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QUESTION ONE (12 marks)

a) Evaluate **2**

$$\int_0^3 \frac{dx}{x^2+9}.$$

b) Find the acute angle between lines $2x - y = 0$ and $x + 3y = 0$. **3**
Give your answer correct to the nearest minute.

c) Solve $\dfrac{x}{2x-1} \leq 5$. **3**

d) Differentiate $\cos^4 x$ with respect to x and hence find $\int \sin x \cos^3 x \, dx$. **2**

e) A is the point (–2,–1) and B is the point (1,5). **2**
Find the coordinates of Q which divides AB internally in the ratio 5:3.

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QUESTION TWO (12 marks)

a) Evaluate $\displaystyle\int_0^{\frac{\pi}{3}} \sin^2(2x)\, dx$. **3**

b) Use the substitution $u = \tan x$, to evaluate $\displaystyle\int_0^{\frac{\pi}{3}} \tan^2 x \sec^2 x \, dx$. **3**

c) A biased coin has a probability of 0.6 of showing a head when it is tossed. The biased coin is tossed five times.

(i) What is the probability of tossing exactly three heads and two tails? **2**

(ii) What is the probability of getting less than three heads? **2**

(iii) What is the most likely outcome and what is the probability that this event will occur? **2**

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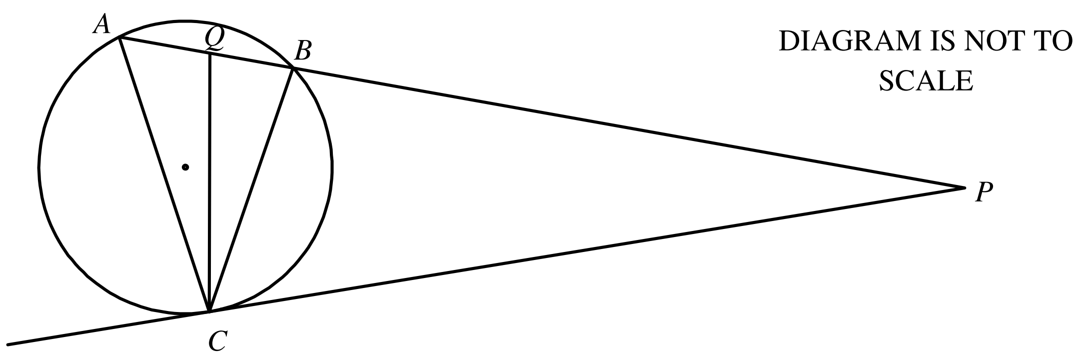
QUESTION THREE (12 marks)

a) Find the exact value of $\sin\left(\sin^{-1}\left(\frac{4}{5}\right)-\tan^{-1}\left(\frac{-5}{12}\right)\right)$. **3**

b) In the diagram below PC is a tangent to the circle at C and QC bisects $\angle ACB$.



(i) Copy the diagram into your answer booklet and prove that $PC = PQ$. **2**

(ii) If $PC = 10$cm and $AQ = 2$cm, Find the length of QB. **2**

c) Molten plastic at a temperature of 250° C, is poured into a mould to form a car part. After 20 minutes the plastic has cooled to 150° C . If the temperature after t minutes, is T ° C, and the surrounding temperature is 30° C, then the rate of cooling is given by:

$$\frac{dT}{dt} = -k\left(T - 30\right), \text{ where } k \text{ is constant.}$$

(i) Show that $T = 30 + Ae^{-kt}$, where A is a constant, satisfies this equation. **1**

(ii) Show that the value of A is 220° C . **1**

(iii) Find the value of k to 2 decimal places. **1**

(iv) The plastic can be taken out of the mould when the temperature drops below 80° C. How long after the plastic has been poured will this temperature be reached? Give your answer to the nearest minute. **2**

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QUESTION FOUR (12 marks)

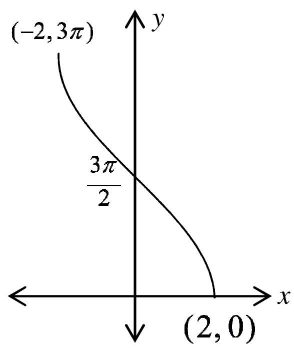
a) Prove by mathematical induction that:
$5^n - 4n - 1 \geq 0$, for all $n \geq 1$. **4**

b) John contributes $800 every quarter into a superannuation fund earning 6% pa, compounded quarterly. Contributions are made for 10 years and then John stops making contributions for 3 years whilst unemployed. He is re-employed, and contributions resume under the previous conditions. John works for a further 7 years until retirement.

