time taken for the arrow to hit the ground.

(iii) Find the distance R metres from base of the wall where the arrow hits the ground.

(iv) Find the acute angle to the horizontal at which the arrow hits the ground.

(b) It is given that:

$$(1+x)^{2n}=\sum_{k=0}^{2n}\binom{2n}{k}x^k$$

(i) Show that $\sum_{k=0}^{2n}\binom{2n}{k}=4^n$

(ii) $\sum_{k=0}^{2n}\binom{2n}{k}\frac{1}{k+1}=\frac{4^{n+1}-2}{4n+2}$

END OF EXAMINATION

Trial HSC EXT1 2003

(a)
$$\int_0^{2\sqrt{3}} \frac{dx}{4+x^2}$$

$= \left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_0^{2\sqrt{3}}$ by table of standard integrals

$= \frac{1}{2}\tan^{-1}\sqrt{3} - 0$

$= \frac{\pi}{6}$ (2)

(b)
$\frac{d}{dx}(\cos^3 x)$

$= 3\cos^2 x \cdot (-\sin x)$

$= -3\cos^2 x \sin x$ (2)

(c)
$(4,6)$, $(13,5)$, $4:-1$

Let (x_r, y_r) be this pt.

$x_r = \frac{-4+52}{3}$

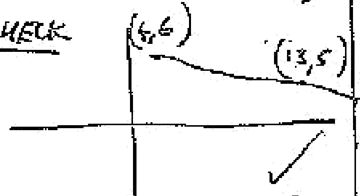
$= \frac{48}{3} = 16$

$y_r = \frac{-6+20}{3}$

$= +\frac{14}{3}$ or $+4\frac{2}{3}$

$\therefore$ pt is $(16, +4\frac{2}{3})$

CHECK



(2)

(d)
$y = \frac{2x}{3x-1}$

Vertical asymptote is $x = \frac{1}{3}$ (1)

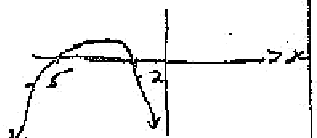
(e)
method 1

$\frac{3}{x+5} \le 1$ $\quad (x \ne -5)$ $\quad \times (x+5)^2$

$3(x+5) \le (x+5)^2$

$(x+5)[3-(x+5)] \le 0$

$(x+5)(-2-x) \le 0$



$x \ne -5$

$\therefore$ $x < -5$ or $x \ge 2$

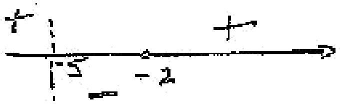
method 2 sign diagram

$\frac{3}{x+5} \le 1$

$\Rightarrow \frac{3}{x+5} - 1 \le 0$

$\frac{-2-x}{x+5} \le 0$

$\frac{x+2}{x+5} \ge 0$



solution is

$x < -5$ or $x \ge 2$ (2)

(f)
$u = x^4$

$du = 4x^3\,dx$

$\Rightarrow \frac{1}{2}du = 2x^3\,dx$

$x = 0 \Rightarrow u = 0$

$x = \frac{1}{\sqrt{2}} \Rightarrow u = \frac{1}{4}$

$I = \int_0^{1/4} \frac{1}{\sqrt{1-u}} \cdot \frac{1}{2}\,du$

$= \left[-\sqrt{1-u}\right]_0^{1/4}$

$= \sqrt{1} - \sqrt{\frac{3}{4}}$

$= 1 - \frac{\sqrt{3}}{2}$ (2)

$\lim_{x \to 0} \frac{\tan 2x}{4x}$

$= \frac{1}{2}\lim_{x \to 0}\frac{\tan 2x}{2x}$

$= \frac{1}{2} \times 1$

$= \frac{1}{2}$ (1)

(b)
method 1

Let $\sin\theta + \sqrt{3}\cos\theta \equiv R\sin(\theta+\alpha)$, $R>0$, $0 \le \alpha \le \pi$

