HORNSBY GIRLS HIGH SCHOOL

Mathematics Extension 1

Year 12 Higher School Certificate
Trial Examination Term 3 2018

STUDENT NUMBER: ______________________________

General Instructions

- Reading Time – 5 minutes.
- Working Time – 2 hours.
- Write using black or blue pen.
  Black pen is preferred.
- NESA-approved calculators and drawing templates may be used.
- A reference sheet is provided separately.
- In Questions 11 – 14, show relevant mathematical reasoning and/or calculations.
- Marks may be deducted for untidy and poorly arranged work.
- Do not use correction fluid or tape.
- Do not remove this paper from the examination room.

Total marks – 70

Section I Pages 3 – 6

10 marks

Attempt Questions 1 – 10

Answer on the Objective Response Answer Sheet provided.

Section II Pages 7 – 14

60 marks

Attempt Questions 11 – 14.

Start each question in a new writing booklet.

Write your student number on every writing booklet.

Question	***1-10***	***11***	***12***	***13***	***14***	***Total***
Total	*/10*	*/15*	*/15*	*/15*	*/15*	*/70*

This assessment task constitutes 45% of the Higher School Certificate Course School Assessment

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the Objective Response answer sheet for Questions 1 – 10

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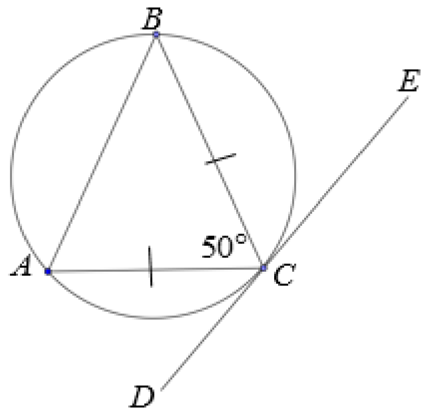
1 *A*, *B* and *C* lie on a circle with $AC = BC$, *DE* is a tangent to the circle at *C*.

The size of $\angle ACB = 50°$. What is the size of $\angle ACD$?



NOT TO SCALE

(A) 50°

(B) 60°

(C) 65°

(D) 80°

2 Which expression is equal to $\sqrt{3}\sin x - \cos x$?

(A) $2\sin\left(x - \frac{\pi}{6}\right)$

(B) $2\sin\left(x + \frac{\pi}{6}\right)$

(C) $\sqrt{10}\sin\left(x - \frac{\pi}{6}\right)$

(D) $\sqrt{10}\sin\left(x + \frac{\pi}{6}\right)$

3 What is the domain of the function $y = 3\sin^{-1}(2 - x)$?

(A) $0 \le x \le 2$

(B) $3 \le x \le 9$

(C) $1 \le x \le 3$

(D) $-3 \le x \le -1$

4 What is the value of the obtuse angle between the lines $5x + 2y - 3 = 0$ and $y = 2x - 1$?

(A) $48°22'$

(B) $175°14'$

(C) $150°21'$

(D) $131°38'$

5 Which group of three numbers could be the roots of the polynomial equation below?

$$x^3 + kx^2 - 17x - 60 = 0$$

(A) $-2, 5, 6$

(B) $1, 2, -30$

(C) $-3, 4, -5$

(D) $-1, 1, 60$

6 What is the derivative of $4\tan^{-1}\dfrac{x}{2}$?

(A) $\dfrac{4}{4+x^2}$

(B) $\dfrac{8}{4+x^2}$

(C) $\dfrac{16}{1+x^2}$

(D) $\dfrac{16}{4+x^2}$

7 The inverse function of $y=\dfrac{4}{x^2}$, $x>0$, is:

(A) $y=\dfrac{2}{\sqrt{x}}$

(B) $y=\dfrac{2}{x}$

(C) $y=-\dfrac{2}{x}$

(D) $y=\pm\dfrac{2}{\sqrt{x}}$

8 A committee of six is to be formed from 7 boys and 5 girls. The committee has to contain at least 3 girls. In how many ways can the committee be chosen?

(A) 210

(B) 350

(C) 462

(D) 924

9 When $P(x) = x^4 + 6x^3 - 5x^2 + 7$ is divided by $x^2 - 1$, the remainder is $6x + k$.

What is the value of k?

(A) -9

(B) -3

(C) 3

(D) 6

10 Given $4\sin^2 18° + 1 = \sqrt{A}$, what is the value of $\cos 36°$ in terms of A?

(A) $\dfrac{3 - \sqrt{A}}{2}$

(B) $\dfrac{1 - A}{2}$

(C) $\dfrac{\sqrt{A} + 1}{4}$

(D) $\dfrac{\sqrt{A}}{4} - 1$

End of Section I

Section II

60 marks
Attempt Questions 11 – 14
Allow about 1 hour and 45 minutes for this section

Answer each question in a new writing booklet. Extra writing booklets are available.

In Questions 11 – 14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Start a new writing booklet

(a) Solve $\dfrac{x^2}{x+12} \le 1$. **3**

(b) Sketch the graph $y = 2\tan^{-1} x + \pi$, clearly indicating all asymptotes. **2**

(c) Evaluate $\displaystyle\int_{0.5}^{1.5} \frac{1}{\sqrt{2x - x^2}}\, dx$, using the substitution $x = u + 1$, giving your answer in simplest exact form. **3**

(d) Differentiate $\dfrac{x \log_e x}{e^x}$. **2**

(e) Find $\displaystyle\lim_{x \to 1} \frac{1 - \frac{1}{x^3}}{1 - \frac{1}{x^2}}$. **3**

(f) Solve for x : $\dfrac{1}{x-1} - 2 = \dfrac{-3}{x+1}$. **2**

End of Question 11

Question 12 (15 marks) Start a new writing booklet

(a) Evaluate $\lim_{x \to 0} \frac{\sin 4x}{7x}$. **1**

(b) Find the volume of the solid of revolution formed when the area bounded by the curve $y = \sin 2x$ and the x-axis is rotated about the x-axis from $x = 0$ to $x = \frac{\pi}{4}$. **3**

(c) The velocity of a particle is $v = 5x + 3 \text{ ms}^{-1}$. **3**

If the initial displacement is 2 m, to the right of the origin, show that the displacement as a function of time is $x = \frac{1}{5}\left(13e^{5t} - 3\right)$.