(i) Show that on retirement the contributions for the first 10 years can be expressed as: **1**
$800(1.015)^{80} + 800(1.015)^{79} + \ldots\ldots + 800(1.015)^{41}$.

(ii) Hence, calculate the total amount in the superannuation fund on retirement. **3**

c) Below is the graph $y = a\cos^{-1} bx$.



(i) Show that the value of a is 3 and that b is $\frac{1}{2}$. **2**

(ii) Find the exact value of the area enclosed by the curve $y = a\cos^{-1} bx$, the x axis and the line $y = \frac{\pi}{2}$. **2**

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QUESTION FIVE (12 marks)

a) A particle moves in a straight line so that its acceleration, a, is given by $a = 4x$. The displacement, x, of the particle is initially 1 metre to the right of the origin with a velocity of $-2ms^{-1}$.

(i) Show that $v = -2x\ ms^{-1}$. **2**

(ii) Express x as a function of t. **2**

(iii) Hence, find the displacement when $t = 2$ seconds to 3 decimal places. **1**

b) (i) Write an expression for $(1+x)^{20}+(1-x)^{20}$. **2**
Leave your answer in ${}^{n}C_{k}$ notation.

(ii) Show that $\sum_{k=0}^{10} {}^{20}C_{2k}\, x^{2k} = \frac{1}{2}\left[(1+x)^{20}+(1-x)^{20}\right]$. **1**

(iii) Integrate both sides of the equation in part (ii). **2**

(iv) Hence, or otherwise show: **2**

$${}^{20}C_0 + \frac{1}{3}\,{}^{20}C_2 + \frac{1}{5}\,{}^{20}C_4 + \ldots\ldots + \frac{1}{21}\,{}^{20}C_{20} = \frac{2^{20}}{21}.$$

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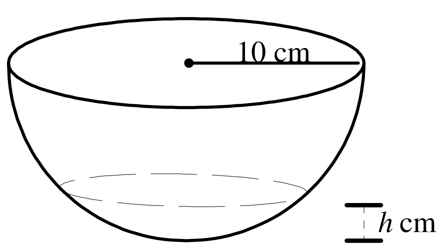
QUESTION SIX (12 marks)

a) (i) The diagram below represents a hemispherical bowl of radius 10 cm.
It is filled with water to a depth of h cm. By finding the volume generated by rotating $x^2+y^2=100$ between $y=10$ and $y=10-h$ about the y axis, show that the volume of water in the bowl is given by:

$$V=\frac{\pi h^2}{3}(30-h)$$ **2**



3

(ii) The hemispherical bowl referred to above is being filled with water at a constant rate of $2\pi\ cm^3\ \min^{-1}$. Find the rate of increase of the depth of the water when the depth is 2 cm.

b) Consider the function $f(x) = \sqrt{x+3}$

(i) Find an expression for $y = f^{-1}(x)$. **1**

(ii) State the domain for $y = f^{-1}(x)$. **1**

(iii) $f(x)$ and $f^{-1}(x)$ intersect at exactly one point P. Let α be the x coordinate of P. Explain why α is a root of the equation $x - \sqrt{x+3} = 0$. **2**

(iv) Show that the value of α occurs when $2 < \alpha < 3$. **1**

(v) Take $x = 2.5$ as the first approximation of α, use one application of Newton's method to find a second approximation for α correct to 3 decimal places. **2**

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<u>QUESTION SEVEN (12 marks)</u>