$\theta = 0$: $R\sin\alpha = \sqrt{3}$

$\theta = \frac{\pi}{2}$: $R\cos\alpha = 1$ $\quad$ *

$\therefore R^2\sin^2\alpha + R^2\cos^2\alpha = 3+1$

$R^2 = 4 \Rightarrow R = 2$

also $0 < \alpha < \frac{\pi}{2}$ from *

and $\tan\alpha = \sqrt{3}$

$\Rightarrow \alpha = \frac{\pi}{3}$

$\therefore$ equation becomes

$2\sin(\theta + \frac{\pi}{3}) = 1$

$\sin(\theta + \frac{\pi}{3}) = \frac{1}{2}$

Now $0 \le \theta \le 2\pi$

$\Rightarrow \frac{\pi}{3} \le \theta + \frac{\pi}{3} \le 2\pi + \frac{\pi}{3}$

So $\theta + \frac{\pi}{3} = \frac{5\pi}{6}, 2\pi + \frac{\pi}{6}$

$\therefore \theta = \frac{\pi}{2}, \frac{11\pi}{6}$ (4)

method 2 t-formulas

Let $t = \tan\frac{\theta}{2}$

TEST $\theta = \pi$ (why?)

LHS $= 0 - \sqrt{3}$

RHS $= 1$

$\theta \ne \pi$

Probably the more difficult method, but it can be done!

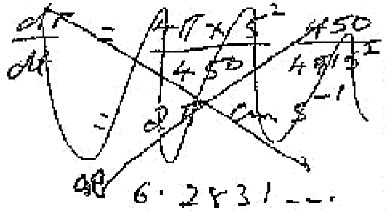
(c)
$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$

$= 4\pi r^2 \frac{dr}{dt}$

$450 = 4\pi \times 15^2 \frac{dr}{dt}$

~~$\frac{dr}{dt} = \frac{4\pi \times 15^2}{450}$~~



$6.2831\ldots$

$\frac{dr}{dt} = \frac{450}{4\pi \times 15^2}$

$= \frac{1}{2\pi}$ or $0.159\ldots$ cm s^{-1} (3)

(d)
(i) $f(0.5) = \cos 0.5 \ldots$

$= 1.57\ldots$

$f(1.5) = \cos 1.5 - \ldots$

$= -0.334$

$f(x)$ is continuous for $0.5 < x < 1.5$

$\Rightarrow$ there exists x in this open interval for $f(x)$ (1)

(ii) Formula is $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

(or derive formula quickly)

$x_1 = 1 - \frac{f(1)}{f'(1)}$

$f(1) = \cos 1 - \ln 1$

$= \cos 1$

$f'(x) = -\sin x - \frac{1}{x}$

$f'(1) = -\sin 1 - 1$

$x_1 = 1 - \frac{\cos 1}{-\sin 1 \ldots}$

$= 1 + \frac{\cos 1}{\sin 1 + 1}$

$= 1.293\ldots$

CHECK is this a better approx (NOT required to show)

$f(1) = \cos 1 > 0$

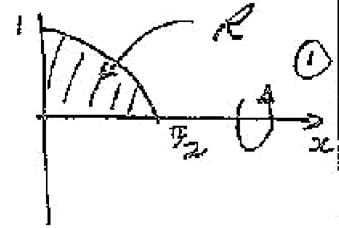
$f''(x) = -\cos x + \frac{1}{x^2}$

$f''(1) = -\cos 1 + 1 > 0$

So $x = 1.293\ldots$ is a better approx.

3.

