Name : ____________________

HURLSTONE AGRICULTURAL HIGH SCHOOL

PRO PATRIA

TRIAL HIGHER SCHOOL CERTIFICATE

1998

MATHEMATICS
3/4 UNIT (COMMON)

Time Allowed – Two hours

(Plus reading time –5 minutes)

Examiner –H. Cavanagh.

DIRECTIONS TO CANDIDATES

- This paper contains 8 questions.
- All questions are to be attempted.
- All necessary working should be shown. Marks may not be awarded for careless or badly arranged work.
- **Start each question on a new sheet of paper.** Ensure that your name is written on each sheet of paper that is submitted.
- Board approved calculators and approved Math-Aids may be used.
- A table of Standard Integrals is supplied for use in the examination.

QUESTION ONE (11 marks) *(Start a new sheet of paper)* Ma

(a) Write $\dfrac{1+\sqrt{7}}{3-\sqrt{7}}$ in the form $a + b\sqrt{7}$ where a and b are rational.

(b) Given $f(x) = \tan^2 x$, find $f'(\frac{\pi}{4})$. 2

(c) A and B are the points $(1, -3)$ and $(6, 7)$ respectively. P divides the interval AB in the ratio $2 : 3$. Find the coordinates of P. 2

(d) Solve for x : $\dfrac{6}{x} > x - 1$ 3

(e) (i) Sketch the graph of $y = |x - 2|$

(ii) For what values of x is $|x - 2| < x$ 2

QUESTION TWO (12 marks) *(Start a new sheet of paper)*

(a) Differentiate $e^{4x}\sin x$. 2

(b) Use the substitution $u = \log_e x$ to evaluate $\displaystyle\int_1^e \frac{(\log_e x)}{x}\,dx$ 3

(c) Solve the equation $\sin 2x = \tan x$ for $0 \le x \le \pi$. 4

(d) Evaluate $\displaystyle\int_1^{\sqrt{2}} \frac{dx}{\sqrt{4-2x^2}}$ 3

Please turn over to Question Three.....

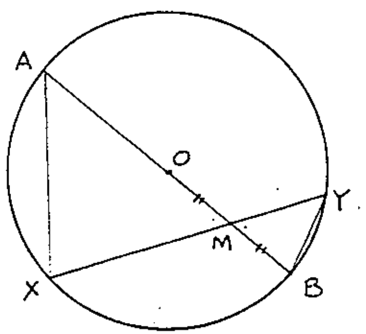
QUESTION THREE (9 marks) *(Start a new sheet of paper)* Marks

(a) (i) Find the domain and range of the function $y = 4\cos^{-1}\frac{x}{3}$ 5

(ii) Sketch the graph of the function $y = 4\cos^{-1}\frac{x}{3}$ showing clearly the intercepts on the coordinate axes and the coordinates of any endpoints.

(iii) Find the area of the region in the first quadrant bounded by the curve $y = 4\cos^{-1}\frac{x}{3}$ and the coordinate axes.

(b) In the diagram below, AB is a diameter of a circle, centre O. The chord XY intersects AB at M, such that OM = BM. 4



(i) Copy, or trace, the diagram carefully onto your exam paper.

(ii) Show that ΔAXM ||| ΔYBM.

(iii) If XM = 8 cms and YM = 6 cms, find the radius of the circle.

Please turn over to Question Four.....

QUESTION FOUR (10 marks) *(Start a new sheet of paper)* Mar

(a) Using the principles of Mathematical Induction, show that $3^{3n} + 2^{n+2}$ is divisible by 5 for all positive integers n greater than or equal to 1. 5

(b) Newton's Law of Cooling states that when an object at temperature T° C is placed in an environment at temperature T_0°C, the rate of temperature loss is given by the equation $\frac{dT}{dt} = k(T - T_0)$ where t is the time in seconds and k is a constant. 5

(i) Show that $T = T_0 + Ae^{kt}$ is a solution to the equation above.