(d) A dinner party is arranged for 16 people. The people will be seated along two sides of a rectangular table with 8 chairs on each side. Four people wish to sit on one particular side and a couple wishes to sit on the other side next to each other. **2**

In how many ways can the 16 people be seated?

Question 12 continues on page 9

(e) A particle is moving in simple harmonic motion about the origin, with displacement x metres. The displacement is given by $x = 2\cos\left(3t + \frac{\pi}{3}\right)$.

(i) What is the total distance travelled by the particle when it first reaches the origin? **2**

(ii) Find the maximum speed of the particle. **1**

(f) A tub of ice cream taken out of the freezer has a temperature of $-18°C$. It is placed in a room of constant temperature $23°C$. **3**

After t minutes the temperature, $T°C$ of the ice cream is given by:

$$T = 23 - Ae^{-0.04t},$$

where A is a positive constant.

How long does it take for the ice cream to reach a temperature of $-12°C$ which is considered to be the ideal temperature for serving ice cream?

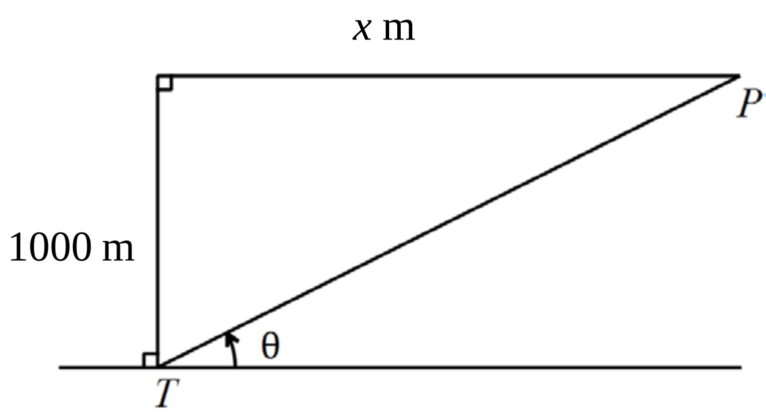
End of Question 12

Question 13 (15 marks) Start a new writing booklet.

(a) Use mathematical induction to prove that $n^3 + 2n$ is divisible by 3 for all integers $n \geq 1$. **3**

(b) Tom, on horizontal ground, is looking at an aeroplane P through a telescope T.

The aeroplane is approaching at a speed of 200 ms^{-1} at a constant altitude of 1000 m above the telescope. When the horizontal distance of the aeroplane from the telescope is x m, the angle of elevation of the aeroplane is θ radians.



NOT TO SCALE

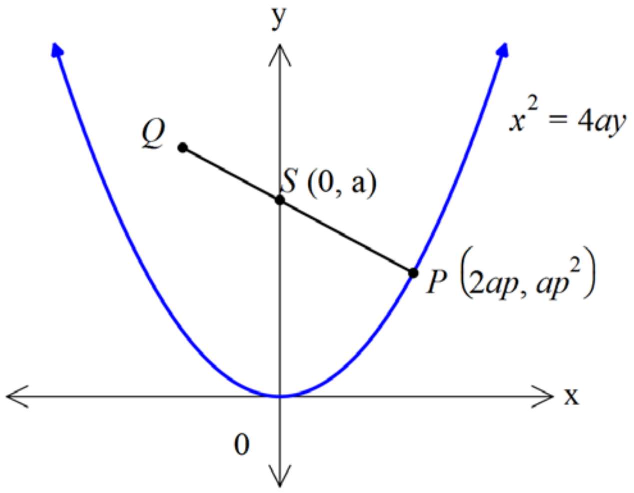
(i) Show that $\dfrac{d\theta}{dt} = \dfrac{2\times10^5}{x^2+10^6}$. **2**

(ii) Find the rate at which θ is changing when $\theta = \dfrac{\pi}{4}$, giving your answer to the nearest degree per second. **2**

Question 13 continues on page 11

(c) In the diagram below, $P\left(2ap, ap^2\right)$ is a point on the parabola $x^2 = 4ay$ with focus $S(0, a)$.

The point Q lies on PS produced such that Q divides PS externally in the ratio $3:1$.



NOT TO SCALE

(i) Show that Q has coordinates $\left(-ap, \dfrac{3a - ap^2}{2}\right)$. **1**

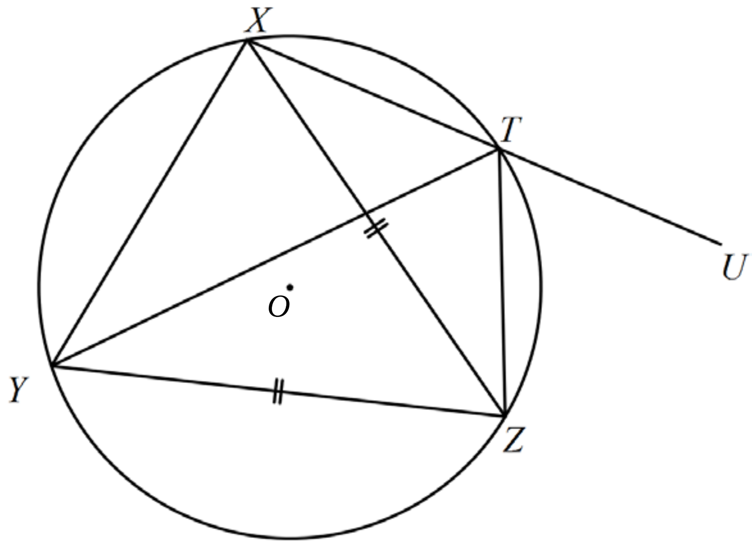
(ii) Show that as P varies, the locus of Q is a parabola with equation **2**

$$x^2 = -2a\left(y - \frac{3a}{2}\right).$$

(iii) Give a geometrical description of the locus of Q, stating all important features. **2**