(i)



(1)

(ii)

$V = \pi \int_0^{\frac{\pi}{2}} \cos^2 x \, dx$

$2\cos^2 x = \cos 2x + 1$

$\therefore V = \pi \int_0^{\frac{\pi}{2}} (1 + \cos 2x) dx$

$= \pi \left[x + \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{2}}$

$= \pi\left(\frac{\pi}{2} + \frac{1}{2}\sin \pi\right)$

$= \frac{\pi^2}{2}$ unit^3 (2)

(b)

$x^3 + 4x^2 - 6x - 8 = 0$

$\alpha + \beta + \gamma = -4$

$\alpha\beta + \beta\gamma + \gamma\alpha = -6$

$\alpha\beta\gamma = 8$

$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

$= \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$

$= \frac{-6}{8}$

$= -\frac{3}{4}$ (3)

(c) $\left(2x^3 + \frac{1}{3x^2}\right)^5$

method 1

General term

$= \binom{5}{r}(2x^3)^{5-r}\left(\frac{1}{3x^2}\right)^r$

$= \binom{5}{r}\frac{2^{5-r}}{3^r} \frac{x^{15-3r}}{x^{2r}}$

$= \binom{5}{r}\frac{2^{5-r}}{3^r} x^{15-5r}$

choose $r = 3$

term $= \binom{5}{3}\frac{2^2}{3^3}$

$= \frac{40}{27}$

method 2

$\left(2x^3 + \frac{1}{3x^2}\right)^5$

$= (x^3)^5\left(2 + \frac{1}{3x^5}\right)^5$

~~Transform~~

$= x^{15}\left(2 + \frac{1}{3x^5}\right)^5$

General term

$= \binom{5}{3}\frac{2^2}{3^3} = \frac{40}{27}$

(d)

$x^3 + ax + b$

$= (x-2)(x+3)Q(x) + 2x + 1$

Let $P(x) = x^3 + ax + b$

$P(2) = 8 + 2a + b$
$= 2 \times 2 + 1$

$\boxed{2a + b = -3}$

$P(-3) = -27 - 3a + b$
$= 2 \times -3 + 1$

$\boxed{-3a + b = 22}$

Solving gives

$a = -5$

$b = 7$

4.

(i)

$x^2 = 8y \Rightarrow y = \frac{1}{8}x^2$

$y' = \frac{1}{4}x$

m_{tangent} at $(4p, 2p^2)$

is $\frac{1}{4} \times 4p = p$

$m_{\perp} = -\frac{1}{p}$

$y - 2p^2 = -\frac{1}{p}(x - 4p)$

$py - 2p^3 = -x + 4p$

$x + py = 4p + 2p^3$ (3)

(ii)

$A = (0, 4 + 2p^2)$ $(p \neq 0)$

$R = \left(\frac{0 + 4p}{2}, \frac{4 + 2p^2 + 2p^2}{2}\right)$

$= (2p, 2p^2 + 2)$ (1)

(iii)

Let $x = 2p$

$y = 2p^2 + 2$

$= 2 \times \left(\frac{x}{2}\right)^2 + 2$

$= \frac{1}{2}x^2 + 2$

$\therefore y - 2 = \frac{1}{2}x^2$

parabola

Vertex $= (0, 2)$

The focus of $x^2 = 8y$

is $x^2 = 4 \times 2 \times y$

$\therefore$ focus $= (0, 2)$ (3)

Hence result.

(b)

(i) $\int_1^3 \frac{dx}{x}$

$= \left[\ln x\right]_1^3$

$= \ln 3$ (1)

(ii)

x	1	2	3
$1/x$	1	1/2	1/3

$\int_1^3 \frac{dx}{x} \doteqdot \frac{h}{3}\left[y_0 + 4y_1 + y_2\right]$

$= \frac{1}{3}\left[1 + 4 \times \frac{1}{2} + \frac{1}{3}\right]$

$= \frac{10}{9}$ (2)

(iii) $\ln 3 \doteqdot \frac{10}{9}$

$e^{10/9} = 3$

$\left(e^{10/9}\right)^{9/10} = 3^{9/10}$

$e \doteqdot 3^{0.9}$

$= 2.68787\ldots$

(2)