(ii) A packet of peas, initially at 24° C is placed in a snap-freeze refrigerator in which the internal temperature is maintained at −40°C. After 5 seconds, the temperature of the peas is 19° C. How long will it take for the peas to reach a temperature of 0° C?

QUESTION FIVE (12 marks) *(Start a new sheet of paper)*

(a) (i) Express $2\sin\theta + \cos\theta$ in the form $r\sin(\theta + \alpha)$. 3

(ii) Hence, or otherwise, solve $2\sin\theta + \cos\theta = 1$ for $0° \le \theta \le 360°$.

(b) $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are two points on the parabola $x^2 = 4ay$. 9

(i) If PQ passes through the point $C(2a, 3a)$ show that $pq = p + q - 3$.

(ii) If M is the midpoint of PQ, show that the coordinates of M are $\left(a(pq + 3), \frac{a}{2}(pq + 3)^2 - apq\right)$.

(iii) Hence show that the locus of the midpoint of all chords passing through $(2a, 3a)$ is $x^2 - 2ax = 2a(y - 3a)$.

Please turn over to Question Six.....

QUESTION SIX (11 marks) *(Start a new sheet of paper)*

Marks

(a) In a group X, there are 7 boys and 3 girls, whilst in another group Y, there are 2 boys and 8 girls. 4

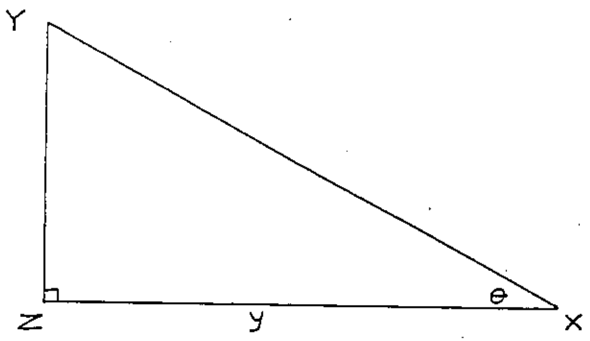
(i) One person is selected at random from each group. Find the probability that

(α) both are boys. (β) one is a boy and one is a girl.

(ii) One group is chosen at random from X and Y and then two people are selected from this group. Find the probability that

(α) both are boys. (β) one is a boy and one is a girl.

(b) In ΔXYZ, $ZX = y$, $\angle YZX = 90°$ and $\angle YXZ = \theta$. 7



(i) Show that the perimeter P of the triangle is given by $P = y(1 + \tan\theta + \sec\theta)$.

(ii) If $y = 10$ cm and θ is increasing at a constant rate of 0.2 radians / sec find the rate at which the perimeter of the triangle is increasing when $\theta = \frac{\pi}{6}$ radians.

Please turn over to Question Seven ....

QUESTION SEVEN (10 marks) *(Start a new sheet of paper)*

Mark

Heavy raindrops are moving horizontally at 36 km/hr in clouds being blown by a steady wind. They then fall 200 metres to the ground below. 10

(Air resistance may be neglected, and the approximate value $g = 10$ m/sec^2 may be assumed.)

(i) Find the time taken for a drop to reach the ground.

(ii) Find the speed and angle at which a drop hits the horizontal ground.

(iii) At what angle does a drop hit the ground when the wind speed is doubled?

Please turn over to Question Eight.....

QUESTION EIGHT (12 marks) *(Start a new sheet of paper)* Mark

(a) The acceleration of a particle, when x metres from the origin on a directed axis, is $(4x - 2x^3)$ ms^{-2}. 4

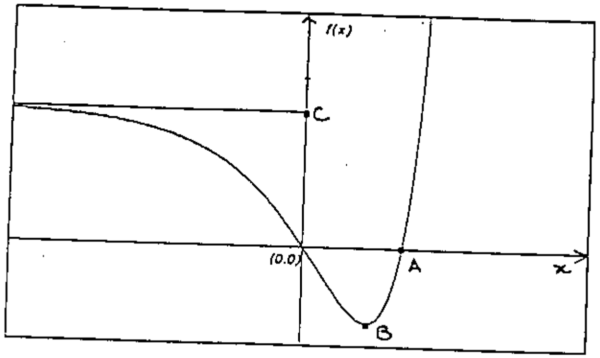
It is released from rest at $x = 2$.