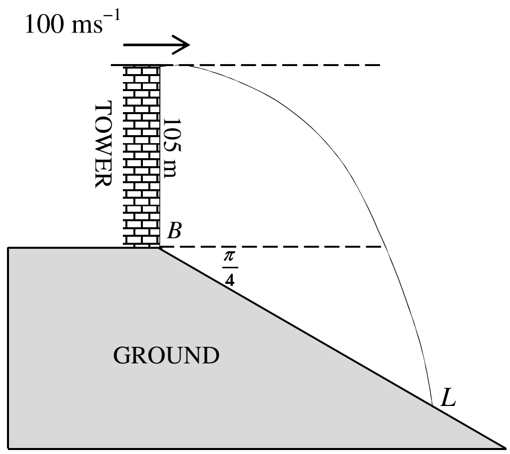
a) On a certain day the depth of water in a harbour is 7 metres at low tide (9:25am) and $10\frac{2}{3}$ metres at high tide (3:40pm). Assume the rise and fall of the surface of the water to be in simple harmonic motion in the form $\ddot{x} = -n^2(x-b)$ where $x = b$ is the centre of motion and $x = a$ is the amplitude.

(i) Show that $x = b - a\cos nt$ satisfies $\ddot{x} = -n^2(x-b)$. **1**

(ii) Find the values of a, b and n. **3**

(iii) Hence, find the earliest time before 3:40pm on this day, a boat may safely enter the harbour if the minimum depth of $9\frac{1}{2}$ metres of water is required. **2**

b) A bullet is fired horizontally with a velocity of $100ms^{-1}$ from the top of a tower 105 metres high.The tower is at the top of a hill, which slopes downwards at an angle of depression of $\frac{\pi}{4}$. The bullet lands at L, on the ground as represented on the diagram below.



(i) Consider B, the base of the tower, as the origin, and using acceleration due to gravity as $10\ ms^{-2}$, show that the expressions for the x and y coordinates of the position of the bullet at time t seconds are: **2**

$x = 100t$ and $y = 105 - 5t^2$.

(ii) Show that the equation of the line BL is $y = -x$. **1**

(iii) Find the time taken for the bullet to hit the ground at L. **2**

(iv) Find the distance BL to the nearest metre. **1**

<u>END OF THE EXAMINATION</u>

QUESTION ONE (12 marks)

a) Evaluate

$$\int_0^3 \frac{dx}{x^2+9} \ .= \left[\frac{1}{3}\tan^{-1}\frac{x}{3}\right]_0^3 = \frac{\pi}{12}$$

b)

$$m_1 = 2\, and\, m_2 = -\frac{1}{3}$$

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = 7$$

$$\theta = 81^\circ 52'$$

c)

$$\frac{x}{2x-1}\times(2x-1)^2 \le 5(2x-1)^2 \quad x \ne \frac{1}{2}$$

$$2x^2 - x \le 20x^2 - 20x + 5$$

$$0 \le 18x^2 - 19x + 5$$

$$0 \le (9x-5)(2x-1)$$

$$x \ge \frac{5}{9}\ and\ \ x < \frac{1}{2}$$

d)

$$\frac{dy}{dx} = 4\cos^3 x \times -\sin x$$

$$\frac{dy}{dx} = -4\cos^3 x \sin x$$

$$-4\int \cos^3 x \sin x dx = \cos^4 x + c$$

$$\int \cos^3 x \sin x dx = -\frac{1}{4}\cos^4 x + c$$

e)

$$\left(\frac{-6+5}{8}, \frac{-3+25}{8}\right)$$

$$\left(\frac{-1}{8}, 2\frac{3}{4}\right)$$

QUESTION TWO (12 marks)

a)

$$\int_0^{\frac{\pi}{3}} \sin^2(2x)dx = \int_0^{\frac{\pi}{3}} \frac{1-\cos 2(2x)}{2}dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{3}} 1-\cos(4x)dx = \frac{1}{2}\left[x - \frac{1}{4}\sin(4x)\right]_0^{\frac{\pi}{3}}$$

$$= \frac{\pi}{6} + \frac{\sqrt{3}}{16}$$

b)

$u = \tan x$

$\frac{du}{dx} = \sec^2 x \quad du = \sec^2 x\,dx \quad when\, x = 0 \;\; u = \tan 0 = 0$

