HORNSBY GIRLS' HIGH SCHOOL

2008
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- Reading Time – 5 minutes
- Working Time – 2 hours
- Write using a black or blue pen
- Approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown for every question
- Begin each question on a fresh sheet of paper

Total marks (84)

- Attempt Questions 1 – 7
- All questions are of equal value

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Total Marks
Attempt Questions 1–7
All Questions are of equal value

Begin each question on a NEW SHEET of paper, writing your student number and question number at the top of the page. Extra paper is available.

Question 1 (12 marks) Use a SEPARATE sheet of paper. **Marks**

(a) Find $\lim_{x \to 0} \frac{\sin 3x}{2x}$. **1**

(b) Find the acute angle between the lines $y = 2x - 9$ and $3y = x + 8$. **2**

(c) Find $\frac{d(3^x)}{dx}$. **1**

(d) State the domain and range of the function $y = 2\cos^{-1} 3x$. **2**

(e) Use the substitution $u = \tan x$ to evaluate $\int_{\pi/6}^{\pi/3} \frac{dx}{\cos^2 x \tan x}$. **3**

(f) Consider the function $y = x\cos^{-1} x - \sqrt{1 - x^2}$,

(i) Show that $\frac{dy}{dx} = \cos^{-1} x$. **2**

(ii) Hence, or otherwise, evaluate $\int_0^1 \cos^{-1} x\, dx$. **1**

Question 2 (12 marks) Use a SEPARATE sheet of paper. **Marks**

(a) Solve $x-5<\frac{14}{x}$. **3**

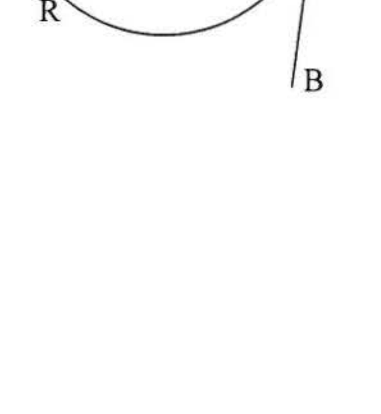
(b) Find the general solution to $\tan 2\theta=\sqrt{3}$. **2**

Express your answer in terms of π.

(c) The polynomial $f(x)=2x^3+ax^2+bx+6$ has a remainder of -6 when divided by $(x-1)$ and $f(-2)=0$. **2**

Find the values of a and b.

(d) Find the exact value of $\int_0^{\frac{\pi}{4}}\cos^2\left(\frac{1}{2}x\right)dx$. **3**

(e) 

Given that $PQ=PR$ and AB is a tangent to the circle PQR at P, prove that $RQ \parallel BA$. **2**

Question 3 (12 marks) Use a SEPARATE sheet of paper. **Marks**

(a) 8 people are to be seated at a round table

(i) How many seating arrangements are possible? 1

(ii) Two people, Sarah and Ken, can not sit together.
How many seating arrangements are then possible? 2

(b) The function $f(x) = x^3 + ax^2 + bx + c$ has a relative maximum at $x = \lambda$ and a relative minimum $x = \beta$.

(i) Prove $\lambda + \beta = -\frac{2}{3}a$. 2

(ii) Show that a point of inflexion occurs at $x = \frac{\lambda + \beta}{2}$. 2
(A check for concavity is not required.)

(c) A roast chicken has been taken from an oven and placed in a room of constant temperature 20°C. At time t minutes its temperature T decreases according to the equation

$$\frac{dT}{dt} = -k(T - 20) \quad \text{where } k \text{ is a positive constant.}$$

The initial temperature of the chicken is 80°C and cools to 50°C after 10 minutes.

(i) Verify that $T = 20 + Ae^{-kt}$ is a solution of this equation where A is a constant. 1

(ii) Find the values of A and k. 2

(iii) How long will it take for the chicken to cool to 30°C? 2

Give your answer to the nearest minute.

Question 4 (12 marks) Use a SEPARATE sheet of paper. **Marks**

(a) A body is in Simple Harmonic Motion and its position at a time t is given by the equation

$$x = R\cos(nt+\alpha)+1.$$

The period of motion is π seconds. Initially the body is at rest 3 units to the left of the origin.