Question 13 continues on page 12

(d) In the diagram below, XYZ is a triangle in which $XZ = YZ$. T is a point on the minor arc XZ of the circle centre O, passing through X, Y and Z. XT is produced to U. **3**



NOT TO SCALE

Copy or trace the diagram into your writing booklet.

Prove that TZ bisects $\angle UTY$.

End of Question 13

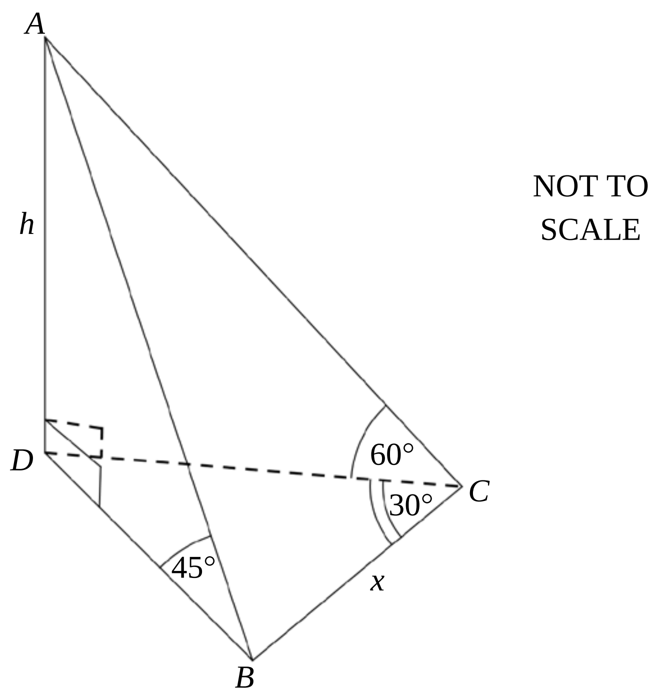
Question 14 (15 marks) Start a new writing booklet

(a) (i) Write down the general solutions to the equation $2\cos 3\theta = 1$. **2**

(ii) Hence, or otherwise, find all solutions of the equation **2**

$$2\cos 3\theta = 1 \quad \text{for} \quad 0 \le \theta \le \pi .$$

(b) In the diagram below, $ABCD$ is a triangular pyramid with base ΔBCD and perpendicular height $AD = h$. $\angle BCD = 30°$, $\angle ABD = 45°$ and $\angle ACD = 60°$.



(i) Use the cosine rule to show that $2h^2 + 3xh - 3x^2 = 0$. **3**

(ii) Hence show that $\dfrac{h}{x} = \dfrac{\sqrt{33} - 3}{4}$. **2**

Question 14 continues on page 14

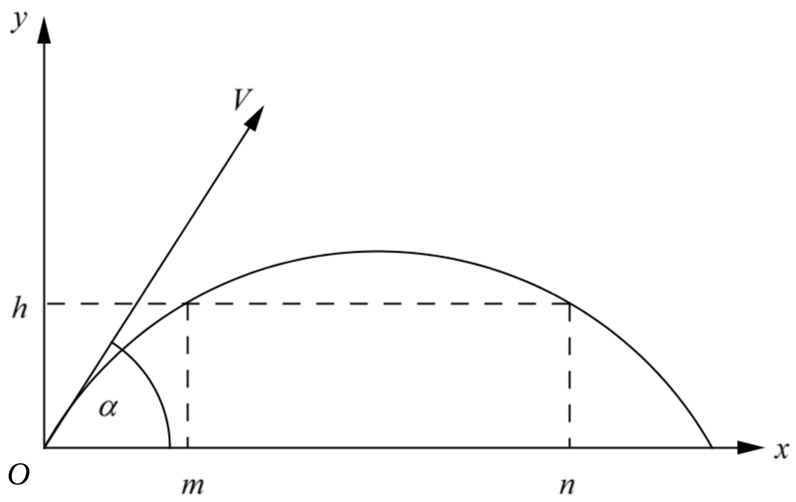
Question 14 (continued)

(c) An object is projected with velocity V ms^{-1} from a point O at an angle of elevation α.

Axes x and y are taken horizontally and vertically through O respectively.

The object just clears two vertical chimneys of height h meters at horizontal distance m metres and n metres from O.

The acceleration due to gravity is taken as $10\ \text{ms}^{-2}$ and air resistance is ignored.



NOT TO SCALE

(i) Show that the expressions of the particle's horizontal and vertical displacements after t seconds are given by **2**

$$x = Vt\cos\alpha \text{ and } y = Vt\sin\alpha - 5t^2.$$

(ii) Show that at $x = m$, $V^2 = \dfrac{5m^2(1+\tan^2\alpha)}{m\tan\alpha - h}$. **2**

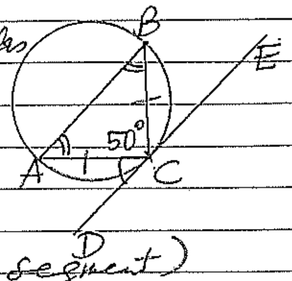
(iii) Show that $\tan\alpha = \dfrac{h(m+n)}{mn}$. **2**