5. (a) $\cos\left(\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right)$

$= \cos\left(-\frac{\pi}{6}\right)$

$= \cos \frac{\pi}{6}$

$= \frac{\sqrt{3}}{2}$ (1)

(b)

(i) LHS $=$

$\frac{dT}{dt} = \frac{d}{dt}\left(5 + Ae^{-kt}\right)$

$= 0 + A \times (-k)e^{-kt}$

$= -Ake^{-kt}$

RHS $=$

$-k(T - 5)$

$= -k(5 + Ae^{-kt} - 5)$

$= -Ake^{-kt}$

(ii)

$\frac{dT}{dt} = -k(T -$

$T = 65$, $\frac{dT}{dt} =$

$-1 = -k(65 -$

$k = \frac{1}{50} = 0$

ii)

$T = 15 + Ae^{-0.02t}$

When $t=0$, $T=65$

$65 = 15 + A$

$A = 50$

$\therefore T = 15 + 50e^{-0.02t}$

$t = 20$

$T = 15 + 50e^{-0.02\times 20}$

$= 15 + 50e^{-0.4}$

$= 48.51\ldots$

$\doteqdot 49°$ (2)

iv)

$T = 15 + 50e^{-0.02t}$

$T = 35 \quad t = ?$

$35 = 15 + 50e^{-0.02t}$

$e^{-0.02t} = \frac{2}{5}$

$t = \frac{-1}{0.02}\ln 0.4$

$= 45.81\ldots$

$\doteqdot 46$ minutes

(3)

(b) $\frac{d^2x}{dt^2} = -4\left(x + \frac{16}{x^3}\right)$

$\frac{d^2x}{dt^2} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -4(x + 16x^{-3})$

$\frac{1}{2}v^2 = -4\left(\frac{1}{2}x^2 + 16\frac{x^{-2}}{-2}\right) + c$

$v^2 = -4\left(x^2 - \frac{16}{x^2}\right) + c'$

$v = 0, \quad x = 2$

$\Rightarrow$

$0 = -4\left(2^2 - \frac{16}{2^2}\right) + c'$

$= 0 + c'$

$\Rightarrow c' = 0$

$\therefore v^2 = -4\left(x^2 - \frac{16}{x^2}\right)$

$= 4\left(\frac{16 - x^4}{x^2}\right)$

(3)

6.

(a) $\frac{1}{1\times 2} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

For $n=1$

$LHS = \frac{1}{1\times 2} = \frac{1}{2}$

$RHS = \frac{1}{1+1} = \frac{1}{2}$

$\therefore$ true for $n=1$

Suppose statement true for $n=k$

ie $\frac{1}{1\times 2} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1}$

Consider $n = k+1$

$\frac{1}{1\times 2} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)}$

$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$ by inductive hypothesis.

$= \frac{k(k+2)+1}{(k+1)(k+2)}$

$= \frac{(k+1)^2}{(k+1)(k+2)}$

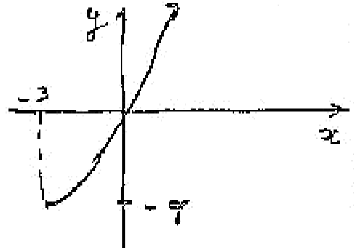
$= \frac{k+1}{k+2}$

So result true for $n=k \Rightarrow$ result true for $n=k+1$

$\therefore$ by M.I. true for $n = 1, 2, 3, \ldots$

5(b)

(i) $f(x) = x(x+6)$



Range: $f(x) \ge -9$ (1)

(ii) $f^{-1}(x)$ exists as for $x \ge -3$ $f(x)$ is increasing. (1)

Domain of $f^{-1}(x)$ ~~[illegible]~~ is $x \ge -9$ (1)