(i) Find the values of R, n and α. **3**

(ii) Find the velocity of the body when $t=\frac{\pi}{6}$. **1**

(b) (i) Consider the equation $x\ln x - 1 = 0$. Show that a solution of this equation lies between $x = 1$ and $x = 2$. **1**

(ii) Using $x = 2$ as a first approximation for a solution, apply Newton's method once to find a better approximation. Give your answer to 1 decimal place. **2**

(c) Prove the identity $\dfrac{2\tan\theta}{1+\tan^2\theta} = \sin 2\theta$. **2**

(d) A 'Wheel of Chance' has 9 equal compartments around its rim.
When this wheel is spun a player can win \$100 on 1 designated compartment.
Grace is given the opportunity to have 25 consecutive spins of the wheel.
Find, giving your answer correct to 4 decimal places, the probability that she will win:

(i) exactly \$200, **1**

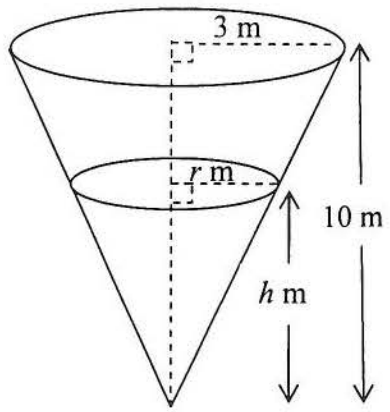
(ii) at least \$200. **2**

Question 5 (12 marks) Use a SEPARATE sheet of paper. **Marks**

(a) Use the principle of mathematical induction to show that

$1^3 + 2^3 + ... + n^3 = (1 + 2 + ... + n)^2$ for all $n \geq 1$ **3**

(b)



The diagram shows a conical wheat flu. The flu is being filled with wheat at the rate of 2m^3 per minute. The height of wheat at time t minutes is h metres and the radius of the wheat's top surface is r metres.

(i) Show that $r = \dfrac{3h}{10}$. **1**

(ii) Find the rate at which the height is increasing when the height of the wheat is 8m. **3**

(Volume of cone $= \dfrac{1}{3}\pi r^2 h$)

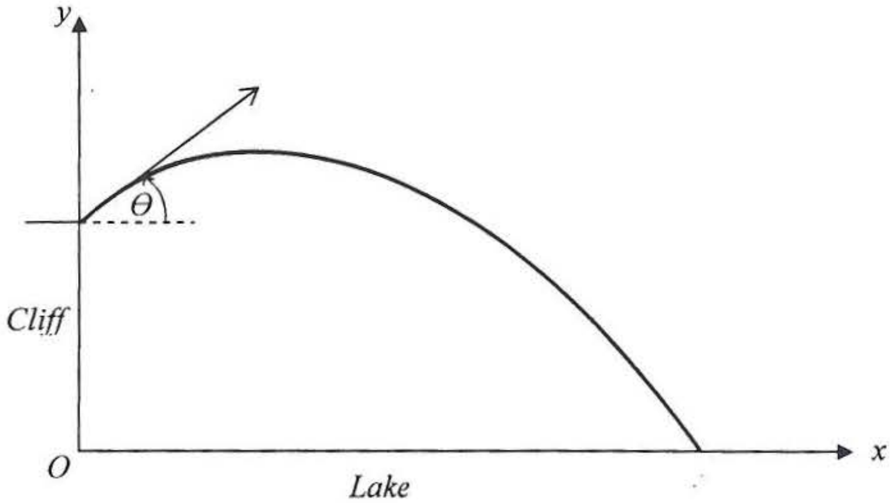
(c) Solve $x^3 - 21x^2 + 126x - 216$ given that the roots form 3 consecutive terms of a geometric series. **3**

(d) Use Simpson's Rule with 3 function values to find an approximation to **2**

$\displaystyle\int_0^{0.4} \sin^{-1} x \, dx$ to one decimal place.

Question 6 (12 marks) Use a SEPARATE sheet of paper. **Marks**

(a) A stone is projected with a velocity of 10 metres per second at an angle of elevation of $\theta = \tan^{-1}\left(\frac{3}{4}\right)$ from the top of a cliff 27 metres high overlooking a lake.



Assume that the equations of motion of the stone are

$$\ddot{x} = 0 \qquad \ddot{y} = -10$$

referred to the coordinate axes shown.

(i) Let (x, y) be the position of the stone at time t seconds after it was thrown, and before the stone hits the lake. **2**
It is known that $x = 8t$.
Show that $y = -5t^2 + 6t + 27$.