$when\, x = \frac{\pi}{3} \;\; u = \tan\frac{\pi}{3} = \sqrt{3}$

$$\int_0^{\sqrt{3}} u^2 du = \left[\frac{u^3}{3}\right]_0^{\sqrt{3}} = \frac{3\sqrt{3}}{3} = \sqrt{3}$$

c)

(i) ${}^5_2C(0.4)^2(0.6)^3 = 0.3456$

(ii) ${}^5_3C(0.4)^3(0.6)^2 + {}^5_4C(0.4)^4(0.6)^1 + {}^5_5C(0.4)^5(0.6)^0 = 0.31744$

(iii) The most likely outcome is 3 heads and 2 tails which has 0.3456 chance of occurring.

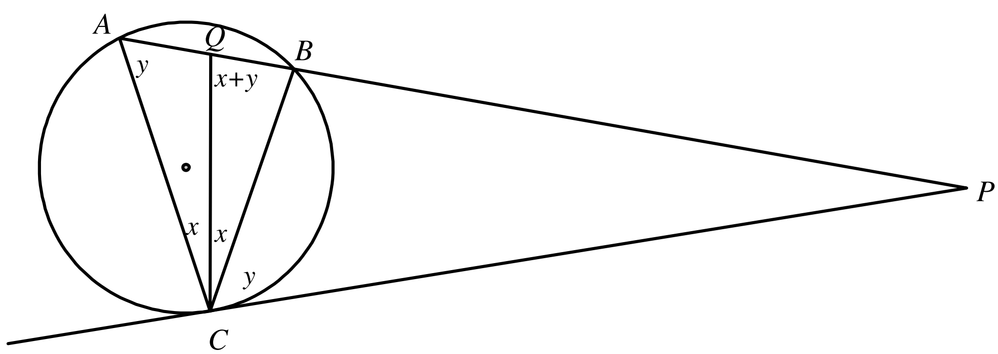
QUESTION THREE (12 marks)

a) $\sin\left(\sin^{-1}\left(\frac{4}{5}\right)-\tan^{-1}\left(\frac{-5}{12}\right)\right)$.

$$\sin(A-B)=sinA\cos B-\sin B\cos A$$
$$=\left(\frac{4}{5}\times\frac{-12}{13}\right)-\left(\frac{5}{13}\times\frac{3}{5}\right)$$
$$=\frac{-63}{65}$$

b)

(i)



Let $\angle ACQ=x, \angle QCB=x$ given that QC bisect $\angle ACB$

Let $\angle BCP=y, \angle CAB=y$ (angle between chord and tangent equal angle in the alternate segment)

$\angle CQP=x+y$ (external angle in ΔAQC is equal to the sum of the opposite internal angles)

$\angle QCP=x+y$

ΔPQC is an isosceles Δ. ($\angle CQP=\angle QCP$)

$QP=PC$ (sides opposite equal angles are equal)

(ii)

$PC^2=PB\times PA$

Let $PB=x, QB=10-x$

$10^2=x\times 12$

$x=100\div 12=8\frac{1}{3}$

$QB=10-8\frac{1}{3}=1\frac{2}{3}$

c)

(i)

$$T = 30 + Ae^{-kt}$$
$$Ae^{-kt} = T - 30$$
$$\frac{dT}{dt} = -kAe^{-kt} = -k(T - 30)$$

(ii) Show that the value of A is $220°$ C .

$$T_0 = 250$$
$$250 = 30 + A$$
$$A = 220$$

(iii) Find the value of k to 2 decimal places.

$$150 = 30 + 220e^{-20k}$$
$$120 = 220e^{-20k}$$
$$\frac{120}{220} = e^{-20k}$$
$$k = \frac{1}{20}\ln\frac{11}{6} = 0.03$$

(iv)

$$80 = 30 + 220e^{-0.03t}$$
$$50 = 220e^{-0.03t}$$
$$\frac{50}{220} = e^{-0.03t}$$
$$t = \frac{\ln\frac{5}{22}}{-0.03} = 49\,\text{minutes}$$

QUESTION FOUR (12 marks)

a) Prove by mathematical induction that:

$5^n - 4n - 1 \geq 0,$ for all $n \geq 1$.