(i) Determine v^2 as a function of x, where v ms^{-1} is the velocity.

(ii) Determine (α) the position at which it next comes to rest,

(β) and its acceleration at that point.

(b) The graph of $f(x) = e^{2x} - 5e^x + 4$ is shown below 8



(i) Write the equation of the horizontal asymptote through C.

(ii) Find the coordinates of A, the point where $y = f(x)$ crosses the x-axis, and B, a stationary point. Leave your answers in exact form.

(iii) Find the equation of the normal through (0,0).

(iv) The normal intersects the curve at another point, N. The x coordinate of N is close to 1.5. Use one application of Newton's Method to find the value of N and round answer correct to three decimal places.

End of Paper.

3U Trial – 1998 – Solutions.

① (a) $\frac{1+\sqrt{7}}{3-\sqrt{7}} \times \frac{3+\sqrt{7}}{3+\sqrt{7}} = \frac{3+4\sqrt{7}+7}{9-7}$

$= \frac{10+4\sqrt{7}}{2}$

$= 5 + 2\sqrt{7}$ 2

(b) $f(x) = \tan^2 x$

$f'(x) = 2\tan x \cdot \sec^2 x$

$f'(\frac{\pi}{4}) = 2\tan\frac{\pi}{4} \cdot \sec^2\frac{\pi}{4}$

$= 2(1)(\sqrt{2})^2$

$= 4$ 2

(c) A(1,−3) 2 : 3 B(6,7) $P\left(\frac{2(6)+3(1)}{5}, \frac{2(7)+3(-3)}{5}\right)$

$P(3,1)$ 2

(d) $\frac{6}{x} > x - 1$

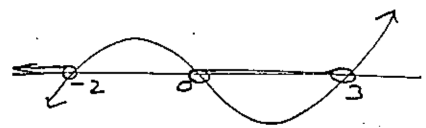
$(\times x^2)$ $6x > x^3 - x^2$

$x^3 - x^2 - 6x < 0$

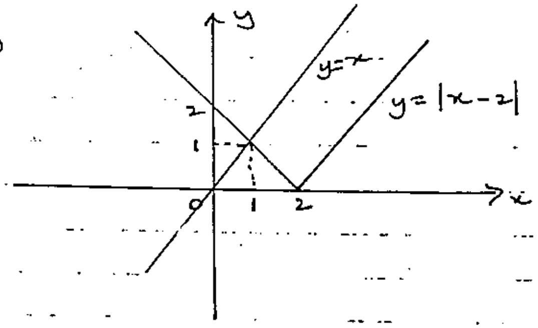
$x(x^2 - x - 6) < 0$

$x(x-3)(x+2) < 0$

i.e. $x < -2$ or $0 < x < 3$



(e)



From graph $|x-2| < x$ for $x >$

2 (a) $\frac{d}{dx} e^{4x}\sin x = 4e^{4x}\sin x + e^{4x}\cos x$

$= e^{4x}(4\sin x + \cos x)$

2

(b) $I = \int_1^e \frac{\log_e x}{x}\, dx$

let $u = \log_e x$ when $x = e,\ u = 1$

when $x = 1,\ u = 0$

$du = \frac{1}{x}\, dx$

$\therefore I = \int_0^1 u\, du$

$= \left[\frac{u^2}{2}\right]_0^1$

$= \frac{1}{2}$

3

(c) $\sin 2x = \tan x \qquad 0 \le x \le \pi$

$2\sin x \cos x = \frac{\sin x}{\cos x}$

$2\sin x \cos^2 x = \sin x$

$\sin x(2\cos^2 x - 1) = 0$

$\therefore \sin x = 0$ or $2\cos^2 x - 1 = 0$

$x = 0, \pi \qquad \cos x = \pm\frac{1}{\sqrt{2}}$

$x = \frac{\pi}{4}, \frac{3\pi}{4}$

$\therefore x = 0, \frac{\pi}{4}, \frac{3\pi}{4}, \pi$

4

(d) $\int_1^{\sqrt{2}} \frac{dx}{\sqrt{4-2x^2}} = \frac{1}{\sqrt{2}}\int_1^{\sqrt{2}} \frac{dx}{\sqrt{2-x^2}}$