(ii) Calculate the time which elapses before the stone hits the lake and find the horizontal distance of the point of contact from the base of the cliff. **3**

(iii) What is the maximum height reached by the stone? **2**

(b) Find the coefficient of x^7 in the expansion of $\left(x^2 - \frac{1}{x}\right)^{12}\left(5 - \frac{1}{x^2}\right)^6$. **3**

(c) Find the Cartesian equation of a curve with the parametric equations **2**

$$x = t + \frac{1}{t} \text{ and } y = t - \frac{1}{t}.$$

Question 7 (12 marks) Use a SEPARATE sheet of paper. **Marks**

(a) Let $(3+2x)^{20} = \sum_{r=0}^{20} a_r x^r$

(i) Write down an expression for a_r. **1**

(ii) Show that $\frac{a_{r+1}}{a_r} = \frac{40-2r}{3r+3}$. **1**

(iii) Hence, or otherwise, find the value of the greatest coefficient in the expansion of $(3+2x)^{20}$. **2**

(b) Consider the function $f(x) = \frac{1}{1+x^2}$,

(i) Sketch the function $y = f(x)$, finding any asymptotes and stationary points. **2**

(ii) Write down the largest domain that contains $x = -1$ for which $y = f(x)$ has an inverse function. **1**

(iii) Find the inverse function $f^{-1}(x)$ for this domain and state the domain of $f^{-1}(x)$. **2**

(iv) Find the area bounded by the curve $f(x) = \frac{1}{1+x^2}$, the x axis and the values $x = -1$ and $x = 1$. **2**

(v) Prove that the area between this curve and the x axis is always less than π units^2. **1**

End of Examination

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STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

2008 HGHS Extension 1 Trial Sol^{ns}

QUESTION 1

a) $\lim_{x\to 0} \frac{\sin 3x}{2x}$

$= \lim_{x \to 0} \frac{3}{2} \frac{\sin 3x}{3x}$

$= \frac{3}{2}$

b) $y = 2x - 9$

$y = \frac{1}{3}x + \frac{8}{3}$

$\tan\theta = \frac{2 - \frac{1}{3}}{1 + 2\times\frac{1}{3}}$

$= \frac{5}{3}\times\frac{3}{5}$

$= 1$

$\theta = 45^\circ$

c) $\frac{d(3^x)}{dx} = \frac{d\, e^{x\ln 3}}{dx}$

$= \ln 3\, e^{x\ln 3}$

$= 3^x . \ln 3$

d) $D: -\frac{1}{3} \le x \le \frac{1}{3}$

$R: 0 \le y \le 2\pi$

e) $I = \int_{\pi/6}^{\pi/3} \frac{dx}{\cos^2 x \tan x}$

$= \int_{\pi/6}^{\pi/3} \frac{\sec^2 x\, dx}{\tan x}$

let $u = \tan x$

$du = \sec^2 x\, dx$

when $x = \frac{\pi}{6}$, $u = \frac{1}{\sqrt{3}}$

$x = \frac{\pi}{3}$, $u = \sqrt{3}$

$\therefore I = \int_{1/\sqrt{3}}^{\sqrt{3}} \frac{du}{u}$

$= \left[\ln u\right]_{1/\sqrt{3}}^{\sqrt{3}}$

$= \ln\sqrt{3} + \ln\sqrt{3}$

$= 2\ln\sqrt{3}$

$= \ln 3$

$= 1.099$

f) i) $y = x\cos^{-1}x - \sqrt{1-x^2}$

$\frac{dy}{dx} = \cos^{-1}x - \frac{x}{\sqrt{1-x^2}} + \frac{1}{2}(1-x^2)^{-\frac{1}{2}}.2x$

$= \cos^{-1}x - \frac{x}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}}$

$= \cos^{-1}x$

ii) $\int_0^1 \cos^{-1}x\, dx = \left[x\cos^{-1}x - \sqrt{1-x^2}\right]_0^1$

$= \cos^{-1}(1) - 0 - 0 + 1$

$= 1$

QUESTION 2

a) $x - 5 < \frac{14}{x}$

$x^2(x-5) < 14x$

$x^2(x-5) - 14x < 0$

$x(x^2 - 5x - 14) < 0$

$x(x-7)(x+2) < 0$

$\therefore x < -2,\ 0 < x < 7$

b) $\tan 2\theta = \sqrt{3}$

$= \tan\frac{\pi}{3}$

$2\theta = n\pi + \frac{\pi}{3}$

$\theta = \frac{1}{2}n\pi + \frac{\pi}{6}$

c) $f(x) = 2x^3 + ax^2 + bx + 6$

$f(1) = 2 + a + b + 6 = -6$

$a + b = -14$ — (1)

