HORNSBY GIRLS HIGH SCHOOL

Mathematics Extension 1

Year 12 Higher School Certificate
2012 Trial Examination

STUDENT NUMBER: ____________________

General Instructions

- Reading Time – 5 minutes
- Working Time – 2 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided
- Show all necessary working
- Marks may be deducted for untidy and poorly arranged work
- Do not use correction fluid or tape
- Do not remove this paper from the examination

Total marks – 70

Section I Pages 2 – 4

10 marks

Attempt Questions 1 – 10

Answer on the Objective Response Answer Sheet provided

Section II Pages 5 – 11

60 marks

Attempt Questions 11 – 14

Start each question in a new writing booklet

Write your student number on every booklet

At the end of the assessment:

- Order you solutions, starting with Objective Response answer sheet, then Questions 11-14
- Place your question paper on top
- Do **NOT** staple through

This assessment task constitutes 45% of the Higher School Certificate Course School Assessment

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the Objective Response answer sheet for Questions 1 – 10

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1 $\lim_{x \to 0} \frac{\sin 3x}{5x} =$

(A) $\frac{3}{5}$

(B) $\frac{5}{3}$

(C) 3

(D) 5

2 If $(x-2)$ is a factor of $P(x) = 2x^3 + x + a$, then $a =$

(A) 18

(B) 9

(C) −18

(D) −9

3 A committee of 3 men and 4 women is to be formed from a group of 8 men and 6 women. The expression which represents the number of ways this can be done is

(A) ${}^8P_3 \times {}^6P_4$

(B) ${}^8P_3 + {}^6P_4$

(C) ${}^8C_3 \times {}^6C_4$

(D) ${}^8C_3 + {}^6C_4$

4 The inverse function of $y = \sin 2x$ is

(A) $y = \sin^{-1} x$

(B) $y = \sin^{-1} \frac{x}{2}$

(C) $y = 2\sin^{-1} x$

(D) $y = \frac{1}{2}\sin^{-1} x$

5 The derivative of $\tan^{-1} \frac{x}{3}$ is

(A) $\frac{3}{9 - x^2}$

(B) $\frac{3}{9 + x^2}$

(C) $\frac{1}{3 + x^2}$

(D) $\frac{1}{3 - x^2}$

6 The coefficient of x^3 in the expansion of $(2 + x)^5$ is

(A) 40
(B) 80
(C) 10
(D) 20

7 The expression $\cos\theta - \sin\theta$ is equivalent to

(A) $2\cos\left(\theta - \frac{\pi}{4}\right)$

(B) $2\cos\left(\theta + \frac{\pi}{4}\right)$

(C) $\sqrt{2}\cos\left(\theta - \frac{\pi}{4}\right)$

(D) $\sqrt{2}\cos\left(\theta + \frac{\pi}{4}\right)$

8 The primitive function of $\sin^2 2x$ is

(A) $\frac{1}{2}x+\frac{1}{2}\sin 4x+c$

(B) $\frac{1}{2}x-\frac{1}{2}\sin 4x+c$

(C) $\frac{1}{2}x+\frac{1}{8}\sin 4x+c$

(D) $\frac{1}{2}x-\frac{1}{8}\sin 4x+c$

9 A particle is moving in simple harmonic motion. Which of the following is true?

(A) the speed is zero at the centre of motion.

(B) the speed is a maximum at the centre of motion.

(C) the acceleration is zero at the extremities of motion.

(D) the acceleration is a maximum at the centre of motion.

10 If $t=\tan\frac{x}{2}$, which of the following is an expression for $\frac{dx}{dt}$?

(A) $\frac{1}{2}(1+t^2)$

(B) $1+t^2$

(C) $\frac{2}{1+t^2}$

(D) $\frac{1}{1+t^2}$

End of Section I

Section II

90 marks
Attempt Questions 11-14
Allow about 1.5 hours for this section

Begin each question in a new writing booklet, indicating the question number.
Extra writing booklets are available.

Question 11 (15 marks) Start a new writing booklet.

