The Hills Grammar School

Founded 1982

2013
Higher School Certificate
Trial Examination

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen.
  Black pen is preferred.
- Board approved calculators may be used.
- A standard table of integrals is provided at the back of this paper.
- In Questions 11-14, show relevant mathematical reasoning and/or calculations.

Total marks – 70

Section I

10 marks

- Attempt Questions 1-10
- Answer on the Multiple Choice Answer Sheet provided.
- Allow about 15 minutes for this section.

Section II

60 marks

- Attempt Questions 11-14
- Answer in the writing booklets provided.
- Start a new booklet for each question.
- Allow about 1 hour and 45 minutes for this Section.

Section 1
10 marks
Attempt Questions 1-10
Allow approximately 20 min for this section

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1. The graph of the function $f(x) = -x(x^2 - 1)(x^2 + 1)$ cuts the x-axis

A. five times   **C.** three times

B. twice   **D.** four times.

2. The solutions to the equation $2\sin\frac{x}{2} = 1$ over the domain $0 \le x \le 2\pi$ are

A. $\frac{\pi}{12}$ *and* $\frac{5\pi}{12}$ **C.** $\frac{\pi}{3}$ *and* $\frac{5\pi}{3}$

B. $\frac{2\pi}{3}$ *and* $\frac{4\pi}{3}$ **D.** $\frac{8\pi}{3}$ *and* $\frac{10\pi}{3}$.

3. If $y = e^{4x}\cos x$ then $\frac{dy}{dx}$ equals

A. $-4e^{4x}\sin x$ **C.** $4e^{4x}\sin x$

B. $4e^{4x}\cos x + e^{4x}\sin x$ **D.** $e^{4x}(4\cos x - \sin x)$.

4. If $\int_m^1 (1-x)\,dx = 8$ and $m \ge 0$, then m is equal to

A. 7   **C.** 0

B. 5   **D.** -3.

5. The displacement x, in metres, of a particle moving in a straight line is given by $x = \tan 3t, \quad 0 \le t \le \frac{\pi}{6}$ where t represents time in seconds. The acceleration, a, of the particle is given by

A. $a = 9\tan 3t$

B. $a = 3\sec^2 t$

C. $a = \frac{-2}{\cos^3 3t}$

D. $a = 18\tan 3t.\sec^2 3t$.

6. A ball is dropped from a height. The acceleration a ms^{-2} and the velocity v ms^{-1} of the ball t seconds after it is dropped is given by $a = 10 - \frac{v}{20}$. An expression for t in terms of v could be obtained by simplifying:

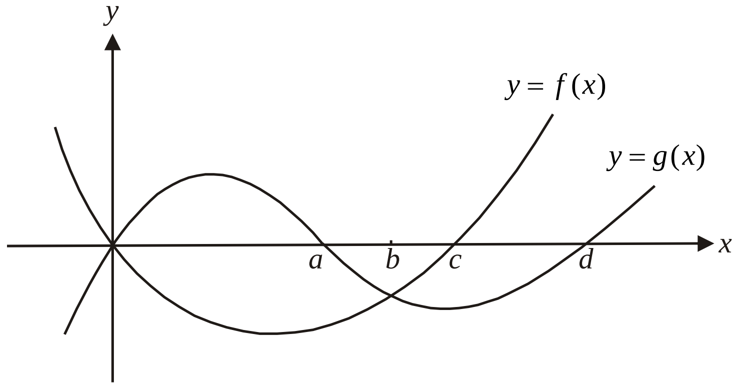
A. $t = 20\int \frac{1}{200 - v}\,dv$

B. $t = \int (10 - \frac{v}{20})\,dv$

C. $t = \int (\frac{v}{20} - 10)\,dv$

D. $t = \int (\frac{1}{10} - \frac{20}{v})\,dv$.

7. The graphs of $y = f(x)$ and $y = g(x)$ are shown below.



The area enclosed by these two graphs is given by:

A. $\int_0^b (g(x) - f(x))\,dx$

B. $\int_0^b (g(x) + f(x))\,dx$

C. $\int_0^a (g(x))\,dx - \int_0^b f(x)\,dx$

D. $\int_0^a (g(x))\,dx - \int_0^c f(x)\,dx$.

8. Rhea made an error proving that $3^{2n}-1$ is divisible by 8 (where *n* is an integer greater than 0) using mathematical induction. Part of the proof is shown below.