$= \frac{1}{\sqrt{2}}\left[\sin^{-1}\frac{x}{\sqrt{2}}\right]_1^{\sqrt{2}}$

$= \frac{1}{\sqrt{2}}\left\{\frac{\pi}{2} - \frac{\pi}{4}\right\}$

$= \frac{\pi}{4\sqrt{2}}$

3

3 (a) (i) $y = 4\cos^{-1}\frac{x}{3}$

$\frac{y}{4} = \cos^{-1}\frac{x}{3}$

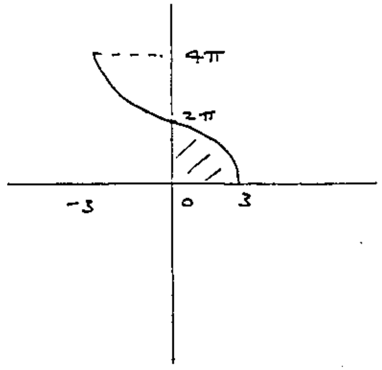
$\frac{x}{3} = \cos\frac{y}{4}$

$x = 3\cos\frac{y}{4}$.

$\therefore D: \{-3 \le x \le 3\}$

$R: \{0 \le y \le 4\pi\}$

(ii)



(iii) $A = \int_0^{2\pi} 3\cos\frac{y}{4}\, dy$

$= \left[12\sin\frac{y}{4}\right]_0^{2\pi}$

$= 12\left(\sin\frac{\pi}{2} - \sin 0\right)$

$= 12$ units^2

5

(b) (ii) In Δ's AXM, YBM

$\angle A = \angle Y$ (angles in same segment, on arc XB)

$\angle AMX = \angle YMB$ (vert. opp. $\angle$'s)

$\therefore \Delta AXM \parallel\!\mid \Delta YBM$ (equiangular)

(iii) $r = 2x$

$\therefore \frac{x}{8} = \frac{6}{x+r}$

$x(x+r) = 48$

$x^2 + x(2x) = 48$

$3x^2 = 48$

$x^2 = 16$

$x = 4 \quad (x > 0)$

$\therefore r = 8$ cm.

④ (a) To prove that $3^{3n} + 2^{n+2}$ is divisible by 5 for all $n \geq 1$

Test for $n=1$,

$$3^3 + 2^3 = 27 + 8$$
$$= 35$$

True for $n=1$ $(35 = 5 \times 7)$

Assume true for $n=k$

i.e. $3^{3k} + 2^{k+2} = 5M$ $(M \in J)$

Consider $3^{3(k+1)} + 2^{(k+1)+2} = 3^3.3^{3k} + 2^{k+3}$

$$= 3^3(5M - 2^{k+2}) + 2^{k+3}$$
(since $3^{3k} = 5M - 2^{k+2}$)
$$= 27.5M - 27.2^{k+2} + 2^{k+3}$$
$$= 27.5M - 25.2^{k+2}$$
$$= 5(27M - 5.2^{k+2})$$
$$= 5P \qquad P \in J$$

Hence, if true for $n=k$, result holds true for $n=k+1$.

Since true for $n=1$, it is true for $n=1+1=2$.
Since true for $n=2$, it is true for $n=2+1=3$.
By principle of mathematical induction, result is true for all positive integral values of n.

