THE HILLS GRAMMAR SCHOOL

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

2004

MATHEMATICS

EXTENSION 1

Teacher Responsible: Mrs P Singh

General Instructions:

- Reading time – 5 minutes
- Working time – 2 hours
- This paper contains 7 questions. ALL questions to be attempted.
- ALL questions are of equal value.
- ALL necessary working should be shown in every question in the booklets provided.
- Start each question in a new booklet.
- A table of standard integrals is supplied at the back of this paper.
- Board approved calculators may be used.
- Hand up your paper in ONE bundle, together with this question paper.
- ALL HSC course outcomes are being assessed in this task. The Course Outcomes are listed on the back of this sheet.

Question One — Marks

(a) Evaluate $\log_8 e$, correct to 2 decimal places. **1**

(b) Differentiate $y = \cos^3 x$. **2**

(c) If $\tan\theta = m$ and $\tan\phi = 3$, find the value of m if $\theta - \phi = \frac{\pi}{4}$. **3**

(d) $P(x) = x^3 - 3x^2 - 3x + 10$

  (i) Show that $x = 2$ is a root of $P(x)$. **1**

  (ii) What is the product of the other 2 roots? **1**

(e) Evaluate $\int_0^{\frac{\pi}{3}} \tan^2 x \, dx$ (leave answer in exact form). **3**

(f) Write down the equation of the vertical asymptote of $y = \frac{3x}{2x-1}$. **1**

$2x - 1 \neq 0$

$2x \neq 1$

$x \neq \frac{1}{2}$

$x = \frac{1}{2}$

$y = \frac{3}{2 - \frac{1}{x}}$

when $x \to \infty$, $y \to \frac{3}{2}$

$x \to -\infty$, $y \to \frac{3}{2}$

$y = \frac{3}{2}$

Question Two

(a) Find the point which divides the line joining (4, 6) to (13, 5) externally in the ratio 4:1. **2**

(b) If the equation $3x^3 - 4x^2 + 2x + 1 = 0$ has roots α, β and λ, find

(i) $2\alpha + 2\beta + 2\lambda$ **1**

(ii) the equation whose roots are 2α, 2β and 2λ. **2**

(c) Solve for x: $\dfrac{x^2 - 5x}{4 - x} \le -3$. **3**

(d) Sand pouring from a pipe at a rate of $16\text{m}^3\ \text{min}^{-1}$ forms a conical pile with height always equal to one quarter of the diameter of the base. (Volume of cone $= \frac{1}{3}\pi r^2 h$). How fast is the height of the pile rising at the instant when the pile is 4m high? **4**

Question Three

(a) Show that the function $f(x) = 5x - \sin 4x - 12$ is increasing for all values of x. **2**

(b) (i) Express $\sin x - \sqrt{3}\cos x$ in the form $A\sin(x - \alpha)$ with $A > 0$ and $0 < \alpha < \frac{\pi}{2}$. **2**

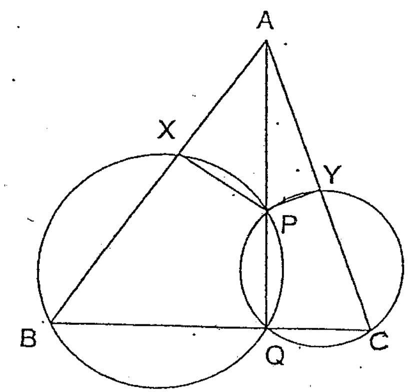
(ii) Find in exact form, the solution to

$\sin x - \sqrt{3}\cos x = \sqrt{2} \quad 0 < x < 2\pi$. **2**

(c) Prove, using the principle of Mathematical Induction,

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}.$$ **2**

(d)



AXB, AYC, APQ and BQC are straight lines. Let $\angle ABC = x$ and $\angle ACB = y$. Copy this diagram in your answer booklet.