Step 1: test for *n* = 1, $3^{2(1)}-1=8$, therefore divisible by 8.
Step 2: Assume the result true for $n=k$

$3^{2k}-1=8P$ where *P* is an integer. Line 1

Hence $3^{2k}=8P+1$

To prove the result is true for $n=k+1$

$3^{2(k+1)}-1=8Q$ where *Q* is an integer. Line 2

$\text{LHS}=3^{2(k+1)}-1$

$=3^{2k}\times 3^2-1$

$=(8P+1)\times 3^2-1$ Line 3

$=72P+1-1$ Line 4

$=72P$

$=8(9P)$

$=8Q$

$=$ RHS

Which line did Rhea make an error?

A. Line 1 **C.** Line 2

B. Line 3 D. Line 4

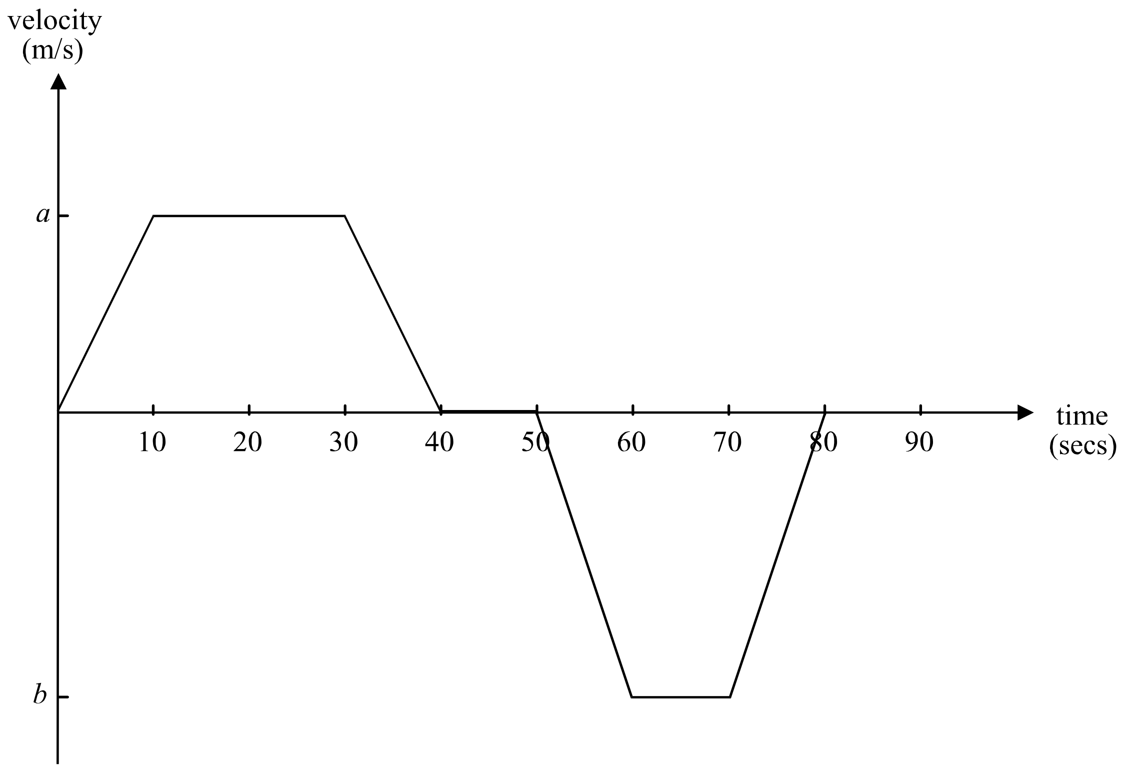
9. A particle moving in simple harmonic motion starts from rest at distance 6 metres from the centre of oscillation. The period is 4π seconds. What is the minimum time taken to move to a point 3 metres from the origin?