(a) Solve for x: $\dfrac{5}{x-1} \leq 4$. **3**

(b) Find the acute angle between the lines $2x + y = 4$ and $y = 5x - 9$, to the nearest degree. **2**

(c) Find the quotient, $Q(x)$, and the remainder, $R(x)$, when the polynomial $P(x) = x^4 - x^2 + 1$ is divided by $x^2 + 1$. **2**

(d) Use the substitution $u = x^2 + 1$ to evaluate $\displaystyle\int_0^1 \frac{2x}{(x^2+1)^3}\,dx$. **3**

(e) Find the coordinates of the point P that divides the interval joining $(-3, 4)$ and $(5, 6)$ internally in the ratio $1:3$. **2**

(f) Ten people are attending a dinner party. The ten people will be seated at a round table.

(i) In how many ways can the ten people be arranged around the table? **1**

(ii) Two guests, Mark and Cecil wish to sit next to each other. In how many ways can the ten people be arranged such that Mark and Cecil are sitting next to each other? **1**

(iii) Hence, find the probability of Mark and Cecil sitting together. **1**

Question 12 (15 marks) Start a new writing booklet.

(a) The velocity $v\,\text{ms}^{-1}$ of a particle moving in simple harmonic motion along the x-axis is given by

$v^2 = 8 + 2x - x^2$.

(i) Between which two points is the particle oscillating? **2**

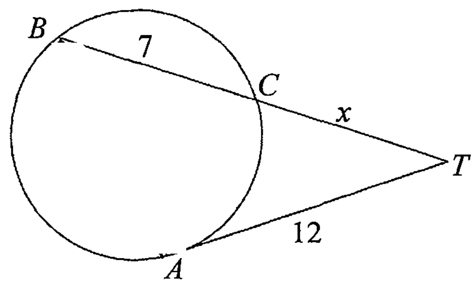
(ii) What is amplitude of the motion? **1**

(iii) Find acceleration of the particle in terms of x. **2**

(iv) Find the period of the oscillation. **1**

(v) Find the maximum speed of the particle. **1**

(b)



NOT TO SCALE

The line AT is the tangent to the circle at A. BT is a secant meeting the circle at B and C. **2**
Given that $AT = 12$, $CD = 7$ and $CT = x$, find the value of x, giving reasons.

Question 12 continues on page 7

Question 12 (continued)

(c) The polynomial $P(x) = 4x^3 + 2x^2 + 1$ has one real root in the interval $-1 < x < 0$.

(i) Find any stationary points of $y = P(x)$ and determine their nature. **2**

(ii) Sketch the graph of $y = P(x)$ for $-1 \leq x \leq 1$. Clearly label any stationary points. **1**

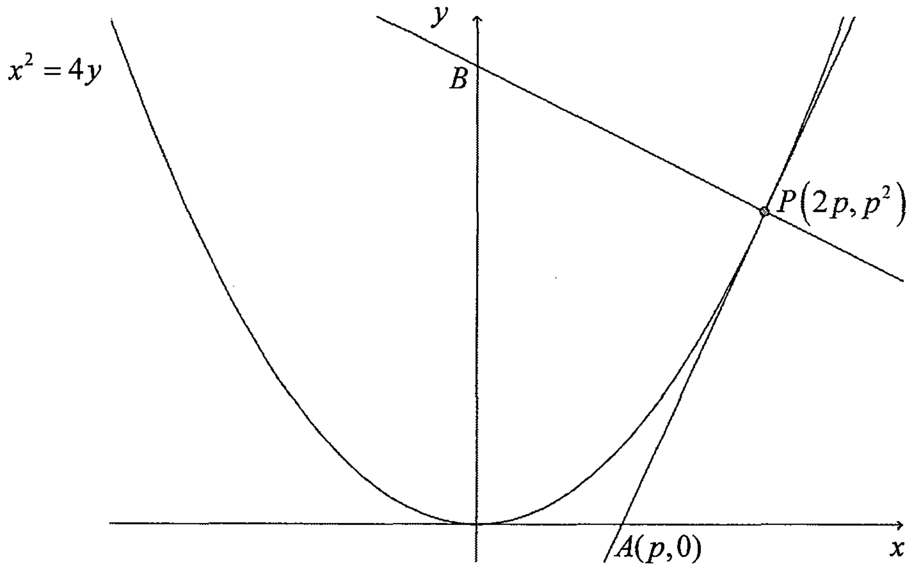
(iii) Let $x = \frac{-1}{4}$ be a first approximation to the root. Apply Newton's method once to obtain another approximation to the root. **2**

(iv) Explain why the application of Newton's method in part (ii) was NOT effective in improving the approximation of the root. **1**

End of Question 12

Question 13 (15 marks) Start a new writing booklet.