Test for $n = 1$

$5^1 - 4(1) - 1 \geq 0,$

$0 \geq 0$

True for $n = 1$.

Assume true for $n = k$, where $k \geq 1$

assume $5^k - 4k - 1 \geq 0,$

Test for $n = k + 1$

$5^{k+1} - 4(k+1) - 1$

$= 5 \times 5^k - 4k - 4 - 1$

$= 5 \times 5^k - 20k + 16k - 5$

$= 5(5^k - 4k - 1) + 16k$

Since $(5^k - 4k - 1) \geq 0$ and $16k \geq 0$

then $5(5^k - 4k - 1) + 16k \geq 0,$ where $k \geq 1$

Since true for $n = 1,$ and true for $n = k + 1,$

then by the process of mathematical induction, it is true for all values of n.

b)

(i)

There are 10 years of contributions, then 3 years without contributions, followed by a further 7 years of contribution. The first contribution occurred 20 years before the last.

Contributions occurred quarterly (80 quarters @ 1.5% per quarter)

First contribution= $800(1.015)^{80}$

Second contribution= $800(1.015)^{79}$

Third contribution= $800(1.015)^{78}$

Last contribution before the 3 year break.= $800(1.015)^{41}$

Therefore total contribution make for first 10 years is

$800(1.015)^{80} + 800(1.015)^{79} + \ldots.. + 800(1.015)^{41}$.

(ii) Hence, calculate the total amount in the superannuation fund on retirement.

Total contribution = first 10 years contribution +Last 7 years contribution

$= 800(1.015)^{80} + 800(1.015)^{79} + \ldots.. + 800(1.015)^{41} + 800(1.015)^{28} + 800(1.015)^{27} + \ldots.. + 800(1.015)^{1}$.

$$\frac{800(1.015)(1.015^{28} - 1)}{1.015 - 1} + \frac{800(1.015)^{41}(1.015^{40} - 1)}{1.015 - 1}$$

$\$27998.96 + \$79935.68$

$= \$107934.64$

c)

(i) Show that the value of a is 3 and that b is $\frac{1}{2}$.

$$y = a\cos^{-1} bx$$

$$x = a\cos^{-1} by$$

$$\frac{x}{a} = \cos^{-1} by$$

$$\cos\frac{x}{a} = by$$

$$\frac{1}{b}\cos\frac{x}{a} = y$$

amplitude $= 2$. Therefore $\frac{1}{b} = 2, b = \frac{1}{2}$

period $= 6\pi$. Therefore $\frac{1}{a} = \frac{2\pi}{6\pi}, a = 3$

(ii)

$$\int_0^{\frac{\pi}{2}} 2\cos\frac{x}{3}\,dx = 2\left[3\sin\frac{x}{3}\right]_0^{\frac{\pi}{2}}$$

$$6\sin\frac{\pi}{6} = 3\,\text{units}^2$$

QUESTION FIVE (12 marks)

a) (i)

$$a = \ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$\frac{1}{2}v^2 = \int 4x\,dx$$

$$\frac{1}{2}v^2 = 2x^2 + c$$

when $x = 1,\ v = -2,$ therefore $c = 0$

$$v^2 = 4x^2$$

$v = \pm\sqrt{4x^2}$ Since velocity is initially negative then

$$v = -\sqrt{4x^2} = -2x$$

(ii) Express x as a function of t.

$$v = \frac{dx}{dt} = -2x$$

$$\frac{dt}{dx} = \frac{1}{-2x}$$

$$t = \int \frac{1}{-2x}dx = \frac{-1}{2}\ln x + c$$

$$when\ \ t = 0, x = 1, \therefore c = 0$$

$$t = -\frac{1}{2}\ln x$$

$$-2t = \ln x$$

$$x = e^{-2t}$$

(iii) Hence, find the displacement when t = 2 seconds to 3 decimal places.