A. $t=\frac{\pi}{3}$ **C.** $t=\frac{2\pi}{3}$

B. $t=\frac{\pi}{6}$ **D.** $t=\frac{5\pi}{6}$

10. A man makes a return trip along the same route by car from his home to a letter box.

The velocity-time graph describing his trip is shown below.



The initial acceleration of the car is $2\text{m}/\text{s}^2$.

The values of *a* and *b* respectively are

A. 10 and −20

B. 10 and −25

C. 20 and −20

D. 20 and −30.

End of Section 1

Section 11

60 marks
Attempt Questions 11-14
Allow approximately 1 hour 40 minutes for this section

Answer each question in a **separate writing booklet**.
Show all necessary working.
Diagrams are not drawn to scale.

Question 11 (15 marks) **Marks**

(a) Find $\int \frac{e^x}{1+e^x}\,dx$. **1**

(b) State the domain and range of $y = 3\sin^{-1}\left(\frac{x}{2}\right)$. **2**

(c) The remainder when *P(x)* is divided by $x^2 - 1$ is $3x - 4$. What is the remainder when *P(x)* is divided by $x + 1$? **2**

(d) Differentiate with respect to *x*: $y = \ln\left(x^3\sqrt{x^2+1}\right)$. **2**

(e) Find $\lim_{x\to 0}\frac{\sin 2x}{\tan 3x}$ showing appropriate working. **2**

(f) Solve the inequality $x \geq \frac{6}{x-1}$. **3**

(g) Write $\sqrt{3}\cos x - \sin x$ in the form $R\cos(x+\alpha)$ and hence find the general solution of the equation $\sqrt{3}\cos x - \sin x = 1$ **3**

Question 12 (15 marks) **Marks**

(a) $P(x)=2x^3-4x^2+3x-10$ has zeros α, β and γ. Determine the value of:

(i) $\alpha^2+\beta^2+\gamma^2$ **2**

(ii) $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$ **2**

(b) The function defined by $f(x)=x^3-\ln(x+1)$ has one real root.

(i) Show that this root lies between $x=0.8$ and $x=0.9$. **1**

(ii) Use the halving the interval method to approximate this root correct to two decimal places. **3**

(c) Evaluate $\int_1^2 \sqrt{4-x^2}\,dx$ using the substitution $x=2\sin\theta$. **4**

(d) Find the area between the curve $y=2\tan^{-1}\left(\frac{x}{2}\right)$ and the lines $x=0$, $x=2$ and the x axis. **3**

Question 13 (15 marks) **Marks**

(a) Consider the curve given by $y = \dfrac{x^2 + 9}{x}$.

(i) Find any stationary points on the curve and determine their nature. **2**

(ii) Find the vertical and oblique asymptotes of the curve. **2**

(iii) Sketch the curve showing the above features. **2**

(b) Find the greatest coefficient in the expansion of $(7 + 3x)^{18}$. Leave your answer in factor form. **3**

(c) The coefficient of x^k in $(1 + x)^n$, where n is a positive integer, is denoted by c_k $\left(\text{where } c_k = {}^nC_k\right)$.