(i) Prove $\angle XPY = x + y$; **2**

(ii) Prove AXPY is a cyclic quadrilateral. **2**

Question Four

(a) (i) Show $\frac{d}{dx}(x\log_e x - x) = \log_e x$. 2

(ii) Hence, or otherwise, evaluate $\int_1^2 \log_e x\,dx$, in exact form. 2

(b) Evaluate $\int_0^1 \sqrt{1-x^2}\,dx$, by using $x = \sin\theta$, leaving answer in exact form. 4

(c) Given: $f(x) = x^2 + 1 \quad x \geq 0$

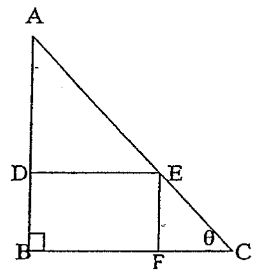
(i) Find $f^{-1}(x)$, stating the domain and range. 2

(ii) Sketch both $f(x)$ and $f^{-1}(x)$ on the same system of axes. 2

Question Five

(a) Prove that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. 3

(b) Find the dimensions of the rectangle of maximum area which can be enscribed in a right angled triangle with sides 12, 16 and 20cm. One corner of the rectangle is to sit in the right angle of the triangle (see below). Let EF = y and DE = x. 5



Copy this diagram into your answer booklet.

(c) (i) Evaluate $\int_1^3 \frac{dx}{x}$. 1

(ii) Use Simpson's Rule with 3 function values to approximate $\int_1^3 \frac{dx}{x}$. 2

(iii) Use your results to parts (i) and (ii) to obtain an approximation for e, to 3 decimal places. 1

Question Six

(a) Given $f(x) = \frac{x}{x^2 - 1}$

(i) Find the asymptotes and intercepts of $f(x)$. 3

(ii) Find the stationary points, if any, and determine their nature. 1

(iii) Find any points of inflection. 2

(iv) Sketch $f(x)$ showing the features from (i), (ii) and (iii). 3

(b) (i) In how many ways can a committee of 3 women and 4 men be chosen from 8 women and 7 men? 1

(ii) What is the number of ways a committee can be formed if woman A refuses to serve on the same committee as woman B. 2

Question Seven

(a) A particle is oscillating in simple harmonic motion such that its displacement x metres from a given origin O satisfies the equation:

$$\frac{d^2x}{dt^2} = -4x \text{ where } t = \text{time in seconds.}$$

(i) Show that $x = a\cos(2t + \beta)$ is a possible equation of motion for the particle when a and β are constants. 1

(ii) The particle is observed at time $t = 0$ to have velocity of 2 m/s and a displacement from the origin of 4 m. Find the amplitude of oscillation. 3

(iii) Determine the maximum velocity of the particle. 1

(b) A cricket ball leaves the bowler's hand two metres above the ground with a velocity of 30 m/s at an angle of 5 degrees below the horizontal. The equations of motion for the ball are

$\ddot{x} = 0$ and $\ddot{y} = -10$.

Take the origin to be at the point where the ball leaves the bowler's hand, at $t = 0$.

(i) Using calculus, prove that the coordinates of the ball at time t are given by

$x = 30t\cos(5^\circ)$, and
$y = -30t\sin(5^\circ) - 5t^2$. 2

(ii) Find the time at which the ball strikes the ground. 2

(iii) Calculate the angle at which the ball strikes the ground. 2

(c) Find the derivative of $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$. 1

END OF PAPER

Soln Marks

Q1 [12]

(a) $\frac{\ln x}{\ln 8} = 0.48$ ✓ (1) (PE1)

(b) $y' = 3\cos^2 x \cdot -\sin x = -3\cos^2 x \sin x$ ✓✓ (2) (PE5)

(c)

$\tan\theta = m$ $\tan(\theta - \phi) = 1$ ✓

$= \frac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi} = 1$

$\frac{m - 3}{1 + 3m} = 1$ ✓

$m = -2$ ✓ (3) (HE7)

(d) (i) $P(2) = 2^3 - 3(2^2) - 3(2) + 10 = 0$ ✓ (1) (PE3)

$\therefore x = 0$ is a root

(ii) $2\alpha\beta = \frac{-10}{1}$ ✓ (1)