(iii) $y = x(x+6)$

$= x^2 + 6x$

$x \ge -3, \quad y \ge -9$

$f^{-1}(x): \quad x \leftrightarrow y$

$x = y^2 + 6y$

$x \ge -9$ and $y \ge -3$

$y^2 + 6y - x = 0$

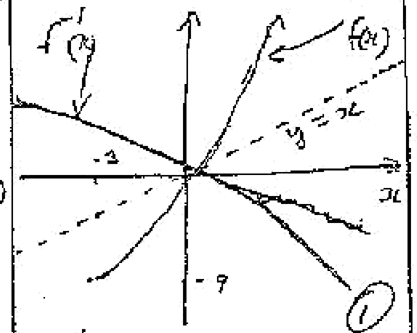
$y = \frac{-6 \pm \sqrt{36 - 4x}}{2}$

$= -3 \pm \sqrt{9 - x}$

but $y \ge -3$

so $y = -3 + \sqrt{9 - x}$

$\Rightarrow f^{-1}(x) = -3 + \sqrt{9 - x}$ (2)



(1)

(c) $y = 3\cos^{-1}\left(\frac{x}{2} - 1\right)$

Domain

$-1 \le \frac{x}{2} - 1 \le 1$

$0 \le \frac{x}{2} \le 2$

$0 \le x \le 4$

Range

$0 \le y \le 3\pi$

7

(i) $\ddot{x} = 0$

$\therefore \dot{x} = c$

at $t=0$ $\dot{x} = 60$

$\therefore \dot{x} = 60$

$x = 60t + c$

at $t=0$, $x=0$

$\Rightarrow c = 0$

$\therefore \boxed{x = 60t}$

$\ddot{y} = -10$

$\dot{y} = -10t +$

at $t=0$ $\dot{y} =$

$\Rightarrow c = 0$

$\therefore \dot{y} = -10t$

$y = -5t^2 +$

at $t=0$, y

$\Rightarrow c = 20$

$\therefore y = 20 - 5$

(ii) at ground

$(x, y) = (R, 0)$

Solve $y = 0$

$20 - 5t^2 =$

$t^2 = 4$

$t > 0 \Rightarrow$

(iii) at $(R, 0)$

$\therefore x(2)$

v) at ground
vector diagram

$t = 2$

$\dot{y} = -10 \times 2$
$= -20$

$\dot{x} = 60$

$\tan\theta = \left|\frac{\dot{y}}{\dot{x}}\right|$

$= \frac{20}{60}$

$= \frac{1}{3}$

$\theta = \tan^{-1}\frac{1}{3}$ (2)

$\doteq 18.43\ldots$

$\underline{\underline{18^\circ}}$

(b)

$(1+x)^{2n} = \sum_{0}^{2n} \binom{2n}{k} x^{k}$

(i) put $x = 1$

$2^{2n} = \sum_{0}^{2n} \binom{2n}{k} 1^{k}$

$4^{n} = \sum_{0}^{2n} \binom{2n}{k}$ (1)

(ii) integrating

$\frac{(1+x)^{2n+1}}{2n+1} + c = \sum_{0}^{2n} \binom{2n}{k} \frac{x^{k+1}}{k+1}$

put $x = 0$

$\frac{1}{2n+1} + c = 0$

$\Rightarrow c = -\frac{1}{2n+1}$

$\therefore \sum_{0}^{2n} \binom{2n}{k} \frac{x^{k+1}}{k+1}$

$= \frac{(1+x)^{2n+1}}{2n+1} - \frac{1}{2n+1}$

put $x = 1$

$\sum_{0}^{2n} \binom{2n}{k} \frac{1}{k+1}$

$= \frac{2^{2n+1} - 1}{2n+1}$

$= \frac{2 \times 4^{n} - 1}{2n+1} \times \frac{2}{2}$

$= \frac{4 \times 4^{n} - 2}{4n+2}$

$= \frac{4^{n+1} - 2}{4n+2}$ (3)