Show that

$$c_0 + 2c_1 + 3c_2 + ... + (n+1)c_n = (n+2)2^{n-1}.$$

(Hint you will need to differentiate) **3**

(d) Prove using mathematical induction that $n^3 - n$ is divisible by 6 for all integers $n \geq 1$. **3**

Question 14 (15 marks) **Marks**

(a)

(i) Evaluate $\int_{-2\sqrt{3}}^{2} \frac{1}{4+x^2}\,dx$. **2**

(ii) Prove: $\tan^{-1} a - \tan^{-1} b = \tan^{-1}\left(\frac{a-b}{1+ab}\right)$. **2**

(b) A particle P is moving in simple harmonic motion about the origin. The particle is initially at rest a units to the right of the origin. If v is the velocity, x is the displacement and $\frac{2\pi}{n}$ is the period of the motion of the particle P at time t :

(i) Show that $v^2 = n^2(a^2 - x^2)$. **2**

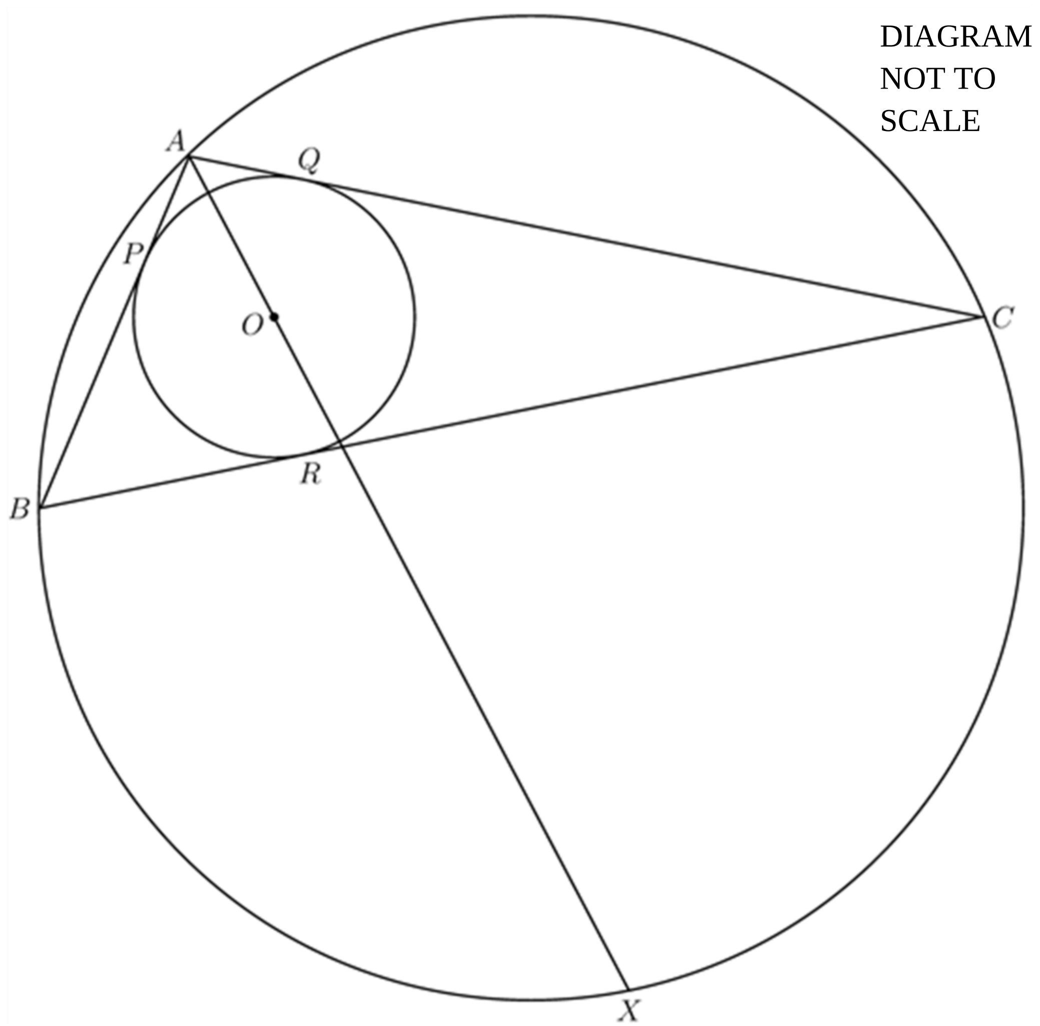
(ii) Hence show that $x = a\cos nt$ **2**