End of Paper

Maths Extension 1 Trial 2018

Section I

1) In $\triangle ABC$, $\angle A = \angle B$ (base $\angle$s, isosceles $\triangle ABC$)



$\angle A + \angle B + \angle C = 180°$ ($\angle$ sum of $\triangle$)

$\therefore \angle A = \angle B = (180° - 50°) \div 2$
$= 65°$

$\angle ACD = \angle B$ ($\angle$ in alternate segment)
$= 65°$

Ans. C

2) $\sqrt{3}\sin x - \cos x$ $\quad \sqrt{(\sqrt{3})^2 + 1^2} = 2$

$= 2\left(\frac{\sqrt{3}}{2}\sin x - \frac{1}{2}\cos x\right)$

$= 2\sin(x - \alpha)$, where $\tan\alpha = \frac{1/2}{\sqrt{3}/2}$

$= 2\sin\left(x - \frac{\pi}{6}\right)$ $\quad = \frac{1}{\sqrt{3}}$

$\alpha = \frac{\pi}{6}$

Ans. A

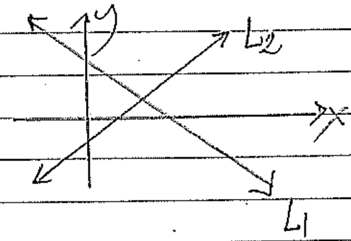
3) $y = 3\sin^{-1}(2 - x)$

D: $-1 \le (2 - x) \le 1$
$-3 \le -x \le -1$
$\therefore 3 \ge x \ge 1$
$1 \le x \le 3$

Ans. C

4) $L_1: 5x + 2y - 3 = 0$, gradient $= -5/2 = m_1$

$L_2: y = 2x - 1$, gradient $= 2 = m_2$



Let θ be the $\angle$ between L_1 and L_2,

$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$

$= \frac{-5/2 - 2}{1 + (-5/2)(2)}$

$= \frac{-9}{-8}$

$= 9/8$

$\therefore \theta = 48° 22'$ (nearest min.)

$\therefore$ the obtuse $\angle = 180° - 48° 22'$
$= 131° 38'$

Ans. D

5) Let α, β, γ be the roots of $x^3 + kx^2 - 17x - 60 = 0$

$\alpha\beta\gamma = -(-60)/1 = 60$

In (C), $(-3)(4)(-5) = 60$

Ans. C

6) $\frac{d}{dx} 4\tan^{-1}\frac{x}{2} = 4 \times \frac{1}{1 + \left(\frac{x}{2}\right)^2} \times \frac{1}{2}$

$= 4 \times \frac{4}{4 + x^2} \times \frac{1}{2}$

$= \frac{8}{4 + x^2}$

Ans. B

7) $y = \frac{4}{x^2}, x > 0$

To find the inverse, interchange x and y,

$x = \frac{4}{y^2}, y > 0$

$y^2 = \frac{4}{x}$

$y = \pm\sqrt{\frac{4}{x}} = \pm\frac{2}{\sqrt{x}}$

Since $y > 0$, $y = \frac{2}{\sqrt{x}}$ is the inverse.

Ans. A

8) From 7 boys and 5 girls, a committee of 6 is to be formed.

The committee has to have at least 3 girls

Number of ways

= ways to have 3 girls and 3 boys

+ ways to have 4 girls and 2 boys

+ ways to have 5 girls and 1 boy

$= {}^5C_3 \times {}^7C_3 + {}^5C_4 \times {}^7C_2 + {}^5C_5 \times {}^7C_1$

$= 350 + 105 + 7$

$= 462$

Ans. C.

9) When $P(x) = x^4 + 6x^3 - 5x^2 + 7$ is divided by $x^2 - 1$, the remainder is $6x + k$.

$P(x) = (x^2-1)Q(x) + 6x + k$, where $Q(x)$ is the quotient.

Put $x = 1$, $P(1) = (1^2-1)Q(1) + 6 + k$

$1^4 + 6\times1^3 - 5\times1^2 + 7 = 6 + k$

$k = 3$

Ans. C

10) $4\sin^2 18° + 1 = \sqrt{A}$

$2\sin^2 18° = \frac{1}{2}(\sqrt{A} - 1)$

$\cos 36° = 1 - 2\sin^2 18°$, (By $\cos 2\theta = 1 - 2\sin^2\theta$, and $\theta = 18°$)

$= 1 - \frac{1}{2}(\sqrt{A} - 1)$

$= 1 - \frac{1}{2}\sqrt{A} + \frac{1}{2}$

$= \frac{3}{2} - \frac{1}{2}\sqrt{A}$

$= \frac{3 - \sqrt{A}}{2}$

Ans. A.

Section II

Question 11

(a) $\frac{x^2}{x+12} \le 1 \quad , \quad x \ne -12$

$x^2(x+12) \le (x+12)^2$

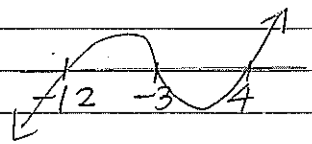
$x^2(x+12) - (x+12)^2 \le 0$

$(x+12)\{x^2 - (x+12)\} \le 0$

$(x+12)(x^2 - x - 12) \le 0$

$(x+12)(x-4)(x+3) \le 0$

$x \le -12 \quad$ or $\quad -3 \le x \le 4$



Since $x \ne -12$,

$\therefore x < -12$ or $-3 \le x \le 4$.