$f(-2) = -16 + 4a - 2b + 6 = 0$

$4a - 2b = 10$ — (2)

$4\times(1) - (2) \Rightarrow 6b = -66$

$b = -11$

$\therefore a = -3$

$\therefore a = -3,\ b = -11$

d) $\int_0^{\pi/4} \cos^2\left(\frac{x}{2}\right) dx = \frac{1}{2}\int_0^{\pi/4} \cos x + 1\ dx$

~~$= \frac{1}{2}\left[\sin x\right]_0^{\pi/4}$~~ $= \frac{1}{2}\left[\sin x + x\right]_0^{\pi/4}$

~~$= \frac{1}{2\sqrt{2}}$~~

$= \frac{1}{2}\left(\frac{1}{\sqrt{2}} + \frac{\pi}{4}\right)$

$= \frac{1}{2\sqrt{2}} + \frac{\pi}{8} = \frac{2\sqrt{2} + \pi}{8}$

e) $\angle APQ = \angle PRQ$ ($\angle$ in alt segment)

In $\triangle PQR$ $\angle PRQ = \angle PQR$ ($\angle$'s opp equal sides)

$\therefore \angle APQ = \angle PQR$

$\therefore RQ \parallel BA$ (alt $\angle$'s equal)

QUESTION 3.

a) i) $7! = 5040$

ii) $7! - 6! \times 2 = 3600$

b) i) $f(x) = x^3 + ax^2 + bx + c$

$f'(x) = 3x^2 + 2ax + b$

~~$f''(x) = 6x + 2a$~~

now $3\lambda^2 + 2a\lambda + b = 0$

and $3\beta^2 + 2a\beta + b = 0$

$\therefore 3\lambda^2 + 2a\lambda = 3\beta^2 + 2a\beta$

$\lambda^2 - \beta^2 = \frac{2}{3}a\beta - \frac{2}{3}a\lambda$

$(\lambda-\beta)(\lambda+\beta) = -\frac{2}{3}a(\lambda-\beta)$

$\therefore \lambda+\beta = -\frac{2}{3}a$

→ or $\lambda + \beta$ are both roots of $f'(x)$

$\therefore \lambda+\beta = \frac{-2a}{3}$ sum of the roots.

ii) pt of inflexion when $f''(x) = 0$

$f''(x) = 6x + 2a$

$= 6\left(\frac{\lambda+\beta}{2}\right) + 2a$

$= 3\left(-\frac{2}{3}a\right) + 2a$

$= -2a + 2a$

$= 0$

c) i) $T = 20 + Ae^{-kt}$

$\frac{dT}{dt} = -kAe^{-kt}$

$= -k(T-20)$

ii) when $t=0$, $T = 80$

$\therefore 80 = 20 + A$

$A = 60$

when $t = 10$, $T = 50$

$\therefore 50 = 20 + 60e^{-10k}$

$60e^{-10k} = 30$

$e^{-10k} = 0.5$

$-10k = \ln 0.5$

$k = 0.0693$

iii) $T = 20 + 60e^{-0.0693t}$

when $T = 30$

$30 = 20 + 60e^{-0.0693t}$

$10 = 60e^{-0.0693t}$

$e^{-0.0693t} = 1/6$

$-0.0693t = \ln(1/6)$

$t = 25.86$ min

$= 26$ min

QUESTION 4.

a) i) $x = R\cos(nt + \alpha) + 1$

$T = \frac{2\pi}{n}$

$\pi = \frac{2\pi}{n}$

$\therefore n = 2$

$\therefore x = R\cos(2t + \alpha) + 1$

$\frac{dx}{dt} = -2R\sin(2t+\alpha)$

when $t = 0$ $\frac{dx}{dt} = 0$

$\therefore 0 = -2R\sin\alpha$

$\alpha = 0$

$\therefore x = R\cos(2t) + 1$

when $t = 0$, $x = -3$

$-3 = R + 1$

$R = -4$

$\therefore R = -4$

$n = 2$

$\alpha = 0$

ii) $V = 8\sin(2 \times \frac{\pi}{6})$

$= 8\frac{\sqrt{3}}{2}$

$= 4\sqrt{3}$

b) let $f(x) = x\ln x - 1$

$f(1) = 1\ln 1 - 1$

$= -1$

$f(2) = 2\ln 2 - 1$

$= 0.4$

$\therefore$ soln lies between $x = 1, 2$.