(iii) Hence find an expression for v in terms of t. **1**

(iv) Find the greatest speed of the particle P. **1**

(v) Find the greatest magnitude of the acceleration of the particle P. **1**

(c) The circle inscribed in triangle *ABC* has centre *O* and meets the sides of the triangle *ABC* at *P*, *Q* and *R*. The circumcircle (ie. the circle that passes through each of the vertices of the triangle ABC) meets *AO* produced at *X*.



Let $\angle OAP = \alpha$ and $\angle OBR = \beta$.

Prove that:

(i) $\Delta OPA \equiv \Delta OQA$ **2**

(ii) *X* is the midpoint of arc *BC*. **2**

END OF ASSESMENT

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \quad \text{if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

ANSWER SHEET FOR MULTIPLE CHOICE SECTION

Student Exam number:____________________________

Teacher: ________________________________

1. **A** ⬭ **B** ⬭ **C** ⬭ **D** ⬭

2. **A** ⬭ **B** ⬭ **C** ⬭ **D** ⬭

3. **A** ⬭ **B** ⬭ **C** ⬭ **D** ⬭

4. **A** ⬭ **B** ⬭ **C** ⬭ **D** ⬭

5. **A** ⬭ **B** ⬭ **C** ⬭ **D** ⬭

6. **A** ⬭ **B** ⬭ **C** ⬭ **D** ⬭

7. **A** ⬭ **B** ⬭ **C** ⬭ **D** ⬭

8. **A** ⬭ **B** ⬭ **C** ⬭ **D** ⬭

9. **A** ⬭ **B** ⬭ **C** ⬭ **D** ⬭

10. **A** ⬭ **B** ⬭ **C** ⬭ **D** ⬭

Year 12 Ext 1 Trial 2013

MC Q

1, C	6, A
2, C	7, A
3, D	8, D
4, B	9, C
5, D	10, D

Q11

(a) $\int \frac{e^x}{1+e^x}\,dx = \ln(1+e^x) + C$ (1)

(b) $y = 3\sin^{-1}\left(\frac{x}{2}\right)$

Domain $-1 \le \frac{x}{2} \le 1$

$\therefore -2 \le x \le 2$ (1)

Range $-\frac{3\pi}{2} \le y \le \frac{3\pi}{2}$ (1)

(c) From info:

$P(x) = (x^2-1)Q(x) + 3x - 4$ (1)

Rem $= P(-1) = -7$ (1)

(d) $y = \ln\left(x^3\sqrt{x^2+1}\right)$

$= \ln x^3 + \ln(x^2+1)^{1/2}$ (1)

$y' = \frac{3}{x} + \frac{1}{2}\cdot\frac{2x}{x^2+1}$ (1)

$= \frac{3(x^2+1) + x^2}{x(x^2+1)}$

$= \frac{4x^2+3}{x(x^2+1)}$

(1)

(e) $\lim_{x\to 0} \frac{\sin 2x}{\tan 3x} = \frac{2}{3}\cdot \lim_{x\to 0}\frac{3x}{\tan 3x}\cdot \lim_{x\to 0}\frac{\sin 2x}{2x}$

$= \frac{2}{3}\cdot 1\cdot 1$

$= 2/3$ (1)

(f) $x \ge \frac{6}{x-1}$, $x \ne 1$

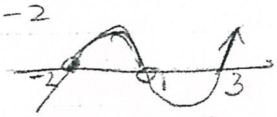
$x(x-1)^2 \ge 6(x-1)$ (1)

$x(x-1)^2 - 6(x-1) \ge 0$

$(x-1)(x^2-x-6) \ge 0$

$(x-1)((x-3)(x+2)) \ge 0$ (1)

$x = 1, 3, -2$



$\therefore SS = \{-2 \le x < 1\}$ or $x \ge 3$. (1)

↑ NB

Q11 cont.