(b) $-\frac{\pi}{2} < \tan^{-1}x < \frac{\pi}{2}$

$-\pi < 2\tan^{-1}x < \pi$

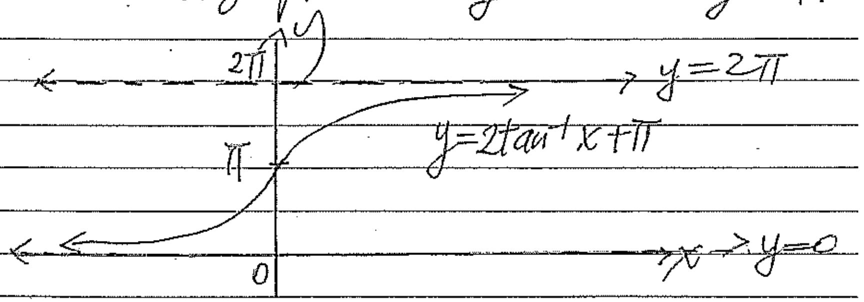
$0 < 2\tan^{-1}x + \pi < 2\pi$

For $y = 2\tan^{-1}x + \pi$,

$0 < y < 2\pi$

When $x = 0$, $\tan^{-1}x = 0$, $y = \pi$

Horizontal asymptotes : $y = 0$ and $y = 2\pi$.



(c) $\int_{0.5}^{1.5} \frac{1}{\sqrt{2x-x^2}}\,dx$, Put $x = u+1$

$dx = du$

When $x = 0.5, u = -0.5$

When $x = 1.5, u = 0.5$

$2x - x^2 = x(2-x)$

$= (u+1)[2-(u+1)]$

$= (u+1)(1-u)$

$= 1-u^2$

$= \int_{-0.5}^{0.5} \frac{1}{\sqrt{1-u^2}}\,du$

$= \left[\sin^{-1}u\right]_{-0.5}^{0.5}$

$= \sin^{-1}0.5 - \sin^{-1}(-0.5)$

$= \frac{\pi}{6} - \left(-\frac{\pi}{6}\right)$

$= \frac{\pi}{3}$

(d) $\frac{d}{dx}\frac{x\ln x}{e^x} = \frac{e^x \frac{d}{dx}(x\ln x) - (x\ln x)\frac{d}{dx}e^x}{(e^x)^2}$

$= \frac{e^x\left(x \times \frac{1}{x} + 1 \times \ln x\right) - (x\ln x) \times e^x}{(e^x)^2}$

$= \frac{1 + \ln x - x\ln x}{e^x}$

(e) $\lim_{x\to 1} \frac{1 - \frac{1}{x^3}}{1 - \frac{1}{x^2}} = \lim_{x\to 1} \frac{x^3 - 1}{x(x^2-1)}$, (multiply by $\frac{x^3}{x^3}$)

$= \lim_{x\to 1} \frac{(x-1)(x^2+x+1)}{x(x-1)(x+1)}$

$= \lim_{x\to 1} \frac{x^2+x+1}{x(x+1)}$

$= \frac{1^2+1+1}{1(1+1)}$

$= 1\frac{1}{2}$

(f) $\frac{1}{x-1} - 2 = \frac{-3}{x+1}$, $x \neq 1, -1$.

Multiply by $(x-1)(x+1)$,

$(x+1) - 2(x-1)(x+1) = -3(x-1)$

$x+1 - 2(x^2-1) + 3x - 3 = 0$

$x+1 - 2x^2 + 2 + 3x - 3 = 0$

$2x^2 - 4x = 0$

$2x(x-2) = 0$

$\therefore x = 0$ or 2

Question 12

(a) $\lim_{x\to 0} \frac{\sin 4x}{7x} = \lim_{x\to 0} \frac{4}{7}\frac{\sin 4x}{4x}$

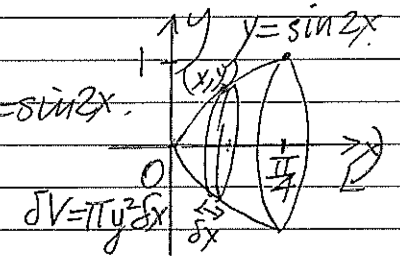
$= \frac{4}{7} \lim_{x \to 0} \frac{\sin 4x}{4x}$

$= \frac{4}{7} \times 1$

$= \frac{4}{7}$

(b) Volume of the solid

$= \int_0^{\pi/4} \pi y^2 dx$, where $y = \sin 2x$.



$= \pi \int_0^{\pi/4} \sin^2 2x \cdot dx$

$= \pi \int_0^{\pi/4} \frac{1-\cos 4x}{2} dx$, (By $\sin^2\theta = \frac{1-\cos 2\theta}{2}$, $\theta = 2x$)

$= \frac{\pi}{2}\left[x - \frac{1}{4}\sin 4x\right]_0^{\pi/4}$

$= \frac{\pi}{2}\left[\frac{\pi}{4} - \frac{1}{4}\sin(4\times\frac{\pi}{4})\right] - \frac{\pi}{2}\left[0 - \frac{1}{4}\sin 0\right]$

$= \frac{\pi}{2}\left[\frac{\pi}{4} - \frac{1}{4}\sin\pi\right]$

$= \frac{\pi^2}{8}$ units^3

(c) $v = 5x + 3$, where v is velocity, x is displacement. t is time.

$\frac{dx}{dt} = 5x + 3$

$\int dt = \int \frac{dx}{5x+3}$

$t = \frac{1}{5}\ln(5x+3) + C$,

At $t=0$, $x=2$,

$$0 = \frac{1}{5}\ln(5\times2+3)+C$$

$$\therefore C = -\frac{1}{5}\ln 13$$

$$\therefore t = \frac{1}{5}\ln(5x+3) - \frac{1}{5}\ln 13$$

$$= \frac{1}{5}\ln\frac{5x+3}{13}$$

$$5t = \ln\frac{5x+3}{13}$$

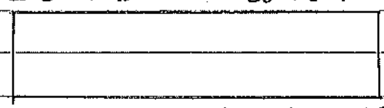
$$\frac{5x+3}{13} = e^{5t}$$

$$5x+3 = 13e^{5t}$$

$$5x = 13e^{5t} - 3$$

$$\therefore x = \frac{1}{5}(13e^{5t}-3)$$

(d) Number of ways for the 4 people, 4 people on this side and 4 others
$= 8\times7\times6\times5$