ii) $f'(x) = \ln x + 1$

$f'(2) = \ln 2 + 1$

$= 1.693$

$x = 2 - \frac{0.386}{1.693}$

$= 1.772$

$\therefore x = 1.8$

c) $L.H.S. = \frac{2\tan\theta}{1+\tan^2\theta}$

$= \frac{2\sin\theta}{\cos\theta} \times \cos^2\theta$

$= 2\sin\theta\cos\theta$

$= \sin 2\theta$

$= R.H.S.$

d) i) ${}^{25}C_2 \left(\frac{1}{9}\right)^2 \left(\frac{8}{9}\right)^{23} = 0.2467$

ii) $1 - {}^{25}C_0 \left(\frac{8}{9}\right)^{25} - {}^{25}C_1 \left(\frac{1}{9}\right)^1 \left(\frac{8}{9}\right)^{24}$

$= 1 - 0.05262 - 0.16445$

$= 0.7829$

QUESTION 5

a) • Prove true for $n=1$

$LHS = 1^3 = 1$

$RHS = 1^2 = 1$

$\therefore$ True for $n=1$

• Assume true for $n=k$

ie $1^3+2^3+\dots+k^3 = (1+2+3+\dots+k)^2$

• Prove true for $n=k+1$

ie $1^3+2^3+\dots+k^3+(k+1)^3 = (1+2+\dots+k+(k+1))^2$

$L.H.S = (1+2+\dots+k)^2+(k+1)^3$

$= \left[\frac{k}{2}(k+1)\right]^2 + (k+1)^3$ (sum of AP k terms)

$= (k+1)^2\left(\frac{k^2}{4}+k+1\right)$

$= \frac{(k+1)^2(k^2+4k+4)}{4}$

$= \frac{(k+1)^2(k+2)^2}{4}$

$R.H.S = \left[\frac{k+1}{2}(k+1+1)\right]^2$ (sum of AP k+1 terms)

$= \frac{(k+1)^2(k+2)^2}{4}$

$= L.H.S$

$\therefore$ True for $n=k+1$

QUESTION 5

b) i) $\frac{r}{3} = \frac{h}{10}$ similar $\triangle$'s

$r = \frac{3h}{10}$

ii) $\frac{dv}{dt} = 2$

$V = \frac{\pi r^2 h}{3}$

$= \pi \cdot \frac{9h^2}{300} \cdot h$

$= \frac{9\pi h^3}{300}$

$\frac{dv}{dh} = \frac{27\pi h^2}{300}$

$\therefore \frac{dh}{dt} = \frac{dh}{dv} \times \frac{dv}{dt}$

$= \frac{100}{9\pi h^2} \times 2$

$= \frac{200}{9\pi h^2}$

when $h=8$

$\frac{dh}{dt} = \frac{100}{\pi \times 8^2}$

$= ~~0.497~~$ 0.1105 m/min.

$= \frac{25}{72\pi}$

c) Let roots be

$\frac{a}{r}, a, ar$

$\frac{a}{r} + a + ar = 21$

$a^3 = 216$

$a = 6$.

$\therefore \frac{1}{r} + 1 + r = \frac{7}{2}$

$7r = 2 + 2r + 2r^2$

$2r^2 - 5r + 2 = 0$

$(2r-1)(r-2) = 0$

$\therefore r = \frac{1}{2}, 2$

$\therefore$ roots are, 3, 6, 12

d) $\int_0^{0.4} \sin^{-1}x \, dx = \frac{0.4}{6}\left[0 + 4\times 0.201 + 0.41\dots\right.$

$= 0.081$

$= 0.1$

QUESTION 6.