(a)



NOT TO SCALE

The diagram above shows the graph of the parabola $x^2 = 4y$. The tangent to the parabola at $P(2p, p^2)$, where $p > 0$, intersects the x-axis at $A(p,0)$.

The normal to the parabola at P intersects the y-axis at B.

The equation of the tangent at P is $y = px - p^2$. **(Do not prove this result.)**

(i) Show that the equation of the normal at P is $x + py = 2p + p^3$. **2**

(ii) Find the coordinates of B. **1**

(iii) Let C be the midpoint of AB. Find the Cartesian equation of the locus of C. **2**

(b) Prove by Mathematical Induction that $4^n > 2n+1$ for any positive integer n. **3**

(c) The function $h(x)$ is given by $h(x) = \sin^{-1} x + \cos^{-1} x$, where $0 \le x \le 1$.

(i) Find $h'(x)$. **1**

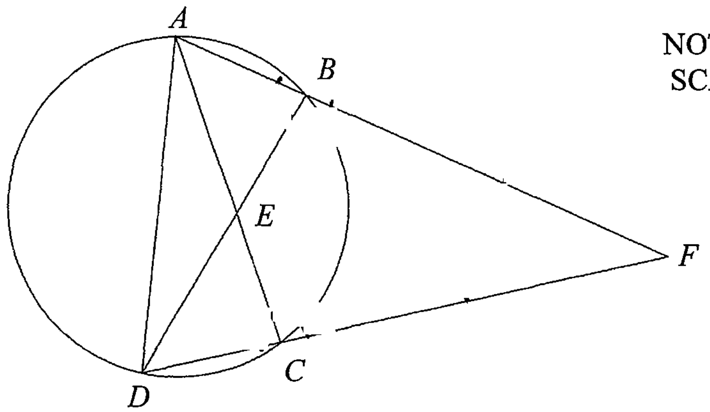
(ii) Sketch the graph of $y = h(x)$ for the given domain. **1**

Question 13 continues on page 9

Question 13 (continued)

(d) The polynomial $P(x) = x^3 - 2x^2 + kx + 24$ has three real roots.
It is known that two of the roots are equal but opposite in sign. Find the value of k. **2**

(e)



NOT TO SCALE

The points A, B, C and D are placed on a circle of radius r such that AC and BD intersect at E. The lines AB and DC are produced to meet at F, and $BECF$ is a cyclic quadrilateral.

Copy or trace the diagram into your writing booklet.

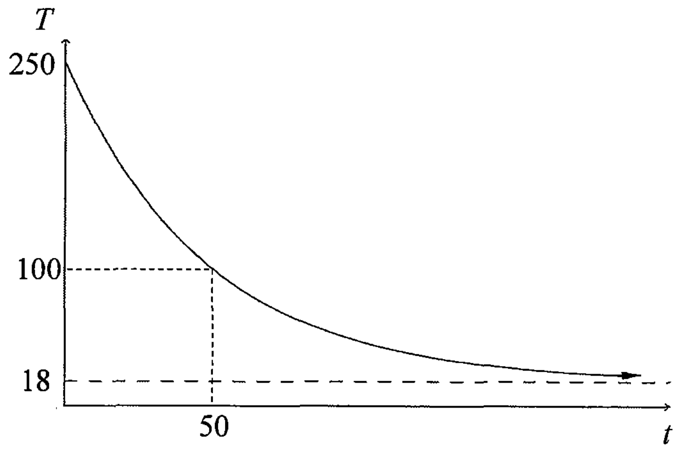
(i) Find the size of $\angle DBF$, giving reasons for your answer. **2**

(ii) Find an expression for the length of AD in terms of r, giving reasons. **1**

End of Question 13

Question 14 (15 marks) Start a new writing booklet.