(g) $\sqrt{3}\cos x - \sin x = 1$

$R = \sqrt{3+1} = 2$

$2\cos(x+\alpha) = 1$

$\cos\alpha = \frac{\sqrt{3}}{2}$, $\sin\alpha = \frac{1}{2}$

$\therefore \tan\alpha = \frac{1}{\sqrt{3}}$ $\quad \alpha = \frac{\pi}{6}$ (1)

$\therefore \cos\left(x + \frac{\pi}{6}\right) = \frac{1}{2}$

$x + \frac{\pi}{6} = 2\pi n \pm \frac{\pi}{3}$ (1)

$x = 2\pi n + \pi/6$ or

$2\pi n - \frac{\pi}{2}$, $n \in \mathbb{Z}$. (1)

Q12

(a) i) $P(x) = 2x^3 - 4x^2 + 3x - 10$ (1)

$\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha)$

$= \left(\frac{-4}{2}\right)^2 - 2\left(\frac{3}{2}\right)$

$= 4-3$

$= 1$ (1)

(ii) $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} = \frac{\beta\gamma+\alpha\gamma+\alpha\beta}{\alpha\beta\gamma}$ (1)

$= \frac{3/2}{--10/2}$

$= 3/10$ (1)

(b)(i) $f(x) = x^3 - \ln(x+1)$

$f(0.8) = -0.075.. < 0$

$f(0.9) = 0.059. > 0$ (1)

Since signs opp, root lies bet 0.8 and 0.9.

(ii) $x_1 = \frac{0.8+0.9}{2} = 0.85$ (1) 1st step

$f(0.85) = -0.00106.. < 0$

∴ root bet 0.85 and 0.9 (1) method

$x_2 = \frac{0.85+0.9}{2} = 0.875$

$f(0.875) = 0.0413.. > 0$

∴ root lies bet 0.875 + 0.85

∴ $x_3 = \frac{0.875+0.85}{2} = 0.8625$

$f(0.8625) = 0.0196 > 0$

∴ root bet 0.85 and 0.8625.

∴ $x_4 = \frac{0.85+0.8625}{2} = 0.85625$

$f(0.85625) = 0.00921 > 0$

∴ root lies bet 0.85 + 0.85625

∴ root $\frac{0.85+0.86525}{2} = 0.851325$ $f(0.851325) = 0.001$ ∴ root between 0.85 and 0.851

∴ $x = 0.85$ (2 dec) (1) answer

(c) $I = \int_1^2 \sqrt{4-x^2}\,dx$ $x = 2\sin\theta$

$\frac{dx}{d\theta} = 2\cos\theta$

$x=1,\ \theta = \pi/6$

$x=2,\ \theta = \frac{\pi}{2}$ (1) details

$\therefore I = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sqrt{4-4\sin^2\theta}\,.\,2\cos\theta\,d\theta$ (1) subst.

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 2\cos\theta . 2\cos\theta\,d\theta$

$= 4\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos^2\theta\,d\theta$ (1)

$= 2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 1+\cos 2\theta\,d\theta = 2\left[\theta+\frac{1}{2}\sin 2\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$

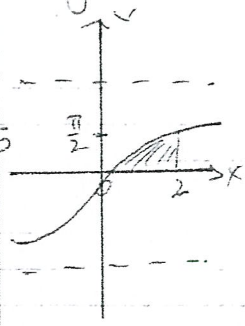
$= 2\left[\left(\frac{\pi}{2}+\frac{1}{2}\sin\pi\right) - \left(\frac{\pi}{6}+\frac{1}{2}\sin\frac{\pi}{3}\right)\right]$