④ (b) (i) $\frac{dT}{dt} = k(T - T_0)$

If $T = T_0 + Ae^{kt}$

Then $\frac{dT}{dt} = Ake^{kt}$

$$= k(Ae^{kt})$$
$$= k(T - T_0) \quad \underline{\text{Q.E.D.}}$$

(ii) When $t=0$, $T=24$, $T_0 = -40$

$\therefore$ $24 = -40 + A$

$\underline{A = 64}$

When $t=5$, $T=19$

$\therefore$ $19 = -40 + 64e^{5k}$

$59 = 64e^{5k}$

$e^{5k} = \frac{59}{64}$

$\underline{k = \frac{1}{5}\log\frac{59}{64}}$ $[k = -0.0162691]$

For $T=0$

$0 = -40 + 64e^{kt}$ (given k above)

$64e^{kt} = 40$

$e^{kt} = \frac{40}{64}$

$kt = \log\frac{5}{8}$

$t = \frac{1}{k}\log\frac{5}{8}$

$= 28.889\ldots$

$\doteqdot$ $\underline{29 \text{ seconds}}$

⑤ (a)(i) $2\sin\theta + \cos\theta = \sqrt{5}\left(\frac{2}{\sqrt5}\sin\theta + \frac{1}{\sqrt5}\cos\theta\right)$

$$= \sqrt5\left(\sin\theta\left(\tfrac{2}{\sqrt5}\right) + \cos\theta\left(\tfrac{1}{\sqrt5}\right)\right)$$

$$= \sqrt5 \sin(\theta + \alpha) \quad \text{where } \cos\alpha = \frac{2}{\sqrt5}$$

$$\sin\alpha = \frac{1}{\sqrt5}$$

i.e. $\underline{\alpha = 26°34'}$

(ii) $\therefore 2\sin\theta + \cos\theta = 1$ for $0° \le \theta \le 360°$

$\sqrt5\sin(\theta + 26°34') = 1$ for $26°34' \le \theta + 26°34' \le 386°34'$

$\sin(\theta + 26°34') = \frac{1}{\sqrt5}$

S	A
T	C

b.a. $\theta + 26°34' = 26°34'$

$\theta + 26°34' = 26°34',\ 153°26',\ 386°34'$

$\therefore \theta = 0°,\ 126°52',\ 360°$

or (ii) $2\sin\theta + \cos\theta = 1$

Using $\sin\theta = \frac{2t}{1+t^2}$ and $\cos\theta = \frac{1-t^2}{1+t^2}$ where $t = \tan\frac{\theta}{2}$

$$\frac{4t}{1+t^2} + \frac{1-t^2}{1+t^2} = 1$$

$$4t + 1 - t^2 = 1 + t^2$$

$$2t^2 - 4t = 0$$

$$2t(t-2) = 0$$

$$t = 0, 2$$

i.e. $\tan\theta/2 = 0$ $\quad \tan\theta/2 = 2$

$\frac{\theta}{2} = 0°, 180°$ $\quad \theta/2 = 126°52'$

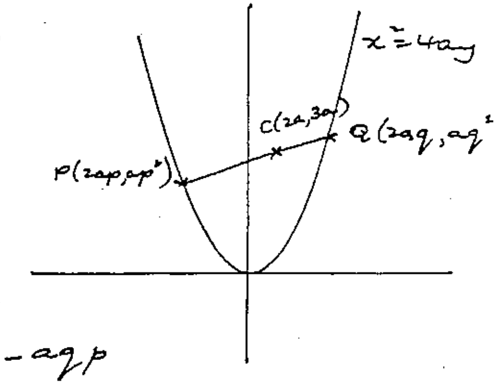
$\theta = 0°, 360°$

also check $\theta = 180°$ as a soln.

$2(0) - 1 \ne 1 \quad \therefore \theta \ne 180°$.

⑤ (b) (i) grad. PQ, $m = \frac{a(q^2-p^2)}{2a(q-p)}$

$$= \frac{q+p}{2}$$



$\therefore$ Equ. PQ:

$$y - ap^2 = \frac{p+q}{2}(x - 2ap)$$

$$= \tfrac12(p+q)x - ap^2 - aqp$$

$$\underline{y = \tfrac12(p+q)x - apq}$$

If $C(2a, 3a)$ satisfies,

$$3a = \tfrac12(p+q)2a - apq$$

$$3a = ap + aq - apq$$

i.e. $\underline{pq = p + q - 3}$ $\quad$ Q.E.D.