Number of ways for the couple, couple and 6 others
$= 7\times2!$

Number of ways of arranging the other 10
$= 10!$

$\therefore$ Number of ways of seating the 16 people
$= (8\times7\times6\times5)\times(7\times2!)\times10!$
$= 23520\times10!$

(e) (i) $x = 2\cos\left(3t+\frac{\pi}{3}\right)$

At $t=0$, $x = 2\cos\frac{\pi}{3} = 1$

When $x=0$, $2\cos\left(3t+\frac{\pi}{3}\right)=0$

$$\cos\left(3t+\frac{\pi}{3}\right)=0$$

$$3t+\frac{\pi}{3}=\frac{\pi}{2}$$

$$3t=\frac{\pi}{6}$$

$$t=\frac{\pi}{18}$$

Particle starts at $x=1$ and moves towards the origin.
$\therefore$ Distance travelled by the particle when it first reaches the origin
$= 1$ m

(ii) $x = 2\cos\left(3t+\frac{\pi}{3}\right)$

$$\frac{dx}{dt} = \left[-2\sin\left(3t+\frac{\pi}{3}\right)\right]\times3$$

$$= -6\sin\left(3t+\frac{\pi}{3}\right)$$

maximum speed
$= \text{maximum of } \left|\frac{dx}{dt}\right|$
$= \text{maximum of } \left|-6\sin\left(3t+\frac{\pi}{3}\right)\right|$
$= 6\times\text{maximum of } \left|\sin\left(3t+\frac{\pi}{3}\right)\right|$
$= 6\times1$
$= 6$ m/s

(f) $T = 23 - Ae^{-0.04t}$

At $t=0$, $T = -18$,

$-18 = 23 - Ae^{-0.04\times 0}$

$-18 = 23 - Ae^{0}$

$-18 = 23 - A$

$A = 23 + 18$

$= 41$

$\therefore T = 23 - 41e^{-0.04t}$

When $T = -12$,

$-12 = 23 - 41e^{-0.04t}$

$41e^{-0.04t} = 23 + 12$

$= 35$

$e^{0.04t} = \frac{41}{35}$

$0.04t = \ln\left(\frac{41}{35}\right)$

$t = \frac{1}{0.04}\ln\left(\frac{41}{35}\right)$

$= 3.9556\ldots$ minutes

$= 3.956$ minutes (3 dp)

Question 13

(a) Prove by Mathematical Induction that $n^3 + 2n$ is divisible by 3 for all integers $n \geq 1$.

Proof:

For $n=1$, $n^3 + 2n = 1^3 + 2\times 1$

$= 3$

$= 3 \times 1$

which is divisible by 3.

$\therefore$ the statement is true for $n=1$.

For $n=k$, assume $k^3 + 2k$ is divisible by 3,

i.e., $k^3 + 2k = 3p$, where p is an integer.

For $n=k+1$,

$(k+1)^3 + 2(k+1)$

$= k^3 + 3k^2 + 3k + 1 + 2k + 2$

$= (k^3 + 2k) + 3k^2 + 3k + 3$

$= (k^3 + 2k) + 3(k^2 + k + 1)$

$= 3p + 3(k^2 + k + 1)$, (By assumption)

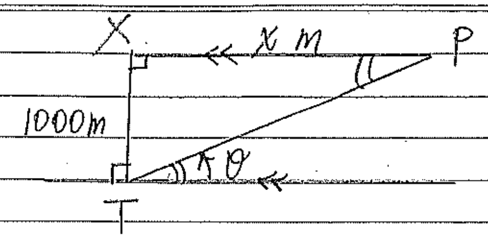
$= 3\{p + (k^2 + k + 1)\}$

which is divisible by 3 since p, k are integers, $\therefore p + (k^2 + k + 1)$ is an integer.

$\therefore$ the statement is true for $n = k+1$.

By the Principle of Mathematical Induction, $n^3 + 2n$ is divisible by 3 for all integers $n \geq 1$.

(b)



(i) In $\triangle PXT$, $\tan\theta = \frac{1000}{x}$

Differentiate $\tan\theta = \frac{1000}{x}$ w.r.t. t,

$\sec^2\theta \frac{d\theta}{dt} = -\frac{1000}{x^2}\frac{dx}{dt}$ —— ①

$PT^2 = PX^2 + TX^2$ (By Pythagoras' th.)

$PT^2 = x^2 + 1000^2$

$\sec^2\theta = \frac{PT^2}{PX^2} = \frac{x^2+1000^2}{x^2}$

$\frac{dx}{dt} = -200\ ms^{-1}$

Put $\sec^2\theta$, $\frac{dx}{dt}$ into ①,

$\frac{x^2+1000^2}{x^2}\frac{d\theta}{dt} = -\frac{1000}{x^2}\times(-200)$

$\therefore \frac{d\theta}{dt} = \frac{1000\times 200}{x^2}\times\frac{x^2}{x^2+1000^2}$

$= \frac{1000\times 200}{x^2+1000^2}$

$= \frac{10^3\times 2\times 10^2}{x^2+(10^3)^2}$

$= \frac{2\times 10^5}{x^2+10^6}$ —— ②

(ii) When $\theta = \frac{\pi}{4}$, $PX = TX = 1000\ m = x$

$\therefore x = 10^3$,

Put $x = 10^3$ into ② in (i),

$\frac{d\theta}{dt} = \frac{2\times 10^5}{(10^3)^2+10^6}$

$= \frac{2\times 10^5}{2\times 10^6}$

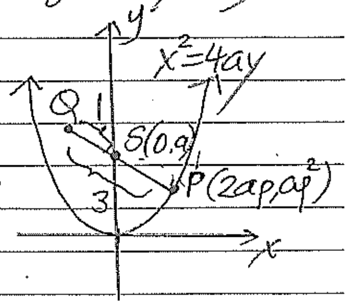
$= \frac{1}{10}{}^c/s$

$= \frac{1}{10}\times\frac{180}{\pi}{}^\circ/s$

$= 5.729\ldots{}^\circ/s$

$= 6^\circ/s$ (nearest degree/s)

(c)(i) Q divides PS externally in the ratio 3 : 1.