$= 2\left[\frac{\pi}{2}-\frac{\pi}{6}-\frac{1}{2}.\frac{\sqrt{3}}{2}\right]$

$= 2\left[\frac{\pi}{3}-\frac{\sqrt{3}}{4}\right]$

$= \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$ (1)

(d) $y = 2\tan^{-1}\left(\frac{x}{2}\right)$; $2\tan\frac{y}{2} = x$



Area = Area rect $- \int_0^{\frac{\pi}{2}} x\,dy$ (1)

$= 2.\frac{\pi}{2} - \int_0^{\frac{\pi}{2}} 2\tan\frac{y}{2}\,dy$

$= \pi + 4\left[\ln\left|\cos\frac{y}{2}\right|\right]_0^{\frac{\pi}{2}}$ (1)

$= \pi + 4\left[\ln\frac{1}{\sqrt{2}} - \ln 1\right]$

$= (\pi - 2\ln 2)\ u^2$ (1)

Solutions TRIAL THGS 2013 2 UNIT Ext 1

13 a) /6

(i) $y = x + \frac{9}{x}$

$y' = 1 - \frac{9}{x^2}$

$y' = 0$

$\therefore x = \pm 3$ (1)

$(3, 6), (-3, -6)$

$y'' = \frac{18}{x^3}$

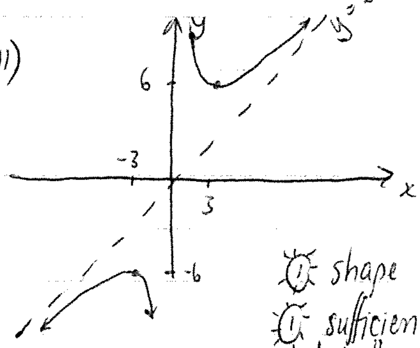
$x = 3, y'' > 0$ ∪ min

$x = -3, y'' < 0$ ∩ max (1)

(ii) vertical: $x = 0$ (1)

oblique: $y = x$ (1)

(iii)



(1) shape
(1) sufficient labelling

(b) /3

$\frac{t_{k+1}}{t_k} = \frac{{}^{18}C_{k+1}(7)^{17-k}(3)^{k+1}}{{}^{18}C_k(7)^{18-k}(3)^k}$ (1)

$= \frac{\frac{18!}{(k+1)!(17-k)!} \cdot 3}{\frac{18!}{k!(18-k)!} \cdot 7}$

$= \frac{18-k}{k+1} \cdot \frac{3}{7}$

$\therefore \frac{54-3k}{7k+7} > 1$ (1)

$\therefore k < 4.7$

So $k = 4$

$\therefore$ Choose the 5th

Ans $= {}^{18}C_5 \, 7^{13} 3^5$ (1)

(c) /3

$(1+x)^n = C_0 + C_1 x + C_2 x^2 + \cdots + C_n x^n$ ①

Differentiating ①

(1) $n(1+x)^{n-1} = C_1 + 2C_2 x + \cdots + nC_n x^{n-1}$ ②

In ① sub $x = 1$

(1) $2^n = C_0 + C_1 + C_2 + \cdots + C_n$ ③

In ② sub $x = 1$

$n\,2^{n-1} = C_1 + 2C_2 + \cdots + nC_n$ ④

(1) ③ + ④

$n2^{n-1} + 2.2^{n-1} = C_1 + 2C_2 + 3C_3 + \cdots + (n+1)C_n$

ie $(n+2).2^{n-1} = C_1 + 2C_2 + 3C_3 + \cdots + (n+1)C_n$

as required.