(ii) $M = \left(\frac{2a(p+q)}{2},\ \frac{a(p^2+q^2)}{2}\right)$

$$= \left(a(p+q),\ \tfrac{a}{2}(p^2+q^2)\right)$$

from (i) $pq + 3 = p + q$ and $p^2 + q^2 = (p+q)^2 - 2pq$
$= (pq+3)^2 - 2pq$.

$$\therefore M = \left(a(pq+3),\ \tfrac{a}{2}\left[(pq+3)^2 - 2pq\right]\right)$$

$$= \underline{\left(a(pq+3),\ \tfrac{a}{2}(pq+3)^2 - apq\right)} \quad \text{Q.E.D.}$$

(iii) Since at M,

$x = a(pq+3)$

$pq + 3 = \frac{x}{a}$

$pq = \frac{x}{a} - 3$

Then $y = \frac{a}{2}\left(\frac{x}{a}\right)^2 - (x - 3a)$

$= \frac{x^2}{2a} - x + 3a$.

$2ay = x^2 - 2ax + 6a^2$

$x^2 - 2ax = 2ay - 6a^2$

(6) (a) X : 7B, 3G

Y : 2B, 8G

(i) (α) $P(BB) = \frac{7}{10} \times \frac{2}{10} = \underline{\frac{7}{50}}$

(β) $P(B,G) + P(G,B) = \frac{7}{10} \times \frac{8}{10} + \frac{3}{10} \times \frac{2}{10}$

$= \frac{56+6}{100} = \underline{\frac{31}{50}}$

(ii) (α) $P(X, BB) + P(Y, BB) = \frac{1}{2} \times \frac{7}{10} \times \frac{6}{9} + \frac{1}{2} \times \frac{2}{10} \times \frac{1}{9}$

$= \frac{42+2}{180} = \underline{\frac{11}{45}}$

(β) $P(X, BG) + P(X, GB) + P(Y, BG) + P(Y, GB)$

$= \frac{1}{2} \times \frac{7}{10} \times \frac{3}{9} + \frac{1}{2} \times \frac{3}{10} \times \frac{7}{9} + \frac{1}{2} \times \frac{2}{10} \times \frac{8}{9} + \frac{1}{2} \times$

$= \frac{21+21+16+16}{180} = \underline{\frac{37}{90}}$

4

(b) (i) $\tan\theta = \frac{YZ}{y}$

$\therefore YZ = y\tan\theta$

$\cos\theta = \frac{y}{XY}$

$\therefore XY = \frac{y}{\cos\theta} = y\sec\theta$

$P = XZ + YZ + XY$

$= y + y\tan\theta + y\sec\theta$

$= y(1 + \tan\theta + \sec\theta)$ Q.E.D.

(ii) $y = 10$, $\frac{d\theta}{dt} = 0.2$

$\frac{dP}{dt} = \frac{dP}{d\theta} \times \frac{d\theta}{dt}$

$\frac{dP}{d\theta} = 10(\sec^2\theta + \sec\theta\tan$

When $\theta = \frac{\pi}{6}$,

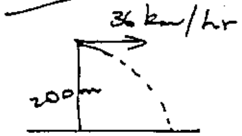
$\frac{dP}{d\theta} = 10\left(\left(\frac{2}{\sqrt{3}}\right)^2 + \frac{2}{\sqrt{3}} \cdot\right.$