$\alpha\beta = -5$

(e) $\int_0^{\pi/3} \tan^2 x \, dx$

$= \int_0^{\pi/3} [\sec^2 x - 1] \, dx$ ✓

$= [\tan x - x]_0^{\pi/3}$ (3) (HE4)

$= (\sqrt{3} - \pi/3) - 0$ ✓

$= \sqrt{3} - \pi/3$

f) $2x - 1 = 0$

$x = 1/2$ ✓ (1) (PE3)

-1-

Q2 (12 marks)

(a) $X_r = \frac{-4 + 52}{3}$ $Y_r = \frac{-6 + 20}{3}$

$= 16$ ✓ $= 4\frac{2}{3}$ ✓ [2, PE3]

$\therefore$ pt $(16, 4\frac{2}{3})$

(b)(i) $\alpha + \beta + \gamma = 4/3$

$\alpha\beta + \alpha\gamma + \beta\gamma = 2/3$

$\alpha\beta\gamma = -1/3$

$\therefore 2(\alpha + \beta + \gamma) = 8/3$ ✓ [1, PE3]

(ii) $x^3 - (2\alpha + 2\beta + 2\gamma)x^2 + (4\gamma\beta + 4\alpha\gamma + 4\beta\gamma)x = x^3 - \frac{8}{3}x^2 + \frac{8x}{3} + \frac{8}{3} =$ ✓

$\therefore 3x^3 - 8x^2 + 8x + 8 = 0$ ✓ (2)

(c) $\frac{x^2 - 5x}{4 - x} \le -3$ $(x \ne -4)$

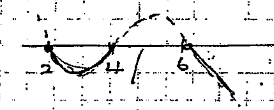
$(4 - x)(x^2 - 5x) \le -3(4 - x)^2$ ✓

$(4-x)(x^2 - 5x) + 3(4 - x)^2 \le 0$

$(4-x)[x^2 - 5x + 12 - 3x] \le 0$

$(4-x)(x^2 - 8x + 12) \le 0$

$(4-x)(x-2)(x-6) \le 0$



[3, HE1]

$\therefore 2 \le x \le 4$ ✓ and $x \ge 6$ ✓ (3)

(d) we need $\frac{dh}{dt}$

$h = 1/4 D$

$= 1/4 \cdot 2r$

$= 1/2 r$ ✓

$\therefore r = 2h$

$V = 1/3 \pi (2h)^2 h$

$V = 4/3 \pi h^3$ ✓

$\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$

$= 4\pi h^2 \cdot \frac{dh}{dt}$ ✓

$16 = 4\pi(4)^2 \cdot \frac{dh}{dt}$ [4,

$\therefore \frac{dh}{dt} = \frac{1}{4\pi}$ m³/s ✓

$\doteqdot 0.080$ m³/s (2dp)

-2-

Question 3. (12 marks)

(a) $f'(x) = 5 - 4\cos 4x$
$\geq 5 - 4(1)$ since $-1 \leq \cos 4x \leq 1$

$f'(x) \geq 1 > 0 \quad \forall x$

$\therefore f(x)$ increases $\forall x$ [2, HE 7]

b (i)

$A\sin(x-\alpha) = A\cos\alpha \sin x - A\sin\alpha\cos x$
$= \sin x - \sqrt{3}\cos x$

$\Rightarrow A\cos\alpha = 1$
$A\sin\alpha = \sqrt{3}$

$\therefore \tan\alpha = \sqrt{3}$
$\alpha = \pi/3$

$A = \sqrt{1^2 + (\sqrt{3})^2}$
$A = 2$

$\therefore 2\sin(x - \frac{\pi}{3})$

(ii) $2\sin(x - \pi/3) = \sqrt{2}$

$\sin(x - \pi/3) = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

$\therefore (x - \pi/3) = \pi/4$ or $3\pi/4$

$\therefore x = \frac{7\pi}{12}$ or $\frac{13\pi}{12}$ [4, HE 1, 7]