$$when\ \ t = 2$$

$$x = e^{-4} = 0.018$$

b) (i)

$$(1+x)^{20} + (1-x)^{20}$$

$$= {}^{20}_{0}C + {}^{20}_{1}Cx + {}^{20}_{2}Cx^2 + \ldots\ldots {}^{20}_{20}Cx^{20} + {}^{20}_{0}C - {}^{20}_{1}Cx + {}^{20}_{2}Cx^2 - \ldots\ldots + {}^{20}_{20}Cx^{20}$$

$$= 2\left({}^{20}_{0}C + {}^{20}_{2}Cx^2 + {}^{20}_{4}Cx^4 + \ldots\ldots {}^{20}_{20}Cx^{20}\right)$$

$$= 2\sum_{k=0}^{10} {}^{20}_{2k}Cx^{2k}$$

(ii)

$$(1+x)^{20} + (1-x)^{20} = 2\sum_{k=0}^{10} {}^{20}_{2k}Cx^{2k}$$

$$\sum_{k=0}^{10} {}^{20}_{2k}Cx^{2k} = \frac{1}{2}\left((1+x)^{20}+(1-x)^{20}\right)$$

(iii) Integrate both sides of the equation in part (ii).

$$\frac{1}{2}\left((1+x)^{20}+(1-x)^{20}\right) = \sum_{k=0}^{10} {}^{20}_{2k}Cx^{2k}$$

$$\frac{1}{2}\frac{1}{21}\left((1+x)^{21}-(1-x)^{21}\right) + c = \sum_{k=0}^{10} \frac{1}{2k+1} {}^{20}_{2k}Cx^{2k+1} + c$$

(iv) Hence, or otherwise show:

$${}^{20}C_0 + \frac{1}{3}\,{}^{20}C_2 + \frac{1}{5}\,{}^{20}C_4 + ... + \frac{1}{21}\,{}^{20}C_{20} = \frac{2^{20}}{21}.$$

Let $x = 10 \therefore c = 0$

When $x = 1$

$$\frac{1}{2}\frac{1}{21}\left((1+1)^{21}-(1-1)^{21}\right) = \sum_{k=0}^{10} \frac{1}{2k+1} {}^{20}_{2k}C1^{2k+1}$$

$$\frac{1}{21}\frac{1}{2}\left(2^{21}\right) = \frac{1}{21}\left(2^{20}\right) = \sum_{k=0}^{10} \frac{1}{2k+1} {}^{20}_{2k}C$$

$$\frac{2^{20}}{21} = \sum_{k=0}^{10} \frac{1}{2k+1} {}^{20}_{2k}C = \left({}^{20}_{0}C + \frac{1}{3}\,{}^{20}_{2}Cx^2 + \frac{1}{5}\,{}^{20}_{4}Cx^4 + ... \frac{1}{21}\,{}^{20}_{20}Cx^{20}\right)$$

QUESTION SIX (12 marks)

(i)

$$x^2 + y^2 = 100$$
$$x^2 = 100 - y^2$$
$$V = \pi \int_{10-h}^{10} 100 - y^2 dy$$
$$V = \pi \left[100y - \frac{1}{3}y^3 \right]_{10-h}^{10}$$
$$V = \pi \left[\left(100(10) - \frac{1}{3}(10)^3 \right) - \left(100(10-h) - \frac{1}{3}(10-h)^3 \right) \right]$$
$$V = \pi \left[\left(1000 - \frac{1}{3}(1000) \right) - \left(1000 - 100h - \frac{1}{3}(1000 - 300h + 30h^2 - h^3 \right) \right]$$
$$V = \frac{\pi}{3}\left[30h^2 - h^3 \right]$$
$$V = \frac{\pi h^2}{3}\left[30 - h \right]$$

(ii)

$$\frac{dV}{dt} = 2\pi$$
$$V = \frac{\pi h^2}{3}\left[30 - h \right] = 10\pi h^2 - \frac{\pi h^3}{3}$$
$$\frac{dV}{dh} = 20\pi h - \pi h^2$$
$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$
$$\frac{dh}{dt} = \frac{1}{20\pi h - \pi h^2} \times 2\pi$$
when $h = 2$
$$\frac{dh}{dt} = \frac{1}{40\pi - 4\pi} \times 2\pi = \frac{2\pi}{36\pi} = \frac{1}{18} cm \min^{-1}$$

b)

(i) Find an expression for $y = f^{-1}(x)$.

$$x = \sqrt{y+3}$$
$$x^2 = y+3$$
$$y = x^2 - 3$$
$$f^{-1}(x) = x^2 - 3$$

(ii)

Domain $f^{-1}(x)$ is the range for $f(x)$.