$= 20$

i.e. $\frac{dP}{dt} = 20 \times 0.2$

$= 4$ cm/sec

Q.7



$\ddot{y} = -10$

$\dot{y} = -10t + c_1$

$t = 0, \dot{y} = 0, \therefore c_1 = 0$

$\dot{y} = -10t$

$y = -5t^2 + c_3$

$t = 0, y = 200, \therefore c_3 = 200$

$y = -5t^2 + 200$

$\ddot{x} = 0$

$\dot{x} = c_2$

$\dot{x} = 36$ km/hr

$= \frac{36 \times 1000}{60 \times 60}$

$\dot{x} = 10$ m/sec

$x = 10t + c_4$

$t = 0, x = 0, c_4 =$

$x = 10t$

(a)(i) Time for $y = 0$

$0 = -5t^2 + 200$

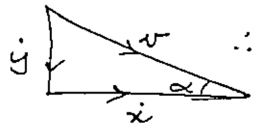
$t^2 = 40$

$t = \pm 2\sqrt{10}$ $(t > 0)$

$\therefore$ drop reaches ground in $\underline{2\sqrt{10}\text{ secs.}}$

(ii) When $t = 2\sqrt{10}$

$\dot{y} = -20\sqrt{10}$, $\dot{x} = 10$.



$\therefore v^2 = (20\sqrt{10})^2 + 10^2$

$= 4100$

$v = 10\sqrt{41}$

$\doteqdot \underline{64.03 \text{ m/sec}}$

$\tan\alpha = \left|\frac{\dot{y}}{\dot{x}}\right|$ for acute ∠

$= \frac{20\sqrt{10}}{10}$

$= 2\sqrt{10}$

$\underline{\alpha \doteqdot 81^\circ}$

(iii) Speed = 20 m/sec.

$\therefore \dot{y} = -20\sqrt{10}$, $\dot{x} = 20$

$\tan\alpha = \sqrt{10}$

$\underline{\alpha = 72^\circ 27'}$

⑧ (a) $a = (4x - 2x^3)\ m/s^2$

when $t=0,\ v=0,\ x=2.$

(i) $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = 4x - 2x^3$

$$\frac{1}{2}v^2 = 2x^2 - \frac{x^4}{2} + c$$

When $x = 2, v = 0$

i.e. $0 = 8 - 8 + c$

$c = 0$

i.e. $\underline{v^2 = 4x^2 - x^4}$

(ii) (α) For $v=0$, $0 = 4x^2 - x^4$

$= x^2(4 - x^2)$

i.e. $x = 0, \pm 2$

Since starts at $x=2$, next comes to rest at $\underline{x=0}$

(β) $\underline{\text{When } x=0,\ a=0}$

④

⑧ (b) (i) $\underline{y = 4}$

(ii) at A, $y=0$

i.e. $e^{2x} - 5e^x + 4 = 0$

let $m = e^x$

Then $m^2 - 5m + 4 = 0$

$(m-4)(m-1) = 0$

$m = 4, 1$

$e^x = 4, 1$

$x = \ln 4, \ \ln 1 (=0)$

$\therefore$ $\underline{\text{At A } (\ln 4, 0)}$

At B, $\frac{dy}{dx} = 0$

i.e. $2e^{2x} - 5e^x = 0$

$e^x(2e^x - 5) = 0$

$e^x = 0, \ 5/2$

$\therefore$ At B, $x = \ln 5/2$, $y = \frac{25}{4} - 5\cdot\frac{5}{2} + 4$

$= -9/4$

i.e. $\underline{B\left(\ln 5/2, -\frac{9}{4}\right)}$

(iii) At $(0,0)$ $\frac{dy}{dx} = 2 - 5 = -3$

$\therefore$ Grad. of normal at $(0,0) = 1/3$

Eqn. of normal : $y = \frac{1}{3}x$

$\underline{x - 3y = 0}$

(iv) At N, $\begin{cases} y = \frac{1}{3}x \\ y = e^{2x} - 5e^x + 4 \end{cases}$

$\therefore \ \frac{1}{3}x = e^{2x} - 5e^x + 4$

$x = 3e^{2x} - 15e^{x} + 12$

$3e^{2x} - 15e^{x} + 12 - x = 0$

let $f(x) = 3e^{2x} - 15e^{x} + 12 - x$.

$f(1.5) = 3e^{3} - 15e^{1.5} + 12 - 1.5$

$f'(x) = 6e^{2x} - 15e^{x} - 1$

$f'(1.5) = 6e^{3} - 15e^{1.5} - 1$

Approx. to $x = 1.5 - \dfrac{f(1.5)}{f'(1.5)}$

$= \underline{1.432}$