(c) Prove $n=1$

LHS: $\frac{1}{1(1+1)} = \frac{1}{2}$ RHS: $\frac{1}{1+1} = \frac{1}{2}$

$\therefore$ True for $n=1$

Assume true for $n=k$ & prove true for $(k+1)$

i.e. $\frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} = \frac{(k+1)}{(k+2)}$

LHS $\frac{k}{(k+1)} + \frac{1}{(k+1)(k+2)}$

$= \frac{k(k+2)+1}{(k+1)(k+2)}$

$= \frac{k^2+2k+1}{(k+1)(k+2)}$

$= \frac{(k+1)^2}{(k+1)(k+2)}$

$= \frac{(k+1)}{(k+2)}$

$=$ RHS

Since $P(k)$ and ~~Hence~~ $P(k+1)$ is true, $\therefore$ true for $n = 1, 2, \ldots$

[2, HE 2]

(3 d)

(i) $\angle XPQ = 180° - x$
$\angle YPQ = 180° - y$ and } (opp $\angle$s of cyclic quad suppl.)

$\therefore \angle XPY = 360° - [180° - x + 180° - y]$ (rev)
$= x + y$
$=$ QED.

(ii) $\angle BAC = 180° - (x+y)$ ... (sum $\angle$s of $\triangle ABC$)

now $\angle XPY + \angle BAC = (x+y) + 180° - (x+y)$
$= 180°$

$\therefore AXPY$ is cyclic (opp $\angle$s suppl) [4, PE3]

Q4 (12 marks)

(a)(i) $f(x) = x\log_e x - x$

$f'(x) = x \cdot \frac{1}{x} + \log_e x - 1$
$= 1 + \log_e x - 1$
$= \log_e x$.
$=$ QED

(ii) $\int_1^2 \log_e x \, dx$

$= \int_1^2 x\log_e x - x \, dx$

$= [x\log_e x - x]_1^2$

$= [2\log_e 2 - 2] - (-1)$

$= \log_e 4 - 1$ [4, HE3]

b) $\int_0^1 \sqrt{1-x^2}\, dx$

$x = \sin\theta$
$\therefore dx = \cos\theta \, d\theta$

Terminals
$x = 0$, $\sin\theta = 0 \therefore \theta =$
$x = 1$ $\sin\theta = 1 \therefore \theta =$

$= \int_0^1 \sqrt{1 - \sin^2\theta}\, \cos\theta\, d\theta$

$= \int_0^{\pi/2} \cos\theta \cdot \cos\theta \, d\theta$

$= \int_0^{\pi/2} \cos^2\theta \, d\theta$

$= \frac{1}{2}\int_0^{\pi/2} (1 + \cos 2\theta)\, d\theta$

$= \frac{1}{2}[\theta + \frac{1}{2}\sin 2\theta]_0^{\pi}$

$\frac{1}{2}[\pi/2]$
$= \pi/4 \; u^2$. [4 HE6]

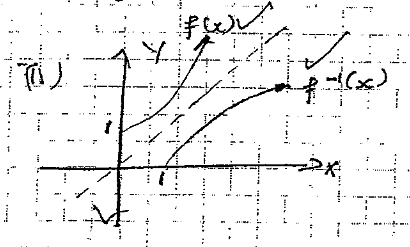
4 (c) (i) $f^{-1}(x) = y^2 + 1$

$y^2 = x - 1$ ✓

$y = +\sqrt{x-1}$

Domain $x \geq 1$ } ✓
Range $y \geq 0$

(ii)



[4, HE4]

Q5 (2 marks)

(a) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

LHS $\cos 3\theta = \cos(2\theta + \theta)$

$= \cos 2\theta \cos\theta - \sin 2\theta \sin\theta$ ✓

$= (\cos^2\theta - \sin^2\theta)\cos\theta - 2[\sin^2\theta \cos\theta]$

$= \cos^3\theta - \cos\theta \sin^2\theta - 2\sin^2\theta \cos\theta$ ✓

$= \cos^3\theta - 3\sin^2\theta \cos\theta$

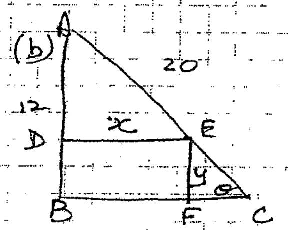
$= \cos^3\theta - 3[1 - \cos^2\theta]$ ✓