$Q = \left(\frac{3\times 0+(-1)2ap}{3+(-1)}, \frac{3a+(-1)ap^2}{3+(-1)}\right)$, $3\to S$, $-1\to P$

$= \left(\frac{-2ap}{2}, \frac{3a-ap^2}{2}\right)$

$= \left(-ap, \frac{3a-ap^2}{2}\right)$

(ii) Let Q be (x_Q, y_Q)

$x_Q = -ap$ —— ①

$y_Q = \frac{3a-ap^2}{2}$ —— ②

From ①, $p = -\frac{x_Q}{a}$

Put p into ②,

$y_Q = \frac{3a - a\left(-\frac{x_Q}{a}\right)^2}{2}$

$$y_Q = \frac{3a - \frac{x_Q^2}{a}}{2}$$
$$= \frac{3a^2 - x_Q^2}{2a}$$
$$2ay_Q = 3a^2 - x_Q^2$$
$$x_Q^2 = 3a^2 - 2ay_Q$$
$$= -2a\left(y_Q - \frac{3a}{2}\right)$$

As P varies, the locus of Q is a parabola with equation
$$x^2 = -2a\left(y - \frac{3a}{2}\right).$$

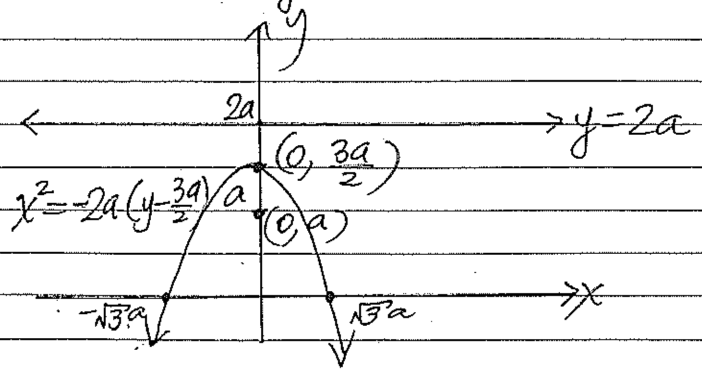
(iii) The locus of Q is
$$x^2 = -2a\left(y - \frac{3a}{2}\right)$$
$$= -4\left(\frac{a}{2}\right)\left(y - \frac{3a}{2}\right)$$

vertex $= \left(0, \frac{3a}{2}\right)$, Put $y=0$, $x^2 = -2a\left(-\frac{3a}{2}\right)$

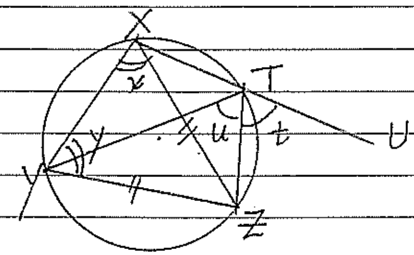
focal length $= \frac{a}{2}$ $\qquad = 3a^2$

focus $= (0, a)$ $\qquad x = \pm\sqrt{3}a$

directrix : $y = 2a$



(d)



Let $\angle XYZ = y$, $\angle YXZ = x$, $\angle YTZ = u$, $\angle UTZ = t$.

Prove : TZ bisects $\angle UTY$.

In ΔXYZ, $XZ = YZ$ (given)
$\therefore x = y$ (base $\angle$s, isosceles ΔXYZ)

$x = u$ ($\angle$s in the same segment)
$y = t$ (interior $\angle$ = exterior $\angle$ of cyclic quadrilateral XYZT)

$\therefore x = y = u = t$.
Hence $u = t$
i.e., TZ bisects $\angle UTY$.

Question 14

(a)(i) $2\cos 3\theta = 1$

$\cos 3\theta = 1/2$

$3\theta = 2n\pi \pm \frac{\pi}{3}$, n is an integer.

$\theta = \frac{2n\pi}{3} \pm \frac{\pi}{9}$, n is an integer.

(ii) $2\cos 3\theta = 1$, $0 \le \theta \le \pi$

$\theta = \frac{2n\pi}{3} \pm \frac{\pi}{9}$, $n = 0, 1, \dots$

$\theta = \pm\frac{\pi}{9}, \frac{2\pi}{3} \pm \frac{\pi}{9}, \dots$

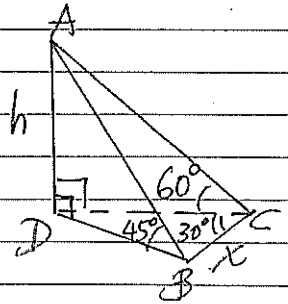
For $0 \le \theta \le \pi$,

$\theta = \frac{\pi}{9}, \frac{2\pi}{3} + \frac{\pi}{9}, \frac{2\pi}{3} - \frac{\pi}{9}$

$= \frac{\pi}{9}, \frac{7\pi}{9}, \frac{5\pi}{9}$.