(a) The graph shown below shows the cooling for a container of paraffin oil which has been heated to a temperature of $250°$C then allowed to cool in air whose temperature was $18°$C.



NOT TO SCALE

It is known that the rate at which the temperature T of the oil is changing is given by $\frac{dT}{dt} = k(T - M)$, where M is the temperature of the surrounding air and t is the time elapsed after cooling begins, in minutes.

(i) Show that $T = M + Ae^{kt}$ is a solution to the given equation. **1**

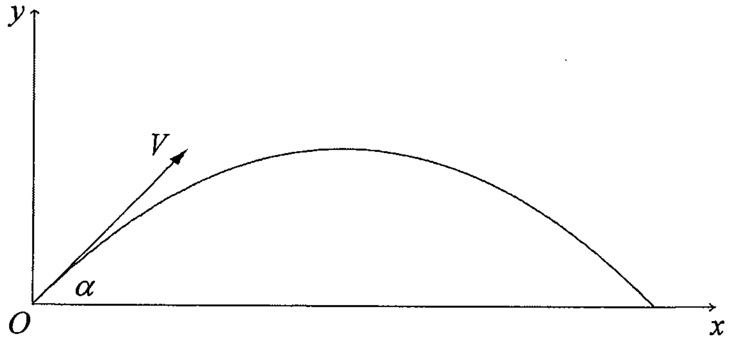
(ii) Find the values of M and A. **2**

(iii) Find the value of k, correct to three significant figures. **2**

Question 14 continues on page 11

Question 14 (continued)

(b) A particle is projected from a point O on horizontal ground, with speed $V\text{ms}^{-1}$ at an angle of elevation to the horizontal of α, as shown in the diagram below.



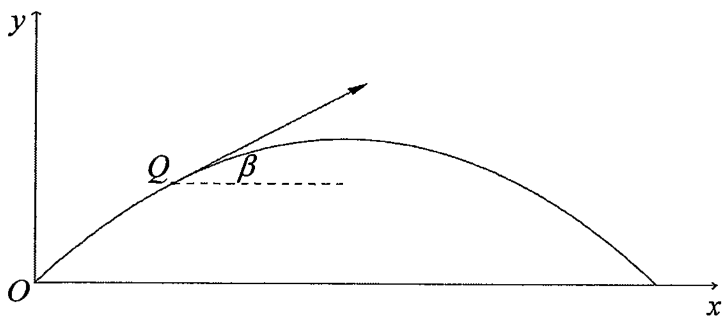
NOT TO SCALE

The particle's equations of motion are $\ddot{x}=0$, $\ddot{y}=-g$.

(i) By integration, show that the equations of motion are $x=Vt\cos\alpha$ and $y=\frac{-gt^2}{2}+Vt\sin\alpha$. **2**

(ii) Show that the time of flight of the particle is $\frac{2V\sin\alpha}{g}$. **1**

The particle reaches a point Q, as shown on the diagram below, where the direction of flight makes an angle of β with the horizontal.



NOT TO SCALE

(iii) Show that $\tan\beta=\frac{V\sin\alpha-gt}{V\cos\alpha}$. **1**

(iv) Hence, show that the time taken for the particle to travel from O to Q is $\frac{V\sin(\alpha-\beta)}{g\cos\beta}$. **2**

(v) Consider the case where $\beta=\frac{\alpha}{2}$. If the time taken to travel from O to Q is one-third of the total time of flight, show that $\tan\frac{\alpha}{2}=\frac{2}{3}\sin\alpha$. **2**

(vi) Hence, find the value of α. **2**

End of paper

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Year 12 Mathematics Extension 1 Hornsby Girls High School HSC Trial Examination Solutions

Objective Response Question 1-10

1. A
2. C
3. C
4. D
5. B
6. A
7. D
8. D
9. B
10. C

Question 11

(a)

$\frac{5}{x-1} \le 4$

$5(x-1) \le 4(x-1)^2$

$0 \le (x-1)(4x-4-5)$

$0 \le (x-1)(4x-9)$

From parabola,

$x \ge \frac{9}{4}, x < 1$

(b)