$= \cos^3\theta - 3\cos\theta + 3\cos^3$

$= 4\cos^3\theta - 3\cos\theta$

= QED. (3)

(b)



$A = xy$

$\tan\theta = 12/16 = 3/4$

$\tan\theta = \dfrac{y}{16 - x}$ ✓

$\therefore y = \frac{3}{4}(16 - x)$

$\therefore A = x \cdot \frac{3}{4}(16 - x)$ ✓

$= 12x - \dfrac{3x^2}{4}$ (5)

$A' = 0$ for max area

$-\frac{6}{4}x = -12$ ✓

$\therefore x = 8\text{cm}$

$\therefore y = 6\text{ cm}$. ✓

Check for max

$A'' = -3/4$ ✓

$\therefore \cap$ < 0

$\therefore$ max area

(PE4)

Q5

(c) (i) $\int_1^3 \frac{dx}{x}$

$= [\ln x]_1^3$

$= \ln 3$. ✓ (1)

(ii)

x	1	2	3
y	1	1/2	1/3

$\int_1^3 \frac{dx}{x} \doteq \frac{h}{3}[y_0 + 4y_1 + y_2]$ ✓ (2)

$\doteq \frac{1}{3}[1 + 4 \times \frac{1}{2} + \frac{1}{3}]$ ✓

$\doteq 10/9$

(iii) $\ln 3 \doteq \frac{10}{9}$

$e^{\frac{10}{9}} \doteq 3$

$e \doteq 3^{9/10}$ (1)

$\doteq 3^{0.9}$ ✓ (HE7)

$\doteq 2.688$ (3dp)

$f(x) = \dfrac{x}{x^2-1}$

(i) vertical asymptote $\lim_{x\to+1} f(x) = \pm\infty$ $\lim_{x\to 1^-} f(x) = \pm\infty$

$\therefore x = \pm 1$ ✓ (3)

Horizontal

$\lim_{x\to\pm\infty} \dfrac{x}{x^2-1} = 0$

$\therefore y = 0$ ✓ intercept $(0,0)$ ✓ (HE4)

(ii) $f'(x) = \dfrac{(x^2-1).1 - x.2x}{(x^2-1)^2}$

$= \dfrac{-x^2-1}{(x^2-1)^2}$

$\therefore f'(x) = 0$

$-x^2 = 1$

$x^2 = -1$ No S.P. ✓ (1)

(iii) $f''(x) = \dfrac{2x(x^2-1)^2 - (-x^2-1)\,2(x^2-1).2x}{(x^2-1)^4}$

$= \dfrac{(x^2-1)(-2x^3+2x+4x^3+4x)}{(x^2-1)^4}$ ✓

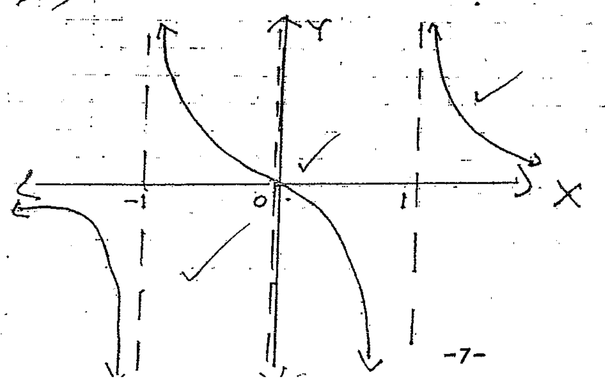
$= \dfrac{2x(x^2+3)}{(x^2-1)^3}$ (2)

Pts inflexion

$f''(x) = 0$ ✓

$\Rightarrow x = 0$ $x^2 = -3$

$(0,0)$ (no soln)



(3)

(HE4)

Q6.