For $f(x) = \sqrt{x+3} \quad x \geq 0$

(iii)

$$f^{-1}\left(f(x)\right) = f\left(f^{-1}(x)\right) = x$$

$y = x$ and $y = \sqrt{x+3}$

$$y = y$$
$$x = \sqrt{x+3}$$
$$x - \sqrt{x+3} = 0$$

(iv) Show that the value of α occurs when $2 < \alpha < 3$.

when $x = 2 \quad 2 - \sqrt{2+3} < 0$

when $x = 3 \quad 3 - \sqrt{3+3} > 0$

$\therefore$ there will be a zero between 2 and 3.

(v) Take $x = 2.5$ as the first approximation of α, use one application of Newton's method to find a second approximation for α correct to 3 decimal places.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$
$$x_1 = 2.5 - \frac{f(2.5)}{f'(2.5)}$$
$$x_1 = 2.303$$

QUESTION SEVEN (12 marks)

a)

(i) Show that $x = b - a\cos nt$ satisfies $\ddot{x} = -n^2(x-b)$.

$x = b - a\cos nt \;\therefore\; a\cos nt = b - x$

$\frac{dx}{dt} = -an\sin nt$

$\frac{d^2x}{dt^2} = an^2\cos nt = n^2(a\cos nt) = n^2(b-x) = -n^2(x-b)$

$\ddot{x} = \frac{d^2x}{dt^2} = -n^2(x-b)$

(ii) Find the values of a, b and n.

$b = 8\frac{5}{6} \quad a = 1\frac{5}{6}$

$\frac{2\pi}{n} = 12\frac{1}{2}$ hours

n$= \frac{4\pi}{25}$

(iii) Hence, find the earliest time before 3:40pm on this day, a boat may safely enter the harbour if the minimum depth of $9\frac{1}{2}$ metres of water is required.

$x = 8\frac{5}{6} - 1\frac{5}{6}\cos\frac{4\pi t}{25}$

when $x = 9\frac{1}{2}$

$9\frac{1}{2} = 8\frac{5}{6} - 1\frac{5}{6}\cos\frac{4\pi t}{25}$

$\cos\frac{4\pi t}{25} = \frac{-4}{11}$

$\frac{4\pi t}{25} = \cos^{-1}\left(\frac{-4}{11}\right)$

$t = 3\text{hours } 51\text{minutes } 56\text{seconds}$

9:25am +3hours 51minutes 56seconds $= 1:17\,pm$

b)

(i)

$$\ddot{x}=0 \qquad \ddot{y}=-g$$

$$\dot{x}=100 \qquad \dot{y}=-gt+c \qquad c=0$$

$$x=100t \qquad y=\frac{-gt^2}{2}+c \qquad \text{when } t=0 \qquad y=105 \;\therefore c=105$$

$$y=\frac{-gt^2}{2}+105=\frac{-10t^2}{2}+105=-5t^2+105$$

(ii) Show that the equation of the line BL is $y = -x$.

$m=\tan\theta$

gradient of BL is -1

When $x=0$, $y=0$

$y=-x$

(iii) Find the time taken for the bullet to hit the ground at L.

$x=100t \qquad y=-5t^2+105$

$$y=-5\left(\frac{x}{100}\right)^2+105$$

$$y=\left(\frac{-5x^2}{10000}\right)+105$$

$y=-x$

$$-x=\frac{-5x^2}{10000}+105$$

$x^2-2000x-210000=0$

$(x-2100)(x+100)=0$

$x=2100$

$$t=\frac{2100}{100}=21\text{minutes}$$

(iv) Find the distance BL to the nearest metre.

$BL=\sqrt{2100^2+2100^2}$

$BL=2970\,\text{cm}$

$\text{BL}\approx 30\text{ m}$