/3 (d)

Try $n = 1$: $1 - 1 = 0$ which is divisible by 6 $\therefore$ True

Assume true for $n = k$ ie $k^3 - k = 6M$, $(M \in I)$

Prove true for $n = k+1$ ie. $(k+1)^3 - (k+1) = 6N$; show $N \in I$

LHS $= (k+1)^3 - (k+1)$

$= k^3 + 3k^2 + 3k + 1 - k - 1$

$= k^3 - k + 3k^2 + 3k$

(1) $= 6M + 3(k^2 + k)$

$= 6\left[M + \frac{k^2+k}{2}\right]$

$= 6N$, provided $k^2 + k$ is even.

= RHS

$\therefore$ True for $n = k+1$

(1) · If k is odd, k^2 is odd, the sum of 2 odds is even

· If k is even, k^2 is even, the sum of 2 evens is even

$\therefore k^2 + k$ is even for all k.

Since true for $n = 1$ it will be true for $n = 2$, and $n = 3$ and so on.

Hence true for all integers $n \geq 1$

(1) structure of proof

Solutions THGS 2013 TRIAL 2UNIT EXT 1

14 a) [4]

(i) $\frac{1}{2}\left[\tan^{-1}\frac{x}{2}\right]_{-2\sqrt{3}}^{2}$ (1)

$= \frac{1}{2}\left[\tan^{-1}\frac{2}{2} - \tan^{-1}\left(\frac{-2\sqrt{3}}{2}\right)\right]$

$= \frac{1}{2}\left[\frac{\pi}{4} + \frac{\pi}{3}\right]$

$= \frac{7\pi}{24}$ (1)

(ii) To prove $\tan^{-1}a - \tan^{-1}b = \tan^{-1}\left(\frac{a-b}{1+ab}\right)$

first take tangents of both sides

So prove $\tan\left[\tan^{-1}a - \tan^{-1}b\right] = \frac{a-b}{1+ab}$ (1)

$LHS = \frac{\tan(\tan^{-1}a) - \tan(\tan^{-1}b)}{1+\tan(\tan^{-1}a).\tan(\tan^{-1}b)}$ (1)

$= \frac{a-b}{1+ab}$

$= RHS$

b) [7] $t=0, v=0, x=a$

(i) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -n^2x$

$\frac{1}{2}v^2 = \frac{-n^2x^2}{2} + C$ (1)

sub $x=a, v=0 \Rightarrow C = \frac{n^2a^2}{2}$

$\therefore \frac{1}{2}v^2 = \frac{-n^2x^2}{2} + \frac{n^2a^2}{a}$ (1)

$v^2 = -n^2x^2 + n^2a^2$

$v^2 = n^2(a^2-x^2)$

(ii) $v^2 = n^2(a^2-x^2)$

$\therefore v = n\sqrt{a^2-x^2}$ (v might be + or −)

$\frac{dx}{dt} = -n\sqrt{a^2-x^2}$

$\frac{dt}{dx} = \frac{1}{-n\sqrt{a^2-x^2}}$ (1)

$t = \frac{1}{n}\int \frac{-1}{\sqrt{a^2-x^2}}\,dx$

$= \frac{1}{n}\cos^{-1}\frac{x}{a} + c$

Sub $t=0, x=0 \Rightarrow c=0$

$t = \frac{1}{n}\cos^{-1}\frac{x}{a}$ (1)

$\cos nt = \frac{x}{a}$ $\therefore x = a\cos nt$

b) continued

(iii) $v = -na\sin nt$ (1)

(iv) greatest speed : na (1)

(v) greatest accel : n^2a (1)

c) [4] (i) various methods eg SSS or RHS or SAS

potential ingredients:

$OP = OQ$ equal radii

$AP = AQ$ equal tangents from external point

OA common

Angles at P, Q being right ∠s

(1) making headway

(1) complete proof.

(ii) $\angle PAO = \angle QAO$ (1)
(corresponding ∠s in congruent △s)

(1) So BX, XC subtend equal angles at circumference

$\therefore BX = XC$

$\therefore X$ is midpoint of BC