(b) (i) ${}^8C_3$ (women) $\times$ ${}^7C_4$ men

$= 1960$ ways. ✓ (1)

(ii) For both women A + B ${}^6C_1$ ways

$\therefore {}^6C_1 \times {}^7C_4 = 210$ ✓

without both serving:

$1960 - 210$

$= 1750$ ways ✓ (2)

(PE3)

Q7

(a)(i) $y = \left[x + \left(x + (x)^{\frac{1}{2}}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}$ ✓ (1)

$y' = \frac{1}{2}\left[x + (x + (x)^{\frac{1}{2}})^{\frac{1}{2}}\right]^{-\frac{1}{2}} . \left[1 + \frac{1}{2}\left(x + (x)^{\frac{1}{2}}\right)^{-\frac{1}{2}}\right] . \left[1 + x^{-\frac{1}{2}}\right]$ (PE5)

(b) (i) $x = a\cos(2t+\beta)$

$\dot{x} = -2a\sin(2t+\beta)$

$\ddot{x} = -4a\cos(2t+\beta)$ ✓ [prove $\ddot{x} = -4x$] (1)

$= -4x$

(ii) $t=0$, $\dot{x} = 2$ m/s $x = 4$ m a?

$4 = a\cos\beta$ ... ①

$2 = -2a\sin\beta$

$1 = -a\sin\beta$ ... ② ✓

(3)

①^2 + ②^2 $4^2 + 1^2 = a^2\cos^2\beta + a^2\sin^2\beta$ ✓ [$\sin^2\beta + \cos^2\beta = 1$]

$17 = a^2$

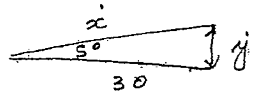
$\therefore a = \sqrt{17}$ ✓

(iii) $V = -2\sqrt{17}\sin(2t+\beta)$ (1)

$\therefore$ max $\dot{x} = \pm 2\sqrt{17}$ ms^{-1} ✓

(HE3)

(b)



when $t = 0$

$\dot{x} = 30\cos(5°)$, $\dot{y} = -30\sin 5°$ ... ①

(i) $\ddot{x} = 0$

$\therefore \dot{x} = C_1$ (constant)

$\therefore \dot{x} = 30\cos 5°$ ... ② from ①

$x = \int 30\cos 5°\, dt$ ✓

$x = 30t\cos 5° + C_2$

when $t = 0$, $x = 0$, $\therefore C_2 = 0$

$\therefore \boxed{x = 30t\cos(5°)}$

$\ddot{y} = -10$

$\therefore \dot{y} = \int -10\, dt$

$= -10t + D_1$ ...

when $t = 0$, $\dot{y} = -30\cos(5°)$ ... from ①

$\therefore D_1 = -30\sin(5°)$

$\dot{y} = -10t - 30\sin(5°)$ ... ③

$y = \int -10t - 30\sin(5°)\, dt$ ✓

$= -5t^2 - 30t\sin(5°) + D_2$ (2)

When $t = 0$, $y = 0$ $\therefore D_2 = 0$

$\therefore \boxed{y = -30t\sin(5°) - 5t^2}$ ... ④

(ii) Ball strikes the ground when $y = -2$

sub $y = -2$ in ④

$-2 = -30t\sin 5° - 5t^2$

$5t^2 + 30t\sin 5° - 2 = 0$

$t = \frac{-30\sin 5° \pm \sqrt{(-30\sin 5°)^2 - 4\times5\times(-2)}}{2\times5}$ ✓

$= \frac{-30\sin 5° + \sqrt{900\sin^2 5° + 40}}{10}$ (ignore −ve) (2)

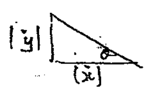
$= 0.4229$ sec

$\doteqdot 0.42$ sec ✓

(iii) $t = 0.42$ sec

$\dot{x} = 30\cos 5°$ fro[m]

$\dot{y} = -4.229 - 30\sin 5°$



$\tan\theta = \frac{\dot{y}}{\dot{x}}$ ✓ (

$\theta \doteqdot 12.9°$

$\therefore \theta \doteqdot 13°$ ✓