$\therefore \theta = \frac{\pi}{9}, \frac{5\pi}{9}$ or $\frac{7\pi}{9}$

(b) (i) In $\triangle ABD$, $\tan 45° = \frac{h}{BD}$



$\therefore BD = h$

In $\triangle ACD$, $\tan 60° = \frac{h}{CD}$

$\therefore CD = \frac{h}{\sqrt{3}}$

In $\triangle BCD$,

$BD^2 = CD^2 + BC^2 - 2 \times CD \times BC \cos\angle BCD$ (By Cosine Rule)

$h^2 = \frac{h^2}{3} + x^2 - 2 \times \frac{h}{\sqrt{3}} \times x \cos 30°$

$= \frac{h^2}{3} + x^2 - \frac{2hx}{\sqrt{3}} \times \frac{\sqrt{3}}{2}$

$h^2 = \frac{h^2}{3} + x^2 - hx$

$3h^2 = h^2 + 3x^2 - 3hx$

$2h^2 + 3xh - 3x^2 = 0$

(ii) $2h^2 + 3xh - 3x^2 = 0$

Divide by x^2,

$\frac{2h^2}{x^2} + \frac{3xh}{x^2} - 3 = 0$

$2\left(\frac{h}{x}\right)^2 + 3\left(\frac{h}{x}\right) - 3 = 0$

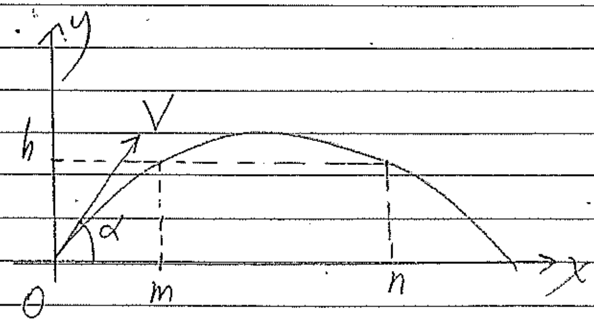
$\frac{h}{x} = \frac{-3 \pm \sqrt{3^2 - 4 \times 2 \times (-3)}}{2 \times 2}$

$= \frac{-3 \pm \sqrt{33}}{4}$

$\therefore \frac{h}{x} = \frac{-3 + \sqrt{33}}{4}$ since $h > 0, x > 0$, $\therefore \frac{h}{x} > 0$;

$\frac{h}{x} = \frac{\sqrt{33} - 3}{4}$ reject $\frac{-3 - \sqrt{33}}{4}$.

(c)(i)



For horizontal motion $\ddot{x} = 0$

$\dot{x} = C_1$

At $t=0$, $\dot{x} = V\cos\alpha$

$\therefore C_1 = V\cos\alpha$

$\dot{x} = V\cos\alpha$

$x = Vt\cos\alpha + C_2$

At $t=0$, $x=0$,

$\therefore 0 = 0 + C_2$

$\therefore C_2 = 0$

$x = Vt\cos\alpha$

<u>For Vertical Motion</u>, $\ddot{y} = -10$

$\dot{y} = -10t + C_3$

At $t=0$, $\dot{y} = V\sin\alpha$

$V\sin\alpha = 0 + C_3$

$C_3 = V\sin\alpha$

$\therefore \dot{y} = -10t + V\sin\alpha$

$y = -5t^2 + Vt\sin\alpha + C_4$

At $t=0$, $y=0$,

$0 = 0 + 0 + C_4$

$C_4 = 0$

$\therefore y = -5t^2 + Vt\sin\alpha$

$y = Vt\sin\alpha - 5t^2$

(II) $x = Vt\cos\alpha$ ——— (1)

$y = Vt\sin\alpha - 5t^2$ ——— (2)

From (1), $t = \dfrac{x}{V\cos\alpha}$

Put $t = \dfrac{x}{V\cos\alpha}$ into (2),

$$\therefore y = V\sin\alpha \times \frac{x}{V\cos\alpha} - 5\left(\frac{x}{V\cos\alpha}\right)^2$$

$$= x\tan\alpha - \frac{5x^2}{V^2\cos^2\alpha}$$

Put $x = m$, $y = h$,

$$\therefore h = m\tan\alpha - \frac{5m^2}{V^2\cos^2\alpha}$$

$$\frac{5m^2}{V^2\cos^2\alpha} = m\tan\alpha - h$$

$$V^2\cos^2\alpha = \frac{5m^2}{m\tan\alpha - h}$$

$$V^2 = \frac{5m^2 \times \frac{1}{\cos^2\alpha}}{m\tan\alpha - h}$$

$$= \frac{5m^2\sec^2\alpha}{m\tan\alpha - h}$$

$$= \frac{5m^2(1+\tan^2\alpha)}{m\tan\alpha - h} \quad \text{——— (3)}$$

(By $\sec^2\alpha - \tan^2\alpha = 1$, $\therefore \sec^2\alpha = 1 + \tan^2\alpha$)

(III) Similarly put $x = n$, $y = h$ into

$$y = x\tan\alpha - \frac{5x^2}{V^2\cos^2\alpha}$$

$$h = n\tan\alpha - \frac{5n^2}{V^2\cos^2\alpha}$$

$$V^2 = \frac{5n^2(1+\tan^2\alpha)}{n\tan\alpha - h} \quad \text{——— (4)}$$

Equating (3) and (4),

$$\frac{5m^2(1+\tan^2\alpha)}{m\tan\alpha - h} = \frac{5n^2(1+\tan^2\alpha)}{n\tan\alpha - h}$$

$$\frac{m^2}{m\tan\alpha - h} = \frac{n^2}{n\tan\alpha - h}$$

$$m^2(n\tan\alpha - h) = n^2(m\tan\alpha - h)$$

$$m^2 n\tan\alpha - m^2 h = mn^2\tan\alpha - n^2 h$$

$$(m^2 n - mn^2)\tan\alpha = m^2 h - n^2 h$$

$$mn(m-n)\tan\alpha = (m^2-n^2)h$$

$$\tan\alpha = \frac{(m^2-n^2)h}{mn(m-n)}$$

$$= \frac{(m-n)(m+n)h}{mn(m-n)}$$

$$= \frac{h(m+n)}{mn}$$