$y = -2x+4$

$m_1 = -2$

$y = 5x-9$

$m_2 = 5$

$\tan\theta = \left|\frac{m_1 - m_2}{1+m_1m_2}\right|$

$\tan\theta = \left|\frac{-2-5}{1+-2\times 5}\right|$

$= \left|\frac{-7}{1-10}\right|$

$= \left|\frac{7}{9}\right|$

$\theta = 37°$

(c)

$P(x) = x^4 - x^2 + 1$

By long division

$Q(x) = x^2 - 2$

$R(x) = 3$

(d)

$u = x^2 + 1$

$\frac{du}{dx} = 2x$

$When\ x = 0, u = 1$

$When\ x = 1, u = 2$

$\int_0^1 \frac{2x}{(x^2+1)^3}dx = \int_1^2 u^{-3}\,du$

$= \left[\frac{u^{-2}}{-2}\right]_1^2$

$= \frac{-1}{8} + \frac{1}{2}$

$= \frac{3}{8}$

(e)

$x_1 = -3 \qquad x_2 = 5$

$y_1 = 4 \qquad y_2 = 6$

$n = 3 \qquad m = 1$

$x = \frac{5\times 1 + 3\times -3}{4} \qquad y = \frac{3\times 4 + 6\times 1}{4}$

$= \frac{5-9}{4} \qquad = \frac{18}{4}$

$= -1 \qquad = \frac{9}{2}$

$P\left(-1, \frac{9}{2}\right)$

(f)

(i) $9! = 326880$

(ii)

$8! \times 2! = 80640$

(iii)

$P(sitting\ together) = \frac{80640}{362880}$

$= \frac{2}{9}$

Question 13

(a)

(i)

Consider tangent:

$y = px - p^2$

$m_{tangent} = p$

$\therefore m_{normal} = \frac{-1}{p}$

Equation of normal is

$y - p^2 = \frac{-1}{p}(x - 2p)$

$py - p^3 = -x + 2p$

$x + py = 2p + p^3$

(ii)

Let $x = 0$

$py = 2p + p^3$

$y = 2 + p^2$

Therefore $B = \left(0, 2 + p^2\right)$.

(iii)

$x = \frac{0 + p}{2}$

$x = \frac{p}{2}$

$2x = p \ldots(1)$

$y = \frac{2 + p^2}{2}$

$= \frac{2 + 4x^2}{2}$

$y = 2x^2 + 1$

(b)

$4^n > 2n + 1$

$Let\ n = 1$

$LHS = 4$

$RHS = 2 + 1$

$LHS > RHS$

$\therefore True\ for\ n = 1$

Assume true for $n = k$

$4^k > 2k + 1$

Let $n = k + 1$

(RTP: $4^{k+1} > 2k + 3$

$4^{k+1} = 4 \times 4^k$

$> 4 \times (2k + 1)$ by assumption

$= 8k + 4$

$> 2k + 3$ for all positiveintegers k

True for n=k+1 if true for $n = k$

Since true for $n = 1$, it is true for $n = 2$, and hence true for $n = 3$, all hence all positive integers n.

(c)

(i)

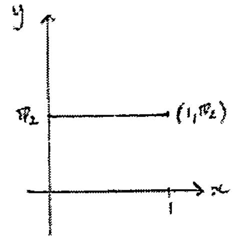
$h(x) = \sin^{-1} x + \cos^{-1} x$

$h'(x) = \frac{1}{\sqrt{1 - x^2}} + \frac{-1}{\sqrt{1 - x^2}}$

$= 0$

(ii) $h(0) = \frac{\pi}{2}$

Hence, $h(x) = \frac{\pi}{2}$, since $h'(x)$ is zero, must be horizontal line



(d)

Let the roots be $\alpha, -\alpha\ and\ \beta$

$\alpha + -\alpha + \beta = 2$ $\qquad$ $-\alpha^2\beta = -24$

$\beta = 2$ $\qquad$ $\alpha = 2\sqrt{3}$

$\alpha \times -\alpha + -\alpha\beta + \alpha B = k$

$k = -\alpha^2$

$k = -12$

(e)

(i)

Let $\angle DBF = \alpha$

$\angle ECD = \alpha$ (exterior $\angle$ of cyclic quadrilateral equal to opposite interior angle)

$\angle ABD = \angle ACD = \alpha$ (angles in the same segment)

$\angle ABD + \angle DBF = 180^\circ$ (angles in straight line)

$2\alpha = 180^\circ$

$\alpha = 90 = \angle DBF$

(ii)

Since AD is a diameter ($\angle$ in the semicircle is a right-angle, $\angle ABD = 90^\circ$)

$\therefore AD = 2r$.

Question 12

(a)

(i) $v^2 = 8 + 2x - x^2$

Let $v = 0$

$x^2 - 2x - 8 = 0$

$(x-4)(x+2) = 0$

$x = -2, x = 4$

Oscillating between $x = -2$ and $x = 4$.

(ii)

Amplitude is $\dfrac{4-(-2)}{2} = 3$ metres

(iii)

$$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$
$$= \frac{1}{2} \times \frac{d}{dx}\left(8 + 2x - x^2\right)$$
$$= \frac{1}{2}(2 - 2x)$$
$$= -(x-1)$$

Acceleration is $\ddot{x} = -(x-1)$

(iv)

$$Period = \frac{2\pi}{n}$$
$$= \frac{2\pi}{1}$$

Period is 2π seconds

(v)

Max speed occurs at the centre of motion.

Let $x = 1$

$$v^2 = 8 + 2 \times 1 - 1^2$$
$$= 8 + 2 - 1$$
$$= 9$$

$|v| = 3$

Maximum speed is 3ms^{-1}

(b)

$x(x+7) = 12^2$ (The square of the length of the tangent from an external point is equal to the product of the intercepts of the secant passing through this point)

$(x+16)(x-9) = 0$

$x = 9\ (x > 0)$

(c)

(i)

$P(x) = 4x^3 + 2x^2 + 1$

$P'(x) = 12x^2 + 4x$

$P'(x) = 0$

$12x^2 + 4x = 0$

Let $4x(3x+1) = 0$

$x = 0, x = \dfrac{-1}{3}$

$P''(x) = 24x + 4$

When $P''(0) = 4$

$P(0) = 1$

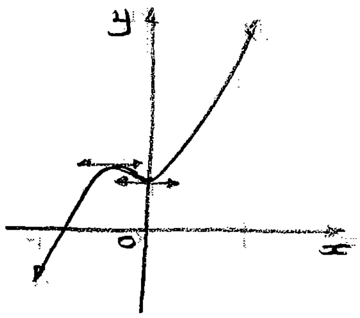
Therefore $(0,1)$ is a minimum turning point.

$P''\left(\dfrac{-1}{3}\right) = -4$

$P\left(\dfrac{-1}{3}\right) = \dfrac{29}{27}$

Therefore $\left(\dfrac{-1}{3}, \dfrac{29}{27}\right)$ is a maximum turning point.

(ii)



(iii)

$$P\left(\frac{-1}{4}\right) = 4 \times \left(\frac{-1}{4}\right)^3 + 2 \times \left(\frac{-1}{4}\right)^2 + 1$$
$$= \frac{17}{16}$$
$$P'\left(\frac{-1}{4}\right) = 12 \times \left(\frac{-1}{4}\right)^2 + 4 \times \left(\frac{-1}{4}\right)$$
$$= \frac{-1}{4}$$
$$x_1 = \frac{-1}{4} - \frac{\frac{17}{16}}{\frac{-1}{4}}$$
$$= 4$$

(iv)

The first approximation is too close to a stationary point, meaning the gradient of the tangent at the point is close to zero. Hence, the tangent will intersect the x-axis at a further distance away from the actual root

Question 14

(a)

(i)

$T = M + Ae^{kt}$

$\frac{dT}{dt} = k \times Ae^{kt}$

$= k(Ae^{kt} + M - M)$

$= k(T - M)$

(ii)

$M = 18$

$T = 18 + Ae^{kt}$

$When\ t = 0, T = 250$

$250 = 18 + A$

$A = 232$

(iii)

$T = 18 + 232e^{kt}$

Sub $t = 50, T = 100$

$100 = 18 + 232e^{50k}$

$\frac{100-18}{232} = e^{50k}$

$\frac{41}{116} = e^{50k}$

$k = \frac{1}{50}\ln\left(\frac{41}{116}\right)$

$k \approx -0.208\ (3sf)$

(b) (i)

$\ddot{x} = 0$

$\dot{x} = \int 0\, dt$

$= c_1$

When

$t = 0, \dot{x} = V\cos\alpha,$

$c_1 = V\cos\alpha$

$\dot{x} = V\cos\alpha$

$x = \int V\cos\alpha\, dt$

$= Vt\cos\alpha + c_2$

When

$t = 0, x = 0, c_2 = 0$

$\therefore x = Vt\cos\alpha$

$\ddot{y} = -g$

$\dot{y} = \int -g\, dt$

$= -gt + c_3$

When

$t = 0, \dot{y} = V\sin\alpha.$

$\therefore c_3 = V\sin\alpha$

$\dot{y} = V\sin\alpha$

$y = \int(-gt + V\sin\alpha)dt$

$y = -\frac{gt^2}{2} + Vt\sin\alpha + c_4$

When $t = 0, y = 0, c_4 = 0$

$y = \frac{-gt^2}{2} + Vt\sin\alpha$

(ii) Method 1: Let $y = 0$

$0 = -\frac{gt^2}{2} + Vt\sin\alpha$

$0 = -t\left(\frac{gt}{2} - V\sin\alpha\right)$

$\frac{gt}{2} = V\sin\alpha \quad (t > 0)$

$t = \frac{2V\sin\alpha}{g}$

Method 2: Let $\dot{y} = 0$

$0 = -gt + V\sin\alpha$

$t = \frac{V\sin\alpha}{g}$

Max height at $\frac{V\sin\alpha}{g}$

By symmetry, time of flight is $\frac{2V\sin\alpha}{g}$

(iii)

At Q, $\dot{x} = V\cos\alpha,\ \dot{y} = V\sin\alpha$ $\quad \tan\beta = \frac{\dot{y}}{\dot{x}}$

$= \frac{V\sin\alpha - gt}{V\cos\alpha}$

(iv)

$\tan\beta = \frac{V\sin\alpha - gt}{V\cos\alpha}$

$\frac{\sin\beta}{\cos\beta} = \frac{V\sin\alpha - gt}{V\cos\alpha}$

$\frac{V\sin\beta\cos\alpha}{\cos\beta} = V\sin\alpha - gt$

$gt = V\sin\alpha - \frac{V\sin\beta\cos\alpha}{\cos\beta}$

$t = \frac{V\sin\alpha\cos\beta - V\sin\beta\cos\alpha}{g\cos\beta}$

$t = \frac{V\sin(\alpha-\beta)}{g\cos\beta}$

(v) Time taken to travel from Q is 2/3 of time of flight

$\frac{V\sin(\alpha-\beta)}{g\cos\beta} = \frac{1}{3} \times \frac{2V\sin\alpha}{g}$

$\frac{V\sin\left(\alpha - \frac{\alpha}{2}\right)}{g\cos\frac{\alpha}{2}} = \frac{2V\sin\alpha}{3g}$

$\frac{V}{g}\tan\frac{\alpha}{2} = \frac{2V\sin\alpha}{3g}$

$\tan\frac{\alpha}{2} = \frac{2}{3}\sin\alpha$

(vi)

Method 1: Let $t = \tan\frac{\alpha}{2}$

$t = \frac{2}{3} \times \frac{2t}{1+t^2}$

$t = \frac{4t}{3+3t^2}$

$3 + 3t^2 = 4$

$3t^2 = 1$

$t^2 = \frac{1}{3}$

$t = \frac{1}{\sqrt{3}}$ (α in 1st quad)

$\tan\frac{\alpha}{2} = \frac{1}{\sqrt{3}}$

$\frac{\alpha}{2} = 30°$

$\alpha = 60°$