a) i) $\ddot{y} = -10$

$\dot{y} = -10t + 10\sin\theta$

$= -10t + \frac{10\times3}{5}$

$= -10t + 6$

$y = \frac{-10t^2}{2} + 6t + 27$

$= -5t^2 + 6t + 27$

ii) when $y = 0$

$5t^2 - 6t - 27 = 0$

$(5t+9)(t-3) = 0$

$\therefore t = 3$

when $t = 3$, $x = 8\times3$

$= 24$

iii) max height when $\dot{y} = 0$

i.e. $-10t + 6 = 0$

$t = \frac{6}{10}$

when $t = 0.6$, $y = -5(0.6)^2 + 6(0.6) + 27$

$= 28.8$

b) $\sum_{k=0}^{12} {}^{12}C_k (x^2)^{12-k} (-1)^k (x^{-1})^k \cdot \sum_{r=0}^{6} {}^{6}C_r\, 5^{6-r} \cdot (-1)^r \cdot (x^{-2})^r$

$\therefore x^{24-2k-k-2r} = x^{24-3k-2r}$

$\therefore 24 - 3k - 2r = 7$

$3k + 2r = 17$

k	r	3k+2r
5	1	17
3	4	17

$\therefore$ coeff's $= {}^{12}C_5 (-1)^5 \cdot {}^6C_1\, 5^5 (-1)^1$

$+ {}^{12}C_3 (-1)^3 \cdot {}^6C_4\, 5^2 (-1)^4$

$= 14850000 - 82500$

$= 14767500$

c) $x = t + \frac{1}{t}$

$y = t - \frac{1}{t}$

$x^2 = t^2 + 2 + \frac{1}{t^2}$ —— (1)

$y^2 = t^2 - 2 + \frac{1}{t^2}$ —— (2)

$x^2 - y^2 = 4$.

QUESTION 7.

a) i) $(3+2x)^{20} = \sum_{r=0}^{20} {}^{20}C_r\, 3^{20-r} \cdot 2^r \cdot x^r$

$\therefore a_r = {}^{20}C_r \cdot 3^{20-r} \cdot 2^r$

ii) $\frac{a_{r+1}}{a_r} = \frac{{}^{20}C_{r+1} \cdot 3^{19-r} \cdot 2^{r+1}}{{}^{20}C_r \cdot 3^{20-r} \cdot 2^r}$

$= \frac{20!}{(r+1)!\,(19-r)!} \times \frac{r!\,(20-r)!}{20!} \times \frac{2}{3}$

$= \frac{20-r}{r+1} \times \frac{2}{3}$

$= \frac{40-2r}{3r+3}$

iii) $\frac{40-2r}{3r+3} \geq 1$

$40 - 2r \geq 3r + 3$

$5r \leq 37$

$r \leq 7\frac{2}{5}$

$\therefore r = 7$

$\therefore a_{r+1} = a_8 = {}^{20}C_8 \cdot 3^{12} \cdot 2^8$

$= 1.71 \times 10^{13}$

QUESTION 7.

b) i) $f(x) = \frac{1}{1+x^2}$

$f'(x) = -(1+x^2)^{-2} \cdot 2x$

$= \frac{-2x}{(1+x^2)^2}$

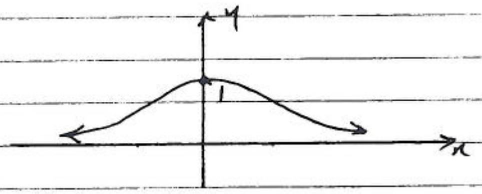
$\lim_{x\to\infty} \frac{1}{1+x^2} = 0^+$

$\lim_{x\to-\infty} \frac{1}{1+x^2} = 0^+$

stat pt when $-2x = 0$

$x = 0$

when $x = 0$, $y = 1$.



ii) $x \le 0$

iii) $x = \frac{1}{1+y^2}$

$1+y^2 = \frac{1}{x}$

$y^2 = \frac{1}{x} - 1$

$y = -\sqrt{\frac{1-x}{x}}$

Domain $0 < x \le 1$

iv) $A = \int_{-1}^{1} \frac{1}{1+x^2}\,dx$

$= \left[\tan^{-1}x\right]_{-1}^{1}$

$= \frac{\pi}{4} + \frac{\pi}{4}$

$= \frac{\pi}{2}$

v) $A_{max} < 2 \lim_{a\to\infty} \int_0^a \frac{1}{1+x^2}\,dx$.

$< 2 \lim_{a\to\infty} \left[\tan^{-1}x\right]_0^a$

$< 2 \lim_{a\to\infty} \left(\tan^{-1}a - 0\right)$

$< 2 \times \frac{\pi}{2}$

$< \pi$.

$\therefore$